

# Efficient evaluation of highly oscillatory acoustic scattering surface integrals <sup>★</sup>

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## Abstract

We consider the approximation of some highly oscillatory weakly singular surface integrals, arising from boundary integral methods with smooth global basis functions for solving problems of high frequency acoustic scattering by three dimensional convex obstacles, described globally in spherical coordinates. As the frequency of the incident wave increases, the performance of standard quadrature schemes deteriorates. Naive application of asymptotic schemes also fails due to the weak singularity. We propose here a new scheme based on a combination of an asymptotic approach and exact treatment of singularities in an appropriate coordinate system. For the case of a spherical scatterer we demonstrate via error analysis and numerical results that, provided the observation point is sufficiently far from the shadow boundary, a high level of accuracy can be achieved with a minimal computational cost.

*Key words:* High frequency scattering, acoustics, oscillatory integrals, method of stationary phase.

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## 1 Introduction

This paper is concerned with the approximation of integrals of the form

$$M\psi(\mathbf{x}) := \int_{\partial D} \frac{m(\mathbf{x}, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{ik|\mathbf{x} - \mathbf{y}| + \hat{\mathbf{d}} \cdot (\mathbf{y} - \mathbf{x})} \psi(\mathbf{y}) \, ds(\mathbf{y}), \quad \mathbf{x} \in \partial D, \quad (1)$$

where  $m(\mathbf{x}, \mathbf{y})$ ,  $\psi(\mathbf{y})$  are smooth and slowly oscillating functions,  $k$  and  $\hat{\mathbf{d}}$  respectively are fixed wavenumber and incident wave direction, and  $\partial D$  is the surface of a three dimensional convex obstacle  $D$ . (We will give precise requirements on  $\partial D$  and the observed direction  $\mathbf{x}$  at the end of this section.) Such integrals arise from boundary integral methods with smooth global basis functions for acoustic scattering problems.

For example, consider scattering of a time-harmonic acoustic plane wave  $u^i$  by a sound soft bounded convex obstacle  $D \subset \mathbb{R}^3$  with smooth surface  $\partial D$  described globally in spherical coordinates. The scattered field  $u$  is the unique radiating solution of the exterior Helmholtz problem

$$\Delta u + k^2 u = 0, \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \quad u = -u^i := -e^{ik\mathbf{x} \cdot \hat{\mathbf{d}}}, \quad \text{on } \partial D,$$

and can be represented as  $u(\mathbf{x}) = -\int_{\partial D} \Phi(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) \, ds(\mathbf{y})$ ,  $\mathbf{x} \in \mathbb{R}^3 \setminus \bar{D}$ , [4, p.59]. Here  $\Phi(\mathbf{x}, \mathbf{y}) := e^{ik|\mathbf{x} - \mathbf{y}|} / (4\pi|\mathbf{x} - \mathbf{y}|)$ , and  $v := \partial(u + u^i) / \partial \mathbf{n} \in C(\partial D)$  is the unique solution of the boundary integral equation

$$\frac{1}{2} v(\mathbf{x}) + \int_{\partial D} \left[ \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{x})} - i\eta \Phi(\mathbf{x}, \mathbf{y}) \right] v(\mathbf{y}) \, ds(\mathbf{y}) = \frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{x}) - i\eta u^i(\mathbf{x}), \quad \mathbf{x} \in \partial D, \quad (2)$$

where  $\mathbf{n}(\mathbf{x})$  is the unit outward normal vector to the surface  $\partial D$  at  $\mathbf{x}$  and  $\eta \in \mathbb{R} \setminus \{0\}$  is a coupling parameter. In this work, we are interested in the high frequency acoustic scattering problem. (For large values of  $k$ , the choice  $\eta = k$  is useful to reduce the condition number of discretized systems [3,11].)

