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SEQUENTIAL LAGRANGIAN DUALITY FOR ABSTRACT CONVEX PROGRAMS WITHOUT REGULARITY CONDITION

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Abstract

In this paper it is shown that in the absence of differentiability and a regularity condition a sequential Lagrangian duality and saddle-point conditions, and sequential stability results hold for abstract convex programs. As an application a sequential Lagrangian duality result is obtained for convex semidefinite programs. The results, which are expressed in terms of a limiting Lagrangian function, yield the known duality results under a simple closed cone condition that is much weaker than the well known constraint qualifications.

1 Introduction and Preliminaries

Consider the cone-convex programming model problem

$$(P) \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & g(x) \in -S, \end{array}$$

where X is a reflexive Banach space, Z is a Banach space, S is a closed convex cone in Z, which does not necessarily have non-empty interior, the mapping $f : X \to I\!R$ is a continuous convex function and $g : X \to Z$ is a continuous and S-convex function.

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Let A be the feasible set of (P), that is $A := \{x \in X \mid g(x) \in -S\}$. A particular case of (P) is the convex semidefinite programming model problem

$$
\begin{array}{lll}
(SDP) & \underset{x \in \mathbb{R}^m}{\text{minimize}} & f(x) \\
 & \text{subject to} & F_0 + \displaystyle\sum_{i=1}^{m} F_i x_i \succeq 0
\end{array}
$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function and, for $i = 0, \ldots, m$, $F_i \in S_n$, the space of symmetric $n \times n$ matrices and $\succeq$ denotes the Löwner partial order of S_n, that is, for $M, N \in S_n$, $M \succeq N$ if and only if $(M - N)$ is positive semidefinite. This model problem is increasingly becoming a basic modelling tool for many important applications in control and signal processing, eigenvalue optimization, and combinatorial optimization (see [17, 18, 19, 22]). It is known that in the absence of a regularity condition (such as a generalized Slater's constraint qualification or a closed cone condition) the standard Lagrangian results do not hold for (SDP) or (P).

The present work was motivated by the recent research in optimization which has shown that sequential forms of the Lagrange multiplier conditions hold in the absence of a constraint qualification (see [10, 21]). Such conditions, expressed in terms of convex subdifferentials at nearby points of a minimizer were obtained by applying sequential convex calculus [21]. More recently authors [10] have given another sequential form of such conditions, where the optimality conditions were given in terms of $\epsilon-$subdifferentials and the convex subdifferentials at the minimizer using the Hahn Banach separation Theorem.

The purpose of this paper is to show that sequential Lagrangian results hold for abstract convex programming problems without any regularity condition. This is achieved by the aid of the powerful conjugate analysis coupled with the geometry of hyperplane separation of convex sets. It is shown that in the absence of differentiability and a regularity condition, a perfect duality holds between (P) and a sequential form of the Lagrangian dual problem. As a special case, corresponding sequential results are obtained for convex semidefinite program (SDP). Sequential Lagrangian saddle-point conditions as well as sequential stability results are also given. These conditions are expressed in terms of the lower limit of a sequential standard Lagrangian function. Our results differ from other Lagrangian results [1, 2, 3, 13, 16] established in the literature, where limiting modified Lagrangian functions are used. The significance of our results is that they yield the standard Lagrangian results under a simple closed cone condition that is much weaker than the well known constraint qualifications. Closed cone conditions are generally employed to study mathematical programs involving sublinear or linear cone constraints. Our results show that a natural extension of such a condition, involving epigraphs of conjugate functions, plays a key role also in abstract convex programs (P). Numerical examples are discussed to illustrate the significance of the sequential results. The main results are presented in reflexive Banach spaces X in order to avoid the use of nets.

However, the results and their proofs continue to hold in locally convex topological spaces.

The continuous dual space of X will be denoted X' and will be endowed with the weak* topology. The (positive) polar of the cone $S \subseteq Z$ is the cone $S^+ = \{\theta \in Z' : \theta(k) \geq 0,\ \forall k \in S\}$. For a subset $D \subset X$, the **indicator function** of D, denoted by δ_D, is defined by $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \in X \setminus D$. Note that if D is a nonempty, closed and convex set then $\delta_D(.)$ is proper convex and lower semi-continuous (l.s.c.) on X. The **support function** of D, denoted by σ_D, is

$$\sigma_D(u) := \sup_{x \in D} u(x), \quad u \in X'.$$

Let $f : X \to I\!R \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then, the **conjugate** function [20] of f, $f^* : X' \to I\!R \cup \{+\infty\}$ is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \text{ dom } f\}$$

where the domain of f, dom f, is given by

$$\text{dom } f = \{x \in X \mid f(x) < +\infty\}.$$

The epigraph of f, epi f, is defined by

$$\text{epi } f = \{(x, r) \in X \times I\!R \mid x \in \text{ dom } f,\ f(x) \leq r\}.$$

If $\tilde{f}(x) = f(x) - k$, $x \in X$, $k \in I\!R$, then epi $\tilde{f}^* =$ epi $f^* + (0, k)$.

The mapping $g : X \to Z$ is **S-convex** if for every u, $v \in X$ and every $t \in [0, 1]$,

$$g(tu + (1-t)v) - tg(u) - (1-t)g(v) \in -S.$$

The weak*convergence of the sequence $\{w_n\}$ of X' to w will be denoted by $w_n \to_* w$. The following lemma plays a key role in establishing sequential Lagrangian results. The lemma was recently given in finite dimensions and then extended to infinite dimensions in [9, 11].

Lemma 1.1 *[9, 11] If $g : X \to Z$ is a continuous and S-convex mapping then each of the following statements holds:*

(i) $g^{-1}(-S) \neq \emptyset \Longleftrightarrow (0, -1) \notin cl\left(\bigcup_{\lambda \in S^+} \text{epi}(\lambda \text{g})^*\right)$.

(ii) $g^{-1}(-S) \neq \emptyset \Longrightarrow \text{epi}\sigma_{\text{A}} = \text{cl}\left(\bigcup_{\lambda \in \text{S}^+} \text{epi}(\lambda \text{g})^*\right)$.

2 Dual Characterizations of Convex Systems

In this section we establish various dual sequential characterizations of cone-convex inequality systems without any regularity condition. We derive such characterizations by employing the epigraph based conjugate analysis and the Hahn-Banach separation theorem. These dual sequential characterizations are then applied to derive sequential Lagrangian results for (P).

Theorem 2.1 *If the system $g(x) \in -S$ is consistent then for any $\alpha \in R$, then the following statements are equivalent:*

(i) $g(x) \in -S \Longrightarrow f(x) \geq \alpha$,

(ii) $(0, -\alpha) \in epi f^* + cl(\cup_{\lambda \in S^+} epi(\lambda g)^*)$.

(iii) $(\exists (\lambda_n) \subset S^+)$ $(\forall x \in X)$ $f(x) + \liminf_{n \to \infty} \lambda_n g(x) \geq \alpha$.

Proof. [(i)$\Longleftrightarrow$(ii)]. Let $A := \{x \in X : g(x) \in -S\}$. Then A is a closed convex subset of X since g is continuous and S-convex. By Lemma 1.1, $\text{cl}\left(\bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^*\right) = \text{epi}\sigma_A$. Hence (ii) means that there exists $(u, \beta) \in \text{epi}\,\sigma_A$ such that $-(u, \alpha + \beta) \in$ epi f^*. This implies that if $x \in A$ then $u(x) \leq \sigma_A\ (u) \leq \beta$ and $-\beta - \alpha \geq -u(x) - f(x)$. So, $f(x) \geq \alpha$ and (i) follows.