As in the classical Kirchoff approximation [2, Section 2.7], for large values of  $k$ , the surface of the convex obstacle  $D$  may be considered locally (in a leading order approximation) as a plane at each point  $\mathbf{x} \in \partial D$  [4, p.54]. This suggests the ansatz

$$v(\mathbf{x}) = \phi(\mathbf{x}) e^{ik\mathbf{x} \cdot \hat{\mathbf{d}}}, \quad (3)$$

where the unknown function  $\phi$  is slowly oscillating compared to  $e^{ik\mathbf{x} \cdot \hat{\mathbf{d}}}$  provided  $\mathbf{x}$  is sufficiently away from the shadow boundary, on which  $\mathbf{n}(\mathbf{x}) \cdot \hat{\mathbf{d}} = 0$ . The ansatz (3) has been used widely in the literature [1,3,6,12]. In particular, we

refer to the discussion in [3] for the fact that  $\phi$  is slowly oscillating only away from the shadow boundary. The analysis in [2, Section 2.7] suggests that the band around the shadow boundary within which  $\phi$  oscillates more rapidly has width of the order of  $k^{-1/3}$ .

Substituting (3) into (2) leads to the second kind boundary integral equation

$$\phi(\mathbf{x}) + \int_{\partial D} \frac{m(\mathbf{x}, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{ik|\mathbf{x} - \mathbf{y}| + i\hat{\mathbf{d}} \cdot (\mathbf{y} - \mathbf{x})} \phi(\mathbf{y}) \, ds(\mathbf{y}) = 2i(k\mathbf{n}(\mathbf{x}) \cdot \hat{\mathbf{d}} - \eta), \quad (4)$$

where  $m(\mathbf{x}, \mathbf{y})$  is a smooth function, given by

$$m(\mathbf{x}, \mathbf{y}) := \frac{1}{4\pi} \left[ \frac{(\mathbf{y} - \mathbf{x})^T \mathbf{n}(\mathbf{x})}{|\mathbf{x} - \mathbf{y}|^2} (1 - ik|\mathbf{x} - \mathbf{y}|) - i\eta \right]. \quad (5)$$

In any numerical scheme to solve (4), with  $\phi(\mathbf{x})$  approximated in a finite dimensional space by  $\phi_L(\mathbf{x}) := \sum_{j=1}^L v_j \rho_j(\mathbf{x})$ , we are faced with the difficulty of evaluation of integrals of the form (1) with density  $\psi$  replaced by the basis function  $\rho_j$ ,  $j = 1, \dots, L$ . In particular, if the functions  $\rho_j$ ,  $j = 1, \dots, L$ , have global support, such as in the scheme of [5], then (1) has a singularity at  $\mathbf{y} = \mathbf{x}$ . Moreover, if the acoustic size  $kA$  is large (where  $A$  is the size of the obstacle), corresponding to the high frequency problem, the integrand will be highly oscillatory. For simulation of scattered acoustic waves, evaluation of (1) is required for many observation directions  $\mathbf{x}$ , and when  $kA$  is large the cost of doing this by standard quadrature schemes may be prohibitive.

Much recent research has focused on the development of efficient schemes for evaluating highly oscillatory integrals. Most of the rigorous results in the literature to date are for one dimensional integrals of the form  $I(k) := \int_a^b g(x) e^{ikf(x)} \, dx$ , where the smooth function  $g$  is slowly oscillating compared to  $e^{ikf}$ . If the moments of  $I(k)$  (i.e.,  $\int_a^b x^n e^{ikf(x)} \, dx$ ,  $n = 0, 1, \dots$ ) are easily obtainable then the scheme in [9] can be used to evaluate  $I(k)$  very efficiently. However, for the integral (1) evaluation of the moments is almost as challenging as the evaluation of the integral itself. Moreover, despite some recent advances [10], even in the case that the moments are known the extension of these schemes to higher dimensions and non-smooth  $g$  poses many challenges. The requirement of moments was relaxed in the recent work [8] using quadrature and ideas from the steepest descent method. However, the scheme in [8] involves a transformation of the path of one dimensional integration into the complex plane, which requires knowledge of both real and complex stationary points. This complicates matters considerably for higher dimensional integrals [7].

The approach considered in this paper is related to that in [3], where ideas from the method of stationary phase are used to replace the integral  $I(k)$  over  $[a, b]$

with several integrals over (smaller) domains around each stationary point, with each of these integrals being evaluated by quadrature. Singularities in  $g$  are dealt with using local expansions. Although very good results have been reported in [3] for circular geometries, it seems that significant breakthroughs may be needed to produce similar results for closed convex surfaces and to justify the results with rigorous error analysis.

Our approach and analysis for evaluating (1), on a restricted class of closed convex surfaces, begins in §2 with an exact treatment of the singularity, using a singularity division technique in an appropriate coordinate system. This gives us an explicit representation of the phase function  $|\mathbf{x} - \mathbf{y}| + \hat{\mathbf{d}} \cdot (\mathbf{y} - \mathbf{x})$  in the new coordinate system, leading in §3 to an explicit nonlinear system to solve for the stationary points. In the case that the scattering obstacle is a sphere, the location of the stationary points is derived analytically in §3.

In §4 we then use ideas from the method of stationary phase to rewrite (1) as a sum of integrals over small regions around each stationary point plus a remainder term. These integrals are defined on significantly smaller size domains and have integrands that are less oscillatory than that in (1). One may evaluate these integrals by the standard quadrature. In this paper, we instead approximate each of these integrals by the leading order terms from an asymptotic expansion, using the exact location of stationary points for a spherical scatterer. This approach leads to simple analytical formulas.

In Theorem 4.3, for the case of a spherical scatterer, we derive an estimate for the approximation of the remainder term. The estimate demonstrates that the error in our approximation depends on the distance from the observation point  $\mathbf{x}$  to the shadow boundary, and that the required minimum distance for convergence of the approximation decreases as  $k$  increases. The algorithm and analysis are demonstrated with numerical results in §5. We wind up this section by summarizing the assumptions and restrictions described above:

- the obstacle  $D$  in (1) is convex (and hence star-shaped), with a smooth surface which can be described globally in spherical coordinates;
- the observation point  $\mathbf{x}$  in (1) is away from the shadow boundary (with distance details in Remark 4.4);
- the analytical formulas for stationary points, and the analysis in this paper are restricted to the sphere.

## 2 Singularity-free formulation

Using the assumption on the obstacle, we write  $\mathbf{x} \in \partial D$  in spherical polar coordinates as  $\mathbf{x} = r(\theta, \phi)\mathbf{p}(\theta, \phi)$ , where for  $\theta \in [0, \pi]$ ,  $\phi \in [0, 2\pi]$ ,  $r(\theta, \phi)$  is a

smooth positive valued function and  $\mathbf{p}(\theta, \phi) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^\top$ . In the following theorem, we derive a singularity-free representation of (1) in a rotated coordinate system, using simple analytic geometry calculations.

**Theorem 2.1** *For a fixed incident and observed direction  $\hat{\mathbf{d}} = \mathbf{p}(\theta_d, \phi_d)$  and  $\mathbf{p}(\theta, \phi)$ , with  $\theta, \theta_d \in [0, \pi]$ ,  $\phi, \phi_d \in [0, 2\pi]$ , the integral  $M\psi(\mathbf{x})$  in (1) can be written as*

$$M\psi(r(\theta, \phi)\mathbf{p}(\theta, \phi)) = \int_{\phi}^{2\pi+\phi} \int_0^{\pi} H(\theta', \phi') e^{ikf(\theta', \phi')\psi(r(\alpha, \beta)\mathbf{p}(\alpha, \beta))} \cos \frac{\theta'}{2} d\theta' d\phi', \quad (6)$$

where  $\alpha \in [0, \pi]$  and  $\beta \in [0, 2\pi]$  are functions of  $\theta, \phi, \theta', \phi'$ , satisfying

$$\sin \alpha \cos \beta = \sin \theta' (\cos \theta \cos \phi \cos(\phi - \phi') + \sin \phi \sin(\phi - \phi')) + \cos \theta' \sin \theta \cos \phi, \quad (7)$$

$$\sin \alpha \sin \beta = \sin \theta' (\cos \theta \sin \phi \cos(\phi - \phi') - \cos \phi \sin(\phi - \phi')) + \cos \theta' \sin \theta \sin \phi, \quad (8)$$

$$\cos \alpha = \cos \theta \cos \theta' - \sin \theta \sin \theta' \cos(\phi - \phi'), \quad (9)$$

$$H(\theta', \phi') := m(r(\theta, \phi)\mathbf{p}(\theta, \phi), r(\alpha, \beta)\mathbf{p}(\alpha, \beta)) \frac{2 \sin(\theta'/2)}{f_1(\theta', \phi')} \mathcal{J}(\mathbf{p}(\alpha, \beta)) \quad (10)$$