Conversely, let $H = \{x \in X : h(x) \geq 0\}$ where $h(x) = f(x) - \alpha$. Clearly, H is a closed convex subset of X and (i) means that $A \subset H$. Now, $A \subseteq H$ if and only if $h + \delta_A \geq 0$. It follows from the definition of epi $(h + \delta_A)^*$ and the inequality $h + \delta_A \geq 0$ that

$$0 \in \text{epi }(h + \delta_A)^*.$$

Since h is a real-valued continuous convex function and A is non-empty, it follows that

$$(h + \delta_A)^* = h^* \oplus \delta_A^*$$

where $\oplus$ denotes the inf-convolution (see Hiriart-Urruty [7, 8]). Moreover for each $x \in dom(h^* \oplus \delta_A^*)$, there exist $x_1, x_2 \in X$ and $x_1 + x_2 = x$ such that

$$(h^* \oplus \delta_A^*)(x) = h^*(x_1) + \delta_A^*(x_2).$$

Now it is easy to see that (see [11])

$$\text{epi }(h^* \oplus \delta_A^*) = \text{ epi } h^* + \text{epi } \delta_A^*.$$

As $\delta_A^* = \sigma_A$, from Lemma 1.1 we get

$$\text{epi } \delta_A^* = \text{cl}\left(\bigcup_{\lambda \in S^+} \text{epi }(\lambda g)^*\right).$$

Thus, if $A \subseteq H$ then

$$0 \in \operatorname{epi} h^* + \operatorname{cl}\left(\bigcup_{\lambda \in S^+} \operatorname{epi} \, (\lambda g)^* \right).$$

This is equivalent to (ii).

[(ii) $\Rightarrow$ (iii)] Suppose that (ii) holds. Then there exist $v \in X', \beta \geq 0$ and sequences $(v_n)_n \subset X', (\alpha_n) \subset R_+, (\lambda_n) \subset S^+$ such that

$$(-v, -f^*(v) - \beta - \alpha) = \lim_{n\to\infty} (v_n, (\lambda_n g)^*(v_n) + \alpha_n).$$

This gives

$$-v = \lim_{n\to\infty} v_n$$
$$-f^*(v) - \beta - \alpha = \lim_{n\to\infty} [(\lambda_n g)^*(v_n) + \alpha_n].$$

Rewriting the last inequality in the form

$$f^*(v) = - \lim_{n\to\infty} [(\lambda_n g)^*(v_n) + \alpha_n] - \beta - \alpha$$

we have, by definition of $f^*(v)$,

$$\langle v, x \rangle - f(x) \leq - \lim_{n\to\infty} [(\lambda_n g)^*(v_n) + \alpha_n] - \beta - \alpha, \quad \forall x \in X,$$

or equivalently, for all $x \in X$,

$$\begin{aligned} f(x) &\geq \lim_{n\to\infty} [(\lambda_n g)^*(v_n) + \alpha_n] + \langle v, x \rangle + \beta + \alpha \\ &\geq \lim_{n\to\infty} [(\lambda_n g)^*(v_n) + \alpha_n] + \langle v, x \rangle + \alpha \end{aligned} \tag{1}$$

Note that $\alpha_n \geq 0$ for all $n \in N$, we have

$$(\lambda_n g)^*(v_n) + \alpha_n \geq (\lambda_n g)^*(v_n), \quad \forall n \in N$$

and

$$\lim_{n\to\infty} [(\lambda_n g)^*(v_n) + \alpha_n] \geq \limsup_{n\to\infty} (\lambda_n g)^*(v_n).$$

Note also that for all $x \in X$,

$$\begin{aligned} \limsup_{n\to\infty} [(\lambda_n g)^*(v_n)] &= \limsup_{n\to\infty} [\sup_{y\in Y} \{\langle v_n, y \rangle - \lambda_n g(y)\}] \\ &\geq \limsup_{n\to\infty} [\langle v_n, x \rangle - \lambda_n g(x)] \end{aligned}$$

and

$$\begin{aligned} \limsup_{n\to\infty} [-\lambda_n g(x)] &\leq \limsup_{n\to\infty} [\langle v_n, x \rangle - \lambda_n g(x)] + \limsup_{n\to\infty} \langle -v_n, x \rangle \\ &\leq \limsup_{n\to\infty} [\langle v_n, x \rangle - \lambda_n g(x)] + \langle v, x \rangle . \end{aligned}$$

Combining this and (1), we get

$$f(x) \geq \limsup_{n\to\infty}[-\lambda_n g(x)] + \alpha, \quad \forall x \in X,$$

which implies that

$$f(x) + \liminf_{n\to\infty} \lambda_n g(x) \geq \alpha, \quad \forall x \in X.$$

[(iii) $\Rightarrow$ (i)] Suppose that there exists $(\lambda_n) \subset S^+$ such that (iii) holds. Note that if $x \in X$ with $-g(x) \in S$ then $\lambda_n g(x) \leq 0$ for all $n \in N$. It follows from (iii) that if $-g(x) \in S$ then

$$f(x) \geq f(x) + \liminf_{n\to\infty} \lambda_n g(x) \geq \alpha.$$

$\square$

The method of proof of [(i)$\Longleftrightarrow$ (ii)] is similar to the one given in [9] whereas (iii) gives a sequential condition of the Lagrangian type which yields sequential Lagrangian duality results for (P). In passing, observe that a sequential condition of the type (iii) which was shown to be equivalent to (i) in the case where $\alpha = 0$ in [5]. However, this equivalence yields a corresponding non-asymptotic condition under a strong closedness condition (see the remark following Theorem 2.2).

We now see that a simple non-sequential dual characterization holds under a general *closed cone condition* that the convex cone $\bigcup_{\lambda\in S^+} epi(\lambda g)^*$ is weak* closed. This constraint qualification holds under the Robinson Regularity Condition, that is, $0 \in$ core $(g(X) + S)$ (see [15]), in the case where Z is a Banach space, or the generalized Slater condition that *int* S is non-empty and $-g(x_0) \in$ int S (see [12, 14]). The closed cone constraint qualification is weaker than the Robinson condition. For instance, let $X = Z = \mathbb{R}$, $S = \mathbb{R}_+$, and $g(x) := 0$ if $x \leq 0$ and $g(x) := x$ if $x > 0$. Clearly, both the Slater and Robinson conditions do not hold. On the other hand, since g is sublinear and it is easy to see that for any $\lambda \geq 0$,

$$epi(\lambda g)^* = \partial(\lambda g)(0) \times \mathrm{R}_+ = [0, \lambda] \times \mathbb{R}_+,$$

and

$$\bigcup_{\lambda\in\mathbb{R}_+} epi(\lambda g)^* = \mathbb{R}^2_+$$

is a closed subset in $\mathbb{R}^2$. Hence, the closed cone condition holds.

Theorem 2.2 *If the system $g(x) \in -S$ is consistent and the convex cone $\cup_{\lambda\in S^+} epi(\lambda g)^*$ is w^*-closed then for any $\alpha \in R$, the following statements are equivalent*

(i) $g(x) \in -S \Longrightarrow f(x) \geq \alpha$,

(ii) $(0, -\alpha) \in epi f^* + \cup_{\lambda\in S^+}\mathrm{epi}(\lambda g)^*$.

(iii) $(\exists \lambda \in S^+)$ $(\forall x \in X)$ $f(x) + \lambda g(x) \geq \alpha$.