is a smooth function in  $\theta', \phi'$ , with  $\mathcal{J}$  the Jacobian of the mapping  $\mathbf{p}(\theta, \phi) \mapsto r(\theta, \phi)\mathbf{p}(\theta, \phi)$ ,  $f_1(\theta', \phi') := |r(\theta, \phi)\mathbf{p}(\theta, \phi) - r(\alpha, \beta)\mathbf{p}(\alpha, \beta)|$ , and

$$f(\theta', \phi') := f_1(\theta', \phi') + \mathbf{p}(\theta_d, \phi_d) \cdot (r(\alpha, \beta)\mathbf{p}(\alpha, \beta) - r(\theta, \phi)\mathbf{p}(\theta, \phi)). \quad (11)$$

(We recall from (7)–(9) that  $\alpha$  and  $\beta$  are nonlinear functions of  $\theta'$  and  $\phi'$ .)

**Proof:** We begin by introducing the orthogonal transformation matrix

$$T(\theta, \phi) := \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which carries  $\mathbf{p}(\theta, \phi)$  to the north pole:  $T(\theta, \phi)\mathbf{p}(\theta, \phi) = [0, 0, 1]^T =: \hat{\mathbf{n}}$ . A little algebra reveals that  $[T(\theta, \phi)]^{-1}\mathbf{p}(\theta', \phi') = \mathbf{p}(\alpha, \beta)$ , and then

$$|\mathbf{p}(\theta, \phi) - \mathbf{p}(\alpha, \beta)| = |[T(\theta, \phi)]^{-1}(\hat{\mathbf{n}} - \mathbf{p}(\theta', \phi'))| = |\hat{\mathbf{n}} - \mathbf{p}(\theta', \phi')| = 2 \sin \frac{\theta'}{2}.$$

We recall that for any integrable  $\Upsilon$  on  $\partial D$ , using the notation  $\partial B$  to denote the unit sphere, we have

$$\int_{\partial D} \Upsilon(\mathbf{y}) \, ds(\mathbf{y}) = \int_{\partial B} \Upsilon(r(\alpha, \beta)\mathbf{p}(\alpha, \beta)) \mathcal{J}(\mathbf{p}(\alpha, \beta)) \, ds(\mathbf{p}(\alpha, \beta)).$$

Substituting  $\mathbf{x} = r(\theta, \phi)\mathbf{p}(\theta, \phi)$ ,  $\mathbf{y} = r(\alpha, \beta)\mathbf{p}(\alpha, \beta)$  in (1), and using the fact that the surface measure on  $\partial B$  is invariant under orthogonal transformation, we get

$$M\psi(r(\theta, \phi)\mathbf{p}(\theta, \phi)) = \int_{\partial B} H(\theta', \phi') e^{ikf(\theta', \phi')} \psi(r(\alpha, \beta)\mathbf{p}(\alpha, \beta)) \frac{ds(\mathbf{p}(\theta', \phi'))}{2 \sin(\theta'/2)}.$$

Since (i)  $ds(\mathbf{p}(\theta', \phi')) = \sin \theta' d\theta' d\phi'$ , (ii) the integrand is  $2\pi$  periodic with respect to  $\phi'$ , and (iii)  $f_1(\theta', \phi') := \sqrt{[r(\theta, \phi)]^2 - 2r(\theta, \phi)r(\alpha, \beta) \cos \theta' + [r(\alpha, \beta)]^2}$ , the result follows by recalling the smoothness of  $m(\cdot, \cdot)$  and  $\mathcal{J}(\cdot)$ .  $\square$

### 3 Evaluation of critical points

It is well known (see e.g. [13]) that the main contribution to the generalized Fourier integral (6) comes only from the values of the integrand at three types of *critical points* [13]:

- (i) Stationary points, where  $\nabla f := \left( \frac{\partial f}{\partial \theta'}, \frac{\partial f}{\partial \phi'} \right)^T = 0$  (we discuss these below);
- (ii) Points on the boundary, at which one of the following equations holds:

$$\frac{\partial f(0, \phi')}{\partial \phi'} = 0; \quad \frac{\partial f(\pi, \phi')}{\partial \phi'} = 0; \quad \frac{\partial f(\theta', \phi)}{\partial \theta'} = 0; \quad \frac{\partial f(\theta', 2\pi + \phi)}{\partial \theta'} = 0. \quad (12)$$

- For closed surface scatterers,  $f(0, \phi')$  and  $f(\pi, \phi')$  are constant functions, and hence the first two equations in (12) hold for all  $\phi' \in [\phi, 2\pi + \phi]$ . Since the phase function  $f(\theta', \phi')$  is  $2\pi$  periodic in  $\phi'$ , the remaining type (ii) critical points, can be found by solving the scalar equation  $\frac{\partial f(\theta', \phi)}{\partial \theta'} = 0$ .
- (iii) Corner points, namely  $(0, \phi)$ ,  $(0, 2\pi + \phi)$ ,  $(\pi, \phi)$ ,  $(\pi, 2\pi + \phi)$ .

We proceed by deriving explicitly a nonlinear system for the critical points of type (i). For fixed incident direction  $(\theta_d, \phi_d)$  and observation direction  $(\theta, \phi)$ , we define  $C_1 := \mathbf{p}(\theta_d, \phi_d) \cdot \mathbf{p}(\theta + \pi/2, \phi)$ ,  $C_2 := \mathbf{p}(\theta_d, \phi_d) \cdot \mathbf{p}(\pi/2, \phi - \pi/2)$ ,  $C_3 := \mathbf{p}(\theta_d, \phi_d) \cdot \mathbf{p}(\theta, \phi)$ ,  $A(\theta', \phi') := \mathbf{p}(\theta, \phi) \cdot \mathbf{p}(\pi/2 - \theta', \phi')$ ,  $B(\phi') := \mathbf{p}(\theta, \phi) \cdot \mathbf{p}(\pi/2, \phi' + \pi/2)$ , and then differentiating (11) reveals that

$$\begin{aligned} \frac{\partial f}{\partial \theta'} &= \left[ \frac{r(\theta, \phi) \sin \theta'}{f_1(\theta', \phi')} + \cos \theta' [C_1 \cos(\phi - \phi') + C_2 \sin(\phi - \phi')] - C_3 \sin \theta' \right] r(\alpha, \beta) \\ &+ \left[ \frac{r(\alpha, \beta) - r(\theta, \phi) \cos \theta'}{f_1(\theta', \phi')} + \sin \theta' [C_1 \cos(\phi - \phi') + C_2 \sin(\phi - \phi')] + C_3 \cos \theta' \right] \\ &\times \left[ \frac{A(\theta', \phi')}{\sqrt{[B(\phi')]^2 + [A(\theta', \phi')]^2}} \frac{\partial}{\partial \alpha} r(\alpha, \beta) - \frac{B(\phi')}{[B(\phi')]^2 + [A(\theta', \phi')]^2} \frac{\partial}{\partial \beta} r(\alpha, \beta) \right], \quad (13) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial \phi'} &= \sin \theta' \{ (C_1 \sin(\phi - \phi') - C_2 \cos(\phi - \phi')) r(\alpha, \beta) \\ &+ \left[ \frac{r(\alpha, \beta) - r(\theta, \phi) \cos \theta'}{f_1(\theta', \phi')} + \sin \theta' [C_1 \cos(\phi - \phi') + C_2 \sin(\phi - \phi')] + C_3 \cos \theta' \right] \\ &\times \left[ \frac{B(\phi')}{\sqrt{[B(\phi')]^2 + [A(\theta', \phi')]^2}} \frac{\partial}{\partial \alpha} r(\alpha, \beta) + \frac{A(\theta', \phi')}{[B(\phi')]^2 + [A(\theta', \phi')]^2} \frac{\partial}{\partial \beta} r(\alpha, \beta) \right] \}. \quad (14) \end{aligned}$$

We recall from (7)–(9) that  $\alpha$  and  $\beta$  are nonlinear functions of  $\theta'$  and  $\phi'$ . For the general class of obstacles considered in this paper, the nonlinear system (13)–(14) must be solved using a numerical scheme such as the Newton method.