Proof. By Theorem 2.1, it is clear that if convex cone $\cup_{\lambda\in S^+} epi(\lambda g)^*$ is w*-closed then (i) is equivalent to (ii). Assume now that (ii) holds. Then there exist $v \in X',\ \beta \geq 0,\ \lambda \in S^+$ such that

$$(-v, -f^*(v) - \beta - \alpha) \in \text{epi}(\lambda g)^*.$$

This means that

$$(\lambda g)^*(-v) \leq -f^*(v) - \beta - \alpha,$$

or equivalently,

$$\langle -v, x\rangle - \lambda g(x) \leq -f^*(v) - \beta - \alpha,\ \forall x \in X.$$

The last inequality implies

$$f^*(v) \leq f^*(v) + \beta \leq \langle v, x\rangle + \lambda g(x) - \alpha,\ \forall x \in X,$$

which, by definition of $f^*(v)$, gives us

$$\langle v, x\rangle - f(x) \leq \langle v, x\rangle + \lambda g(x) - \alpha,\ \forall x \in X.$$

Hence,

$$f(x) + \lambda g(x) \geq \alpha,\ \forall x \in X.$$

We have proved (ii) implies (iii). Observe that if (iii) holds and $x \in X$ with $g(x) \in -S$ then $\lambda g(x) \leq 0$, and hence, $f(x) \geq f(x) + \lambda g(x) \geq \alpha$, which means that (iii) implies (i). $\square$

Note that the equivalence between (i) and (iii) of the previous Theorem was established in [5] under the regularity condition that the set

$$C := \{v \in \mathbb{R}^X\ :\ \exists \lambda \in S^+ \text{ such that } v(x) + \lambda g(x) \geq 0, \forall x \in X\}$$

is closed in the product topology of $\mathbb{R}^X$. It is easy to see that if C is closed in the product topology of $\mathbb{R}^X$ then $(\bigcup_{\lambda\in S^+} epi(\lambda g)^*)$ is w^*-closed in $X' \times \mathbb{R}$. To see this, let $((v_n, \alpha_n)) \subset \bigcup_{\lambda\in S^+} \text{epi}(\lambda g)^*$and $(v_n, \alpha_n) \to_* (v, \alpha)$. Then there exists $\lambda_n \in S^+$ such that for each n, $(v_n, \alpha_n) \in \text{epi}(\lambda_n g)^*$. This condition means that

$$v_n(x) - \lambda_n g(x) \leq \alpha_n,\ \forall x \in X,$$

or equivalently,

$$-v_n(x) + \alpha_n + \lambda_n g(x) \geq 0,\ \forall x \in X.$$

Since $(v_n, \alpha_n) \to_* (v, \alpha)$, $v_n(x) \to v(x)$ for all $x \in X$ and $\alpha_n \to \alpha$ as $n \to \infty$. If we set $\bar{v}_n(x) := -v_n(x) + \alpha_n$ then $\bar{v}_n(x) \to \bar{v}(x) := -v(x) + \alpha$. This means that $\bar{v}_n \to \bar{v}$ in the product topology of $\mathbb{R}^X$.

Now, as C is closed, we have $\bar{v} \in C$. By definition of C, there exists $\lambda \in S^+$ such that

$$-v(x) + \alpha + \lambda g(x) \geq 0, \ \forall x \in X,$$

or

$$v(x) - \lambda g(x) \leq \alpha, \ \forall x \in X.$$

The last inequality implies that

$$(v, \alpha) \in epi(\lambda g)^* \subset \bigcup_{\lambda \in S^+} \mathrm{epi}(\lambda g)^*.$$

However, in general the w^*-closedness of $(\bigcup_{\lambda \in S^+} epi(\lambda g)^*)$ does not necessarily imply that the set C is closed in the product topology. To see this, let $X = \mathbb{R}$, $S = \mathbb{R}_+$, and $g(x) := 0$ if $x \leq 0$ and $g(x) := x$ if $x > 0$. We have seen in the remark that follows Theorem 2.1 that

$$\bigcup_{\lambda \in S^+} \mathrm{epi}(\lambda g)^* = \mathbb{R}^2_+,$$

and this set is closed in $\mathbb{R}^2$ which means that the closed cone condition holds.

However, if we take $v_n(x) := \frac{1}{n}$ if $x \leq 0$ and $v_n(x) := \frac{1}{n} - \sqrt{x}$ if $x > 0$, then it is clear that $v_n \in C$. Let $v(x) := 0$ if $x \leq 0$ and $v(x) := -\sqrt{x}$ if $x > 0$. We have

$$v_n(x) \to v(x), \ \forall x \in \mathbb{R},$$

which means that $v_n \to v$ in the product topology of $\mathbb{R}^{\mathbb{R}}$. Since it is impossible to find $\lambda \geq 0$ such that

$$-\sqrt{x} + \lambda x \geq 0, \ \text{for all } x > 0,$$

x does not belong to C. Thus, C is not closed in the product topology of $\mathbb{R}^{\mathbb{R}}$.

3 Sequential Lagrangian Duality

In this section we develop sequential Lagrangian results, including a perfect duality result, for the convex program (P). We then apply the result to obtain a corresponding duality result for the semidefinite program (SDP).

Sequential Lagrangian Minimax

Recall that for the Problem (P), the standard Lagrange function is defined by

$$L(x, \mu) = f(x) + \mu g(x), \ x \in X, \ \mu \in S^+.$$

The following sequential minimax equality holds for the Lagrangian function.

Theorem 3.1 (Sequential Lagrangian Minimax). *For the Problem (P), assume that the feasible set is nonempty. Then there exists a sequence* $(\bar{\lambda}_n) \subset S^+$ *such that*

$$\inf_{x\in X} \sup_{(\lambda_n)\subset S^+} \liminf_{n\to\infty} L(x,\lambda_n) = \sup_{(\lambda_n)\subset S^+} \inf_{x\in X} \liminf_{n\to\infty} L(x,\lambda_n) = \liminf_{n\to\infty} L(x,\bar{\lambda}_n) = \inf(P). \tag{2}$$

Proof. If $\inf(P) = -\infty$ then it is easy to see that (2) holds for any sequence $(\bar{\lambda}_n) \subset S^+$. Since the feasible set of (P) is nonempty, we have $\inf(P) < +\infty$.

Assume that $\alpha := \inf(P)$ is finite. Then we have

$$g(x) \in -S,\ x \in X \quad \text{implies} \quad f(x) \geq \alpha.$$

By Theorem 2.1, this is equivalent to the condition that there is a sequence $(\bar{\lambda}_n) \subset S^+$ such that

$$f(x) + \liminf_{n\to\infty} \bar{\lambda}_n g(x) \geq \alpha,\ \forall x \in X,$$

which proves that

$$\inf_{x\in X} \liminf_{n\to\infty} L(x,\bar{\lambda}_n) \geq \inf(P) = \alpha. \tag{3}$$

On the other hand, it is clear that

$$\sup_{(\lambda_n)\subset S^+} \liminf_{n\to\infty} L(x,\lambda_n) \geq \inf_{x\in X} \liminf_{n\to\infty} L(x,\bar{\lambda}_n) \text{ and}$$

$$\inf(P) \geq \inf_{x\in X} \sup_{\Lambda\subset S^+} \liminf_{n\to\infty} L(x,\lambda_n). \tag{4}$$

Note that for each sequence $(\lambda_n) \subset S^+$ and for each $y \in X$, we have

$$f(y) + \liminf_{n\to\infty} \lambda_n g(y) \geq \inf_{x\in X} \left\{ f(x) + \liminf_{n\to\infty} \lambda_n g(x) \right\}.$$

Hence,

$$\sup_{(\lambda_n)\subset S^+} \left\{ f(y) + \liminf_{n} \lambda_n g(y) \right\} \geq \sup_{(\lambda_n)\subset S^+} \inf_{x\in X} \left\{ f(x) + \liminf_{n\to\infty} \lambda_n g(x) \right\}.$$

Now by the arbitrariness of y, we get

$$\inf_{y\in X} \sup_{(\lambda_n)\subset S^+} \liminf_{n\to\infty} L(x,\lambda_n) \geq \sup_{(\lambda_n)\subset S^+} \inf_{x\in X} \liminf_{n\to\infty} L(x,\lambda_n).$$

Combining this with (4), we get

$$\inf(P) \geq \inf_{x\in X} \sup_{(\lambda_n)\subset S^+} \liminf_{n\to\infty} L(x,\lambda_n) \geq \sup_{(\lambda_n)\subset S^+} \inf_{x\in X} \liminf_{n\to\infty} L(x,\lambda_n) \geq \inf_{x\in X} \liminf_{n\to\infty} L(x,\bar{\lambda}_n).$$

The desired equality (2) then follows from the last inequality and (3). $\square$

In the following Proposition we derive the standard Lagrangian duality result under the closed cone condition.

Proposition 3.1 *For the problem (P), assume that the feasible set is non-empty. If the closed cone constraint qualification holds then there exists* $\lambda_0 \in S^+$ *such that*

$$\inf_{x\in X} \sup_{\lambda\in S^+} L(x,\lambda) = \sup_{\lambda\in S^+} \inf_{x\in X} L(x,\lambda) = \inf_{x\in X} L(x,\lambda_0) = \inf(P).$$

Proof. If $\inf(P) = -\infty$ then the conclusion of the Proposition holds for any $\lambda_0 \in S^+$. Assume that $\alpha := \inf(P)$ is finite. It follows that

$$g(x) \in -S \implies f(x) \geq \alpha.$$

If the closed cone constraint qualification holds then by Theorem 2.2, there exists $\lambda_0 \in S^+$ such that

$$f(x) + \lambda_0 g(x) \geq \alpha = \inf(P),\ \forall x \in X.$$

This implies

$$\inf_{x\in X} L(x,\lambda_0) \geq \inf(P).$$

The conclusion of the Proposition now follows by the same argument as in that of the proof of Theorem 3.1, simply replace $(\lambda_n) \subset S^+$ by $\lambda \in S^+$ and $(\bar{\lambda}_n)$ by λ_0. $\square$

We now define the **sequential Lagrangian dual problem** for $(P\ )$ as the following

$$(\widetilde{D}) \qquad \max_{(\lambda_n)\subset S^+} \ \inf_{x\in X} \liminf_{n\to\infty} L(x,\lambda_n).$$

Recall that the standard Lagrangian dual problem of (P) is

$$(D) \qquad \max_{\lambda\in S^+} \ \inf_{x\in X} L(x,\lambda).$$

For the sake of convenience, in what follows we use the symbols $[P]$, $[\widetilde{D}]$, and $[D]$ to indicate the values of the Problems (P), $(\widetilde{D})$, and (D), respectively.

As a consequence from the proofs of Theorem 3.1 (Proposition 3.1) we get $[\widetilde{D}] \leq [P]$ ($[D] \leq [P]$) which means that the weak duality holds for (P) and $(\widetilde{D})$ (for (P) and (D), respectively). It is clear that $[D] \leq [\widetilde{D}]$.

Proposition 3.2 *For the problem (P), assume that the feasible set is non-empty. If the closed cone constraint qualification holds then*

$$[P] = [\widetilde{D}] = [D].$$

Moreover, both the Problems (D) *and* $(\widetilde{D})$ *attain their maximums.*

Proof. The conclusion that $[P] = [\widetilde{D}] = [D]$ is a direct consequence of Theorem 3.1 and Proposition 3.1. Also, from the proof of Theorem 3.1, there exists $\Lambda := (\lambda_n) \subset S^+$ such that $\Phi(\Lambda) = \inf_{x\in X} \liminf_{n\to\infty} L(x, \lambda_n) = [P]$. By the weak duality result, $(\widetilde{P})$ attains its maximum at (λ_n). Similarly, the Problem (P) also attains its maximum over S^+. □

We now show that in the absence of the closed cone condition a perfect duality relationship between (P) and $(\widetilde{D})$ holds.

Perfect Duality

Recall that a minimization (maximization) problem is said to be **consistent** if its value is not equal to $+\infty$ ($-\infty$, respectively). We say that the two problems are in **perfect duality** (see [1, 6]) if:

(a) when one problem has finite value, the other has the same value,

(b) when both problems are consistent, they share the same value.

The following theorem is the central result of this section.

Theorem 3.2 *The perfect duality holds between problems (P) and $(\widetilde{D})$.*

Proof. By Theorem 3.1, if (P) and $(\widetilde{D})$ are consistent then $[P\,] = [\,\widetilde{D}]$ and if $[P]$ is finite then we also have $[P]$ =$[\widetilde{D}]$. It remains to prove that this equality still holds if $[\tilde{D}]$ is finite. We will prove that if $[\widetilde{D}]$ is finite then the Problem (P) is consistent. The conclusion then follows from Theorem 3.1.

Suppose that (P) is not consistent then the set $A = g^{-1}(-S)$ is empty. By Lemma 1.1,

$$(0, -1) \in \mathrm{cl}\left(\bigcup_{\lambda\in S^+} \mathrm{epi}(\lambda \mathrm{g})^* \right).$$

So for each $n \in \mathbf{N}$, there exist $\lambda_n \in S^+$, $v_n \in X'$, $\alpha_n \in I\!R$ such that

$$(v_n, \alpha_n) \in \mathrm{epi}\ (\lambda_n g)^*, \quad 0 = \lim_{n\to\infty} v_n, \quad -1 = \lim_{n\to\infty} \alpha_n. \tag{5}$$

It follows from (5) that $(\lambda_n g)^*(v_n) \le \alpha_n$ for each n, which means that

$$\langle v_n, x\rangle - (\lambda_n g)(x) \le \alpha_n, \quad \forall x \in X.$$

Then for each $x \in X$,

$$\limsup_{n\to\infty}(-\lambda_n g)(x) + \liminf_{n\to\infty} \langle v_n, x\rangle \le \limsup_{n\to\infty}[(-\lambda_n g)(x) + \langle v_n, x\rangle] \le -1.$$

Since $\liminf_{n\to\infty} \langle v_n, x\rangle = \lim_{n\to\infty} \langle v_n, x\rangle = 0$, we get

$$\liminf_{n\to\infty} \lambda_n g(x) \geq 1,\ \forall x \in X. \tag{6}$$

Let $\bar{\lambda}_n = n\lambda_n$. Then $\bar{\lambda}_n \in S^+$ for all $n \in \mathbf{N}$. Now by (6)

$$\liminf_{n\to\infty} \bar{\lambda}_n g(x) = +\infty,\ \forall x \in X. \tag{7}$$

In fact, if $\liminf_{n\to\infty} \bar{\lambda}_n g(\bar{x}) = \beta < +\infty$ for some $\bar{x} \in X$ then we first claim that $\beta > -\infty$. By the definition of liminf, there is n_0 such that for any $n \geq n_0$, $\lambda_n g(\bar{x}) > \frac{1}{2}$. So without loss of generality we can assume that

$$\bar{\lambda}_n g(\bar{x}) = n\lambda_n g(\bar{x}) \geq \frac{1}{2},\ \forall n \in \mathbf{N}.$$

Hence,

$$\liminf_{n\to\infty} \bar{\lambda}_n g(\bar{x}) = \beta \geq \frac{1}{2} > -\infty.$$

It follows that there exists a subsequence $(\bar{\lambda}_{n_k})_k$ of the sequence (λ_n) such that

$$\lim_{k\to\infty} \bar{\lambda}_{n_k} g(\bar{x}) = \beta \in \mathbb{R}.$$

Therefore, there is $k_0 \in \mathbf{N}$ large enough such that for $k \geq k_0$,

$$|\bar{\lambda}_{n_k} g(\bar{x}) - \beta| = |n_k \lambda_{n_k} g(\bar{x}) - \beta| < 1.$$