In the remainder of this paper we assume  $r \equiv 1$ . This allows the analytical solution of (13)–(14), simplifying the analysis of §4 considerably. For notational simplicity we also assume  $\hat{\mathbf{d}} = [0, 0, 1]^T$ , but we remark that the analysis of this section and of §4 hold for all unit vectors  $\hat{\mathbf{d}}$ . Using (10)–(11) (and choosing  $\eta = k$  in (5)), we get

$$H(\theta', \phi') = \tilde{H}(k, \theta') := \frac{1}{4\pi} \left[ -\frac{1}{2} + ik \left( \sin \frac{\theta'}{2} - 1 \right) \right], \quad (15)$$

$$f(\theta', \phi') = 2 \sin \frac{\theta'}{2} + \cos \theta (\cos \theta' - 1) - \sin \theta \sin \theta' \cos(\phi - \phi'), \quad (16)$$

$$\frac{\partial f}{\partial \theta'} = \cos \frac{\theta'}{2} - \cos \theta \sin \theta' - \sin \theta \cos \theta' \cos(\phi - \phi'),$$

$$\frac{\partial f}{\partial \phi'} = -\sin \theta \sin \theta' \sin(\phi - \phi').$$

For a sphere, a simple analytic geometry calculation reveals the critical points of type (i), as given by the following theorem.

**Theorem 3.1** *The stationary points  $(\theta', \phi') \in [0, \pi] \times [\phi, 2\pi + \phi)$  of the phase function in (16) are as follows:*

- If  $\theta = 0$  then  $\nabla f = 0$  for  $(\theta', \phi') = (\pi/3, \phi'), (\pi, \phi')$ .
- If  $\theta \in (0, \pi/2)$  then there are five solutions of  $\nabla f = 0$ , given by  $(\theta', \phi') = (\pi - 2\theta, \phi), ((\pi - 2\theta)/3, \phi), ((\pi + 2\theta)/3, \phi + \pi), (\pi, \phi + \pi/2)$  and  $(\pi, \phi + 3\pi/2)$ .
- If  $\theta = \pi/2$  then there are four solutions of  $\nabla f = 0$ , given by  $(\theta', \phi') = (0, \phi)$ ,

$(2\pi/3, \phi + \pi)$ ,  $(\pi, \phi + \pi/2)$  and  $(\pi, \phi + 3\pi/2)$ .

- If  $\theta \in (\pi/2, \pi)$  then there are three solutions of  $\nabla f = 0$ , given by  $(\theta', \phi') = ((\pi + 2\theta)/3, \phi + \pi)$ ,  $(\pi, \phi + \pi/2)$  and  $(\pi, \phi + 3\pi/2)$ .
- If  $\theta = \pi$  then  $\nabla f = 0$  for  $(\theta', \phi') = (\pi, \phi')$ .

**Proof:** For  $\theta = \pi$ ,  $\partial f/\partial\phi' = 0$  for all  $\theta', \phi'$ , and  $\frac{\partial f}{\partial\theta'} = \cos\frac{\theta'}{2} \left(1 + 2\sin\frac{\theta'}{2}\right) = 0$  if and only if  $\theta' = \pi$ . For  $\theta = 0$ ,  $\partial f/\partial\phi' = 0$  for all  $\theta', \phi'$ , and  $\frac{\partial f}{\partial\theta'} = \cos\frac{\theta'}{2} \left(1 - 2\sin\frac{\theta'}{2}\right) = 0$  for  $\theta' = \pi$  or  $\theta' = \pi/3$ . Now, suppose  $\theta \in (0, \pi)$ . Then for  $\partial f/\partial\phi' = 0$  to be satisfied, one of  $\theta' = 0$ ,  $\theta' = \pi$ , or  $\sin(\phi - \phi') = 0$  must hold. If  $\theta' = 0$ ,  $\partial f/\partial\phi' = 0$  for all  $\phi'$ , and  $\partial f/\partial\theta' = 1 - \sin\theta \cos(\phi - \phi') = 0$  if and only if  $\theta = \pi/2$  and  $\cos(\phi - \phi') = 1$ , which is satisfied only for  $\phi' = \phi$ . If  $\theta' = \pi$ ,  $\partial f/\partial\phi' = 0$  for all  $\phi'$ , and  $\partial f/\partial\theta' = \sin\theta \cos(\phi - \phi') = 0$  if and only if  $\cos(\phi - \phi') = 0$ , i.e. if and only if  $\phi' = \phi + \pi/2$  or  $\phi' = \phi + 3\pi/2$ . Finally, suppose  $\theta' \in (0, \pi)$ . Then for  $\partial f/\partial\phi' = 0$  to be satisfied  $\sin(\phi - \phi') = 0$  must hold, and hence  $\phi' = \phi$  or  $\phi' = \phi + \pi$ . If  $\phi' = \phi$ ,  $\frac{\partial f}{\partial\theta'} = \sin\left(\frac{\theta' + \pi}{2}\right) - \sin(\theta + \theta') = 0$  if for  $n = 0, \pm 1, \pm 2, \dots$ , either of