The last inequality implies that $\lim_{k\to\infty} \lambda_{n_k} g(\bar{x}) = 0$. This contradicts $\liminf_{n\to\infty} \lambda_n g(x) \geq 1$. Consequently, (7) holds and hence,

$$\liminf_{n\to\infty} L(x, \lambda_n) = f(x) + \liminf_{n\to\infty} \bar{\lambda}_n g(x) = +\infty,\ \forall x \in X.$$

This implies

$$[\widetilde{D}] = \sup_{(\lambda_n)\subset S^+} \inf_{x\in X} \liminf_{n\to\infty} L(x, \lambda_n) = +\infty.$$

The proof is complete. $\square$

Semidefinite Duality

Consider the following convex semidefinite programming problem

$$\begin{array}{lll} (SDP) & \underset{x\in\mathbb{R}^m}{\text{minimize}} & f(x) \\ & \text{subject to} & F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0 \end{array}$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function and, for $i = 0, \ldots, m$, $F_i \in S_n$, the space of symmetric $n \times n$ matrices and $\succeq$ denotes the Löwner partial order of S_n, that is, for $M, N \in S_n$, $M \succeq N$ if and only if $(M - N)$ is positive semidefinite. We consider S_n as a vector space with the trace inner product $(M, N) := Tr[MN]$, where $Tr[.]$ refers to the trace operation. Let $S = \{M \in S_n \mid M \succeq 0\}$ be the closed convex cone of positive semidefinite $n \times n$ matrices. For convenience, let us denote the sequential form of the Lagrangian for (SDP) by

$$L^\infty(x, (Z_n)) := f(x) - \limsup_{n\to\infty} Tr[Z_n F_0 + \sum_{i=1}^{m} x_i Z_n F_i],\ (Z_n) \subset S.$$

We now deduce from Theorem 3.1 that the following sequential Lagrangian duality holds for semidefinite programs.

Theorem 3.3 *For the problem (SDP), there exists a sequence $(\bar{Z}_n) \subset S$ such that*

$$\inf_{x\in X} \sup_{(Z_n)\subset S} L^\infty(x, (Z_n)) = \sup_{(Z_n)\subset S} \inf_{x\in X} L^\infty(x, (Z_n)) = \inf_{x\in X} L^\infty(x, (\bar{Z}_n)) = \inf(SDP).$$

Proof. The feasible set A is defined by

$$A = \{x \in \mathbb{R}^m \mid F(x) \succeq 0\},$$

where $F(x) = F_0 + \sum_{i=1}^{m} F_i x_i$. The polar cone S^+ of the set $S \subset S_n$ is given by

$$S^+ = \{\theta \in S_n \mid (\theta, s) \geq 0,\ \forall s \in T\}.$$

Then $S^+ = S$ and

$$M \in S \Leftrightarrow Tr[ZM] \geq 0,\ \forall Z \in S.$$

Let $g(x) = -F(x)$. Then the conclusion follows directly from Theorem 3.1. $\square$

For (SDP), the **sequential Lagrangian dual problem** is defined as

$$(\widetilde{DSDP}) \qquad \max_{(Z_n)\subset S} \inf_{x\in X} \left(f(x) - \limsup_{n\to\infty} Tr[Z_n F_0 + \sum_{i=1}^{m} x_i Z_n F_i] \right)$$

It follows from Theorem 3.2 that the Problem (SDP) and $(\widetilde{DSDP})$ are in perfect duality.

The following example shows that the perfect duality duality holds between (SDP) and its sequential dual problem whereas there is a non-zero duality gap between (SDP) and its standard Lagrangian dual problem.

Example 3.1 *Consider the convex semi-definite Problem*

$$\text{(SDP1)} \qquad \text{Minimize} \qquad x_1$$
$$\text{subject to} \qquad \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0.$$

Let $f(x_1, x_2) = x_1$,

$$F_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $F(x) = F_0 + x_1F_1 + x_2F_2,\ x = (x_1, x_2) \in I\!R^2$.

The feasible set A *of* $(SDP1)$ *is*

$$A := \{x \in I\!R^2 \mid F(x) \succeq 0\} = \{(x_1, x_2) \in I\!R^2 \mid x_1 = 0, x_2 \geq 0\}$$

and hence, $\inf(SDP1) = 0$.

For any sequence (Z_n) *with* $Z_n \succeq 0$*, let*

$$\Phi(Z) := \inf_{(x_1,x_2)\in I\!R^2} L^\infty((x_1, x_2), (Z_n)) = \inf_{(x_1,x_2)\in I\!R^2} \left[x_1 - \limsup_{n\to\infty} Tr[(Z_n)(F_0 + \sum_{i=1}^{2} x_iF_i)] \right].$$

Define the sequence $(\hat{Z}_n)$ *by*

$$\hat{Z}_n = \begin{pmatrix} n & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{n} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\hat{Z}_n \succeq 0$ *for all* $n \in \mathbf{N}$ *and* $Tr[\hat{Z}_nF_0] = 0,\ Tr[\hat{Z}_nF_1] = 1,\ Tr[\hat{Z}_nF_2] = \frac{1}{n}$. *Therefore,*

$$\Phi((\hat{Z}_n)) = \inf_{(x_1,x_2)\in I\!R^2} [x_1 - \limsup_{n\to\infty}(x_1 + \frac{1}{n}x_2)] = 0 = \inf(SDP1).$$

By weak duality, we have

$$\sup_{Z\subset S} \Phi(Z) = \Phi(\hat{Z}) = 0 = \inf(SDP1).$$

Thus the primal problem $(SDP1)$ and its sequential dual problem $(\widetilde{DSDP}1)$ have no duality gap. However, there is a non-zero duality gap between the primal problem $(SDP1)$ and its standard Lagrangian dual problem $(DSDP2)$:

$$\text{(DSDP2)} \max_{Z \subset S} \inf_{x \in X} \left(f(x) - Tr[ZF_0 + \sum_{i=1}^{m} x_i Z F_i] \right).$$

To see this, let

$$Z = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_4 & \lambda_5 \\ \lambda_3 & \lambda_5 & \lambda_6 \end{pmatrix} \in S_3.$$

Then $TrZF_0 = \lambda_6,\ x_1 TrZF_1 = 2x_1\lambda_2,\ x_2 TrZF_2 = x_2\lambda_4$. Thus for $Z \succeq 0$,

$$\begin{aligned} \phi(Z) := & \inf_{(x_1,x_2)\in\mathbb{R}^2} L((x_1,x_2),Z) = \inf_{(x_1,x_2)\in\mathbb{R}^2} [-2x_1\lambda_2 - x_2\lambda_4 - \lambda_6] \\ = & \inf_{(x_1,x_2)\in\mathbb{R}^2} [x_1(1-2\lambda_2) + x_2(-\lambda_4) - \lambda_6]. \end{aligned}$$

Note that $Z \succeq 0$ implies $\lambda_4 \geq 0$ and $\lambda_6 \geq 0$. Consider the following possibilities:if $\lambda_4 > 0$ then it is easy to see that $\phi(Z) = -\infty$; if $\lambda_4 = 0$ and $1 - 2\lambda_2 \neq 0$ then we also have $\phi(Z) = -\infty$. If $\lambda_4 = 0$ and $1 - 2\lambda_2 = 0$ then

$$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_4 \end{vmatrix} = \begin{vmatrix} \lambda_1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4} < 0$$

which contradicts the fact that $Z \succeq 0$. Consequently, for all $Z \succeq 0$, $\phi(Z) = -\infty$ and hence,

$$[DSDP2] = \sup_{Z \in S} \phi(Z) = -\infty.$$

4 Sequential Saddle-Point Conditions

In this section we derive a sequential form of the Lagrangian saddle-point conditions for the convex programming problem (P).