$$\frac{\theta' + \pi}{2} = \theta + \theta' + 2n\pi, \quad \text{or} \quad \frac{\theta' + \pi}{2} = \pi - (\theta + \theta') + 2n\pi, \quad (17)$$

hold, i.e. if  $\theta' = \pi(1 - 4n) - 2\theta$ , or if  $\theta' = (\pi(1 + 4n) - 2\theta)/3$ ,  $n = 0, \pm 1, \pm 2, \dots$ . The case  $n = 0$  gives  $\theta' = \pi - 2\theta$  or  $\theta' = (\pi - 2\theta)/3$ , each of which satisfies  $\theta' \in [0, \pi]$  if and only if  $\theta \in [0, \pi/2]$ . For  $n \neq 0$ , all solutions of (17) lie outside  $[0, \pi]$ . Finally, if  $\phi' = \phi + \pi$ ,  $\frac{\partial f}{\partial\theta'} = \sin\left(\frac{\theta' + \pi}{2}\right) - \sin(\theta' - \theta) = 0$  if for  $n = 0, \pm 1, \pm 2, \dots$ , either of

$$\frac{\theta' + \pi}{2} = \theta' - \theta + 2n\pi, \quad \text{or} \quad \frac{\theta' + \pi}{2} = \pi - (\theta' - \theta) + 2n\pi, \quad (18)$$

hold, i.e. if  $\theta' = \pi(1 - 4n) + 2\theta$ , or if  $\theta' = (\pi(1 + 4n) + 2\theta)/3$ ,  $n = 0, \pm 1, \pm 2, \dots$ . The case  $n = 0$  gives  $\theta' = \pi + 2\theta$ , which lies outside  $[0, \pi]$ , or  $\theta' = (\pi + 2\theta)/3$ , which satisfies  $\theta' \in [0, \pi]$ . For  $n \neq 0$ , all solutions of (18) lie outside  $[0, \pi]$ .  $\square$

**Remark 3.2** *Thinking of the sphere as a globe with the incident field directed due north (so the southern hemisphere is illuminated and the northern hemisphere is in shadow, with the shadow boundary at the equator), we can interpret Theorem 3.1 geometrically as follows. If the observation point is at the north pole  $\theta = 0$  then there is a ring of stationary points in the northern hemisphere at  $\theta' = \pi/3$ , with a further line of stationary points at the south pole  $\theta' = \pi$  (a single point on the sphere, but a line of points in  $(\theta', \phi')$  space). As the observation point moves south, the ring of stationary points at  $\theta' = \pi/3$  disappears, and is replaced by two isolated stationary points in the northern hemisphere. One of these moves north as  $\theta$  increases, reaching the north pole and then disappearing as the observation point crosses the equator  $\theta = \pi/2$ , and the*

other moves south, crossing the equator when  $\theta = \pi/4$  and reaching the south pole coincidentally with the observation point. Meanwhile, as  $\theta$  increases from  $\theta = 0$  the line of stationary points at the south pole disappears, and is replaced by three isolated stationary points (in  $(\theta', \phi')$  space). Two of these remain at the south pole, with the other moving north as  $\theta$  increases, crossing the equator when  $\theta = \pi/4$  and reaching the north pole and disappearing as the observation point crosses the equator.