We now consider the following **sequential saddle point problem** of (P)

(SP) Find $\bar{x} \in X$ and $(\bar{\lambda}_n) \subset S^+$ such that

$$\liminf_{n\to\infty} L(x,\lambda_n) \leq \liminf_{n\to\infty} L(\bar{x},\bar{\lambda}_n) \leq \liminf_{n\to\infty} L(x,\bar{\lambda}_n), \forall x \in X, \forall(\lambda_n) \subset S^+. \tag{8}$$

The solution set of (SP) is denoted by sol(SP).

A solution $(\bar{x}, (\bar{\lambda}_n))$ of (SP) is called a **sequential saddle point** of $L(.,.)$. For convenience, we split (8) into following two systems:

$$\liminf_{n\to\infty} L(\bar{x},\lambda_n) \le \liminf_{n\to\infty} L(\bar{x},\bar{\lambda}_n), \forall x \in X, \forall (\lambda_n) \subset S^+ \quad (\alpha)$$

$$\liminf_{n\to\infty} L(\bar{x},\bar{\lambda}_n) \le \liminf_{n\to\infty} L(x,\bar{\lambda}_n), \forall x \in X, \forall (\lambda_n) \subset S^+. \quad (\beta)$$

Theorem 4.1 *Let $\bar{x} \in X$, and let the sequence $(\bar{\lambda}_n) \subset S^+$. Then $(\bar{x},(\bar{\lambda}_n)) \in sol$ (SP) if and only if $(\bar{x},(\bar{\lambda}_n))$ satisfies the following system:*

$$\liminf_{n\to\infty} L(\bar{x},\bar{\lambda}_n) = \inf_{x\in X}\liminf_{n\to\infty} L(x,\bar{\lambda}_n), \ \ \liminf_{n\to\infty}\bar{\lambda}_n g(\bar{x}) = 0 \text{ and } g(\bar{x}) \in -S.$$

Proof. *Necessity.* Suppose that $(\bar{x},(\bar{\lambda}_n)) \in sol$ (SP) then (β) implies that the first equality in the statement of the theorem holds. It follows from (α) that

$$\liminf_{n\to\infty} \lambda_n g(\bar{x}) \le \liminf_{n\to\infty} \bar{\lambda}_n g(\bar{x}), \text{ for all } (\lambda_n) \subset S^+. \quad (9)$$

Letting $\lambda_n = 0$ for each n and letting $(\lambda_n) = (2\bar{\lambda}_n)$ in (9), we get $\liminf_{n\to\infty}\bar{\lambda}_n g(\bar{x}) \ge 0$ and $\liminf_{n\to\infty}\bar{\lambda}_n g(\bar{x}) \le 0$. Note that $\liminf_{n\to\infty}\bar{\lambda}_n g(\bar{x}) < +\infty$ as (β) holds for all $x \in A$. Hence, $\liminf_{n\to\infty}\bar{\lambda}_n g(\bar{x}) = 0$. Now for any $t \in S^+$, let $\lambda_n = t$ for all n, we get from (9)

$$tg(\bar{x}) \le \liminf_{n\to\infty}\bar{\lambda}_n g(\bar{x}) = 0,$$

which proves that $g(\bar{x}) \in -S$.

Sufficiency. Suppose that $(\bar{x},(\bar{\lambda}_n))$ satisfies the conditions in the theorem. Then (β) holds. Since $g(\bar{x}) \in -S$, for all $(\lambda_n)_n \subset S^+$ we have $\lambda_n g(\bar{x}) \le 0$ and hence,

$$\liminf_{n\to\infty}\lambda_n g(\bar{x}) \le 0 = \liminf_{n\to\infty}\bar{\lambda}_n g(\bar{x}).$$

Thus (α) follows. □

Theorem 4.2 *(Sequential Saddle-Point Theorem).The point $a \in A$ is a solution of (P) if and only if there exists $(\bar{\lambda}_n) \subset S^+$ such that $(a,(\bar{\lambda}_n)) \in sol$(SP).*

Proof. *Necessity.* Observe that a is a solution of (P) if and only if

$$g(x) \in -S \implies f(x) \ge f(a).$$

It follows from Theorem 2.1 that there exists a sequence $(\bar{\lambda}_n) \subset S^+$ such that

$$f(x) + \liminf_{n\to\infty}\bar{\lambda}_n g(x) \ge f(a), \quad \forall x \in X. \quad (10)$$

Take $x = a$ in (10), we get

$$f(a) \geq f(a) + \liminf_{n\to\infty} \bar{\lambda}_n g(a) \geq f(a),$$

which implies that $\liminf_{n\to\infty} \bar{\lambda}_n g(a) = 0$. Thus (10) gives

$$\liminf_{n\to\infty} L(a, \bar{\lambda}_n) = \inf_{x\in X} \liminf_{n\to\infty} L(x, \bar{\lambda}_n).$$

By Theorem 4.1, $(a, (\bar{\lambda}_n)) \in \mathrm{sol}(SP)$.

Sufficiency. Suppose that $(\bar{\lambda}_n) \subset S^+$ is a sequence such that $(a, (\bar{\lambda}_n)) \in \mathrm{sol}(SP)$. Then by Theorem 4.1, $g(a) \in -S$, $\liminf_{n\to\infty} \bar{\lambda}_n g(a) = 0$, and for all $x \in A$,

$$\begin{aligned} f(a) = f(a) + \liminf_{n\to\infty} \bar{\lambda}_n g(a) &\leq f(x) + \liminf_{n\to\infty} \bar{\lambda}_n g(x) \\ &\leq f(x), \end{aligned}$$

which proves that a is a solution of (P). □

Theorem 4.3 *The value of the function $\liminf_{n\to\infty} L(.,.)$ is constant over all sequential saddle points $(\bar{x}, \bar{\lambda}_n)$ of $L(.,.)$. Moreover, if $(\bar{x}_1, (\lambda'_n))$ and $(\bar{x}_2, (\lambda''_n))$ are sequential saddle points of $L(.,.)$ then so are $(\bar{x}_1, (\lambda''_n))$ and $(\bar{x}_2, (\lambda'_n))$.*

Proof. Let $(\bar{x}_1, (\lambda'_n)))$ and $(\bar{x}_2, (\lambda''_n))$ are sequential saddle points of $L(.,.)$. Then by definition, for all $x \in X$ and all $(\lambda_n) \subset S^+$,

$$\liminf_{n\to\infty} L(\bar{x}_1, \lambda_n) \leq \liminf_{n\to\infty} L(\bar{x}_1, \lambda'_n)) \leq \liminf_{n\to\infty} L(x, \lambda'_n)), \tag{11}$$

$$\liminf_{n\to\infty} L(\bar{x}_2, \lambda_n) \leq \liminf_{n\to\infty} L(\bar{x}_2, \lambda''_n) \leq \liminf_{n\to\infty} L(x, \lambda''_n). \tag{11$'$}$$

Let $x = \bar{x}_2$ and for $n \in \mathbf{N}$, $\lambda_n = \lambda''_n$ in (11) and then $x = \bar{x}_1$, $\lambda_n = \lambda'_n$ in (11$'$), we get

$$\liminf_{n\to\infty} L(\bar{x}_1, \lambda''_n) \leq \liminf_{n\to\infty} L(\bar{x}_1, \lambda'_n)) \leq \liminf_{n\to\infty} L(\bar{x}_2, \lambda'_n)),$$
$$\liminf_{n\to\infty} L(\bar{x}_2, \lambda'_n)) \leq \liminf_{n\to\infty} L(\bar{x}_2, \lambda''_n) \leq \liminf_{n\to\infty} L(\bar{x}_1, \lambda''_n),$$

which implies that

$$\liminf_{n\to\infty} L(\bar{x}_1, \lambda'_n)) = \liminf_{n\to\infty} L(\bar{x}_1, \lambda''_n) = \liminf_{n\to\infty} L(\bar{x}_2, \lambda''_n) = \liminf_{n\to\infty} L(\bar{x}_2, \lambda'_n)).$$

From this, (11) and (11$'$), we get

$$\liminf_{n\to\infty} L(\bar{x}_1, \lambda_n) \leq \liminf_{n\to\infty} L(\bar{x}_1, \lambda'_n)) = \liminf_{n\to\infty} L(\bar{x}_2, \lambda''_n) \leq \liminf_{n\to\infty} L(x, \lambda''_n)$$

for all $x \in X$ and all sequences $(\lambda_n) \subset S^+$, which proves that $(\bar{x}_1, (\lambda''_n))$ is a sequential saddle point of $L(.,.)$. By a similar argument, $(\bar{x}_2, (\lambda'_n))$ is also a sequential saddle point of $L(.,.)$. The conclusion follows. □

Corollary 4.1 *Suppose that $a \in A$ is a solution of (P) and $(\lambda_n) \subset S^+$ is a sequence such that $(a, (\lambda_n)) \in \mathrm{sol}(SP)$. Then the function $\liminf_{n\to\infty} L(., \lambda_n)$ is constant on the solution set of (P).*

Proof. If $\bar{x}$ is another solution of (P). Then it follows from Theorem 4.2 that there exists a sequence $(\bar{\lambda}_n) \subset S^+$ such that $(\bar{x}, (\bar{\lambda}_n))$ is another sequential saddle point of $L(., .)$. By Theorem 4.3,

$$\liminf_{n\to\infty} L(\bar{x}, (\bar{\lambda}_n)) = \liminf_{n\to\infty} L(a, (\lambda_n)) = \liminf_{n\to\infty} L(\bar{x}, (\lambda_n)).$$

The corollary has been proved. □

5 Stability

Now consider the perturbed problem (P_z) of (P) where z is in a certain neighbourhood of 0 in Z.

$$(P_z) \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & -g(x) - z \in S, \end{array}$$

The optimal value of (P_z) is denoted by $F(z)$. That is,

$$F(z) = \inf\{f(x) \mid -g(x) - z \in S\},\ z \in Z.$$

We set $F(z) = +\infty$ if the feasible set of (P_z) is empty. Note also that the function F is convex [4]. When $z = 0$ then (P_0) is exactly (P). So, for the sake of convenience, in the following we will call it (P_0) instead of (P).

The following theorem gives a relation between the sequential dual problem $(\widetilde{D})$ of (P_0) and the value function F of perturbed Problems (P_z).

Theorem 5.1 *Suppose that a is a solution of (P_0). The following statements are equivalent.*

(i) $\exists(\bar{\lambda}_n) \subset S^+$ such that

$$F(z) \geq F(0) + \limsup_{n\to\infty} \bar{\lambda}_n z \quad \forall z \in Z,$$

(ii) $f(a) = \inf(P_0) = \Phi((\bar{\lambda}_n)) = \mathrm{Max}(\widetilde{D})$,

where $\Phi((\bar{\lambda}_n)) := \inf_{x\in X} \liminf_{n\to\infty} L(x, (\bar{\lambda}_n)) = \inf_{x\in X} \liminf_{n\to\infty} [f(x) + \bar{\lambda}_n g(x)]$.

Proof. [(i) $\Rightarrow$ (ii)]. Suppose (i) holds. Let $x \in X$. Take $z = -g(x) \in Z$ then $-g(x) - z = 0 \in S$. This means that x is feasible for (P_z). It follows from (i) that

$$f(x) \geq F(z) \geq F(0) + \limsup_{n\to\infty} \bar{\lambda}_n z,$$

which implies that (note that $z = -g(x)$)

$$\begin{aligned} F(0) &\leq f(x) - \limsup_{n\to\infty}(-\bar{\lambda}_n g(x)) \\ &= f(x) + \liminf_{n\to\infty} \bar{\lambda}_n g(x). \end{aligned}$$

It follows from this inequality that

$$f(a) = F(0) \leq \Phi((\bar{\lambda}_n)).$$

On the other hand, since $a \in A$, by the weak duality, $\Phi((\bar{\lambda}_n)) \leq f(a)$. Thus

$$f(a) = \inf(\mathrm{P}_0) = \Phi(\bar{\lambda}) = \mathrm{Max}(\widetilde{D}).$$

[(ii) $\Rightarrow$ (i)] Suppose (ii) holds. Take any $z \in Z$. If the feasible set of (P_z) is empty then $F(z) = +\infty$ and (i) holds trivially.

If the feasible set of (P_z) is nonempty, take $x \in X$ such that $-g(x) - z \in S$. It follows from this and (ii) that $\bar{\lambda}_n(g(x) + z) \leq 0$ and

$$F(0) = \mathrm{Min}(\mathrm{P}_0) = \mathrm{Max}(\widetilde{D}) = \Phi(\bar{\Lambda}) \leq f(x) + \liminf_{n\to\infty} \bar{\lambda}_n g(x).$$

These two inequalities imply that

$$F(0) \leq f(x) + \liminf_{n\to\infty}(-\bar{\lambda}_n z) = f(x) - \limsup_{n\to\infty} \bar{\lambda}_n z, \tag{8}$$

and thus,

$$F(0) \leq F(z) - \limsup_{n\to\infty} \bar{\lambda}_n z. \tag{9}$$

Note that in this case $F(z) < +\infty$ and by (8), $\limsup_{n\to\infty} \bar{\lambda}_n z \neq \infty$, and hence, (9) is well defined (i.e., does not have the form $\infty - \infty$). Now, if $\limsup_{n\to\infty} \bar{\lambda}_n z = -\infty$ then the equality in (i) holds. Otherwise (then $\limsup_{n\to\infty} \bar{\lambda}_n z$ is finite), we get from (9)

$$F(z) \geq F(0) + \limsup_{n\to\infty} \bar{\lambda}_n z$$

which means that (i) holds. $\square$

Recall that the Problem (P_z) is called **stable** at $z = 0$ [4] if there exists a constant $k \in I\!R$ such that

$$F(z) - F(0) \geq k\|z\| \text{ for all sufficiently small } \|z\|.$$

The following corollary explains the relation between solutions of the dual problem and the stability of the Problem (P_0) at $z = 0$. Note also that all the conditions (i) - (vii) in the corollary hold with the same λ^* which is the a standard Lagrange multiplier for (P_0) corresponding to the solution $a \in A$.

Corollary 5.1 *For the Problem (P_0), let $a \in A$. The the following statements are equivalent.*

(i) There exist $\lambda^* \in S^+$ such that

$$f(x) + \lambda^* g(x) \geq f(a), \ \forall x \in X,$$

(ii) There exists $\lambda^* \in S^+$ such that

$$\begin{aligned} &0 \in \partial f(a) + \partial(\lambda^* g)(a) \\ &\lambda^* g(a) = 0, \end{aligned}$$

(iii) There exists $u \in \partial f(a)$ such that

$$(-u, -u(a)) \in \bigcup_{\lambda \in S^+} \mathrm{epi}(\lambda g)^*,$$

(iv) There exists $\lambda^* \in S^+$such that, for $(\bar{\lambda}_n)$ with $\bar{\lambda}_n = \lambda^*$, $n \in \mathbf{N}$,

$$[\widetilde{D}] = [D] = \inf_{x \in X} \liminf_{n \to \infty} L(x, \bar{\lambda}_n) = \inf_{x \in X} L(x, \lambda^*) = L(a, \lambda^*) = f(a),$$

(v) The Problem (P_0) attains its minimum at a and there exists $\lambda^* \in S^+$ such that

$$F(z) \geq F(0) + \lambda^* z, \ \forall z \in Z,$$

(vi) (P_0) attains its minimum at a and the Problem (P_z) is stable at $z = 0$,

(vii) (P_0) attains its minimum at a and the subdifferential $\partial F(0)$ of F at 0 is nonempty.

Proof. [(i) $\Longleftrightarrow$ (ii)] It follows from (i) that $\lambda^* g(a) = 0$. This means that the function $L(x, \lambda^*) = f(x) + \lambda^* g(x)$ attains its minimum on X at a. By the standard optimality condition for convex problem we get

$$0 \in \partial_x L(a, \lambda^*) = \partial f(a) + \partial(\lambda^* g)(a).$$

The converse is clear. The equivalence between (ii) and (iii) was established in [10].

[(i) $\Longleftrightarrow$ (iv)] Assume that (i) holds. Then it is clear that $\lambda^* g(a) = 0$ and hence,

$$\inf_{x \in X} \liminf_{n \to \infty} L(x, \bar{\lambda}_n) = L(a, \lambda^*) = \inf_{x \in X} L(x, \lambda^*) = f(a).$$

By weak duality, we get

$$[\widetilde{D}] = [D] = \inf_{x\in X} \liminf_{n\to\infty} L(x, \bar{\lambda}_n) = \inf_{x\in X} L(x, \lambda^*) = L(a, \lambda^*) = f(a).$$

The converse is obvious. The equivalence between (iv) and (v) follows from the same argument as in the proof of Theorem 4.1. The equivalence of (v), (vi) and (vii) was established in [4]. □

The following example illustrates the significance of the closed cone condition.

Example 5.1 *Consider the problem*

$$(P1) \qquad \begin{array}{ll} \textit{minimize} & -x \\ \textit{subject to} & (\max\{0, x\})^2 \le 0. \end{array}$$

Let $f(x) := -x$ *and* $g(x) := (\max\{0, x\})^2$, $x \in I\!R$. *Then* $\inf(P1) = 0$ *and* $a = 0$ *is a minimizer of* $(P1)$.

Note that for this Problem, we have

$$\bigcup_{\lambda\ge 0} \text{epi } (\lambda g)^* = \{(0,0)\} \cup \{(x, y) \in \mathbb{R}^2 \mid y > 0,\ x \ge 0\}.$$

So the condition (*v*) *in Corollary 5.1 does not hold and hence, by Corollary 5.1, the perturbed Problem* $(P1_z)$ *is not stable at* $z = 0$. *Note also that by the last equality, the closed cone constraint qualification does not hold for* $(P1)$ *either.*

However, the perfect duality result (Theorem 3.2) holds and

$$f(a) = \inf(P1) = \text{Max } (\widetilde{DP1}) = [\widetilde{DP1}]$$

where $(\widetilde{DP1})$ *is the sequential Lagrangian dual problem of* $(P1)$.

On the other hand, by an elementary calculation, we get, for each $\lambda \ge 0$,

$$\inf_{x\in I\!R} L(x, \lambda) = \inf_{x\in I\!R} [-x + \lambda(\max\{0, x\})^2] = \begin{cases} -\infty & \text{if } \lambda = 0 \\ -\frac{1}{4\lambda} & \text{if } \lambda > 0. \end{cases}$$

Thus, for the standard dual problem $(DP1)$ *of* $(P1)$, *we have*

$$[DP1] := \sup_{\lambda\ge 0} \inf_{x\in I\!R} L(x, \lambda) = 0.$$

So in this case $[DP1] = [\widetilde{DP1}] = \inf(P1)$.

References

[1] J. M. Borwein, "A note on perfect duality and limiting Lagrangeans", *Mathematical Programming*, 18 (1980), 330-337.

[2] J. M. Borwein and H. Wolkowicz, "Characterizations of optimality without constraint qualification for the abstract convex program", *Mathematical Programing Study 19 (1982), 77-100.*

[3] J. M. Borwein and H. Wolkowicz, "Characterizations of optimality for the abstract convex program with finite dimensional range", *Journal of Australian Mathematical Society, Series A, 30 (1981), 390-411.*

[4] B. D. Craven, *Mathematical Programming and Control Theory*, Chapman and Hall, London, 1978.

[5] J. Gwinner, "Results of Farkas type", *Numerical Functional Analysis and Optimization*, 9(1987), 471-520.

[6] M. Hayashi and H. Komiya, "Perfect duality for convexlike Porgrams", *Journal of Optimization Theory and Applications,* 38(2) (1982) 179-189.

[7] J. B. Hiriart-Urruty, "ϵ-Subdifferential", in *Convex Analysis and Optimization*, Edited by J. P. Aubin and R. Vinter, Pitman, London, England, 1982, 43-92.

[8] J. B. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms*, Volumes I and II, Springer-Verlag, Berlin Heidleberg, 1993.

[9] V. Jeyakumar, "Characterizing set containments involving infinite convex constraints and reverse convex constraints, *SIAM Journal on Optimization*(in press-accepted for publication on Sept. 17, 2002). See http://www.maths.unsw.edu.au/applied/reports/amr02.html

[10] V. Jeyakumar, G. M. Lee and N. Dinh, "New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs", Applied Mathematics Research Report AMR02/9, UNSW, 2002 (submitted for publication). See http://www.maths.unsw.edu.au/applied/reports/amr02.html

[11] V. Jeyakumar, G. M. Lee and N. Dinh, "Solution sets of convex vector minimization problems", Applied Mathematics Research Report AMR02/15, UNSW, 2002 (submitted for publication). See http://www.maths.unsw.edu.au/applied/reports/amr02.html

[12] V. Jeyakumar, A. M. Rubinov, B. M. Glover and Y. Ishizuka, "Inequality systems and global optimization", *Journal of Mathematical Analysis and Applications,* 202(1996), 900-919.

[13] V. Jeyakumar and H. Wolkowicz, "Zero duality gaps in infinite-dimensional programming", *Journal of Optimization Theory and Applications,* 67(1990) 87-108.

[14] V. Jeyakumar and H. Wolkowicz, "Generalizations of Slater's constraint qualification for infinite convex programs, *Mathematical Programing,* $57(1)(1992), 85 - 102$.

[15] V. Jeyakumar and A. Zaffaroni, "Asymptotic conditions for weak and proper optimality in infinite dimensional convex vector optimization", *Numerical Functional Analysis and Optimization*, 17 (1996), 323-343.

[16] K. O. Kortanek and Q. Zhang, "Perfect duality in semi-infinite and semi-definite programming", *Mathematical Programming, Ser. A*, 91 (2001), 127-144.

[17] Nesterov, Y. and Nemirovskii, A. (1994), Interior-point methods in convex programming, Studies in Applied Mathematics, 13, SIAM Publications, Philadelphia, PA.

[18] Ramana, M. V. (1997), An exact duality theory for semidefinite programming and its complexity implications, *Mathematical Programming*, 77, 129-162.

[19] Ramana, M. V., Tunçel, L. and Wolkowicz, H. (1997), "Strong duality for semi-definite programming", *SIAM Journal on Optimization*, 7, 644-662.

[20] R. T. Rockafellar, *Conjugate Duality and Optimization*, SIAM, Pliladelphia, 1974.

[21] L. Thibault, "Sequential convex subdifferential calculus and sequential Lagrange multipliers", *SIAM Journal on Control and Optimization* 35(1997), no. 4, 1434-1444.

[22] Wolkowicz, H., Saigal, R. and Vandenberghe, L., *Handbook of semi-definite programming*, International Series in Opeartions Research and Management Science, 27, Kluwer Academic Publishers, Dordrecht, 2000.