#### 4 Localized method of stationary phase and error analysis

Assuming for simplicity that  $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$ , it follows from Theorem 3.1 that there are three stationary points, at

$$(\theta_s^1, \phi_s^1) := \left( \frac{\pi + 2\theta}{3}, \phi + \pi \right), \quad (\theta_s^2, \phi_s^2) := \left( \pi, \phi + \frac{\pi}{2} \right), \quad (\theta_s^3, \phi_s^3) := \left( \pi, \phi + \frac{3\pi}{2} \right),$$

and if  $\theta \in (0, \pi/2)$  then there are two more stationary points at

$$(\theta_s^4, \phi_s^4) := (\pi - 2\theta, \phi), \quad (\theta_s^5, \phi_s^5) := \left( \frac{\pi - 2\theta}{3}, \phi \right).$$

We isolate these stationary points using a *partition of unity*. Taking pairwise disjoint neighborhoods  $\Omega_j'$  of  $(\theta_s^j, \phi_s^j)$ ,  $j = 1, \dots, N(\theta)$ , where

$$N(\theta) = \begin{cases} 3 & \text{if } \pi/2 < \theta < \pi, \\ 5 & \text{if } 0 < \theta < \pi/2, \end{cases} \quad (19)$$

and letting  $\Omega_j$  be a small neighborhood of  $(\theta_s^j, \phi_s^j)$  such that  $\bar{\Omega}_j \subset \Omega_j'$ , we can construct a  $C^\infty$  neutralizing function  $\chi_j$  (see [13, Ch.V, ex.7]) such that  $\chi_j \equiv 1$  on  $\Omega_j$ ,  $\chi_j \equiv 0$  outside  $\Omega_j'$ . We then rewrite (6) as

$$M\psi(\mathbf{p}(\theta, \phi)) = \sum_{j=1}^{N(\theta)+1} M_j\psi(\mathbf{p}(\theta, \phi)), \quad (20)$$

where with  $G_j(\theta', \phi') := \chi_j(\theta', \phi') \tilde{H}(k, \theta') \psi(\mathbf{p}(\alpha, \beta)) \cos(\theta'/2)$ ,  $j = 1, \dots, N(\theta)$ , and  $g(\theta', \phi') := \left[ 1 - \sum_{j=1}^{N(\theta)} \chi_j(\theta', \phi') \right] \tilde{H}(k, \theta') \psi(\mathbf{p}(\alpha, \beta)) \cos(\theta'/2)$ ,

$$M_j\psi(\mathbf{p}(\theta, \phi)) := \begin{cases} \int_{\phi}^{2\pi+\phi} \int_0^\pi G_j(\theta', \phi') e^{ikf(\theta', \phi')} d\theta' d\phi', & j = 1, \dots, N(\theta), \\ \int_{\phi}^{2\pi+\phi} \int_0^\pi g(\theta', \phi') e^{ikf(\theta', \phi')} d\theta' d\phi', & j = N(\theta) + 1. \end{cases}$$

Thus for  $j = 1, \dots, N(\theta)$  the domain of integration of  $M_j$  is a small region  $\Omega_j$ , and the integrand of  $M_{N(\theta)+1}$  is a  $C^\infty$  function with no stationary points. We approximate  $M_j\psi(\mathbf{p}(\theta, \phi))$ ,  $j = 1, \dots, N(\theta)$  using asymptotic expansions about each stationary point. First we define

$$\widehat{M}_1\psi(\mathbf{p}(\theta, \phi)) := -\frac{2\pi i G_1(\theta_s^1, \phi_s^1) e^{ikf(\theta_s^1, \phi_s^1)}}{k \left[ \frac{3}{2} \cos\left(\frac{\theta-\pi}{3}\right) \sin\theta \sin\left(\frac{\pi+2\theta}{3}\right) \right]^{1/2}}, \quad (21)$$

$$\widehat{M}_2\psi(\mathbf{p}(\theta, \phi)) := 0, \quad \widehat{M}_3\psi(\mathbf{p}(\theta, \phi)) := 0, \quad (22)$$

$$\widehat{M}_4\psi(\mathbf{p}(\theta, \phi)) := \frac{2\pi i G_4(\theta_s^4, \phi_s^4) e^{ikf(\theta_s^4, \phi_s^4)}}{k \left[ \frac{1}{2} \cos\theta \sin\theta \sin 2\theta \right]^{1/2}}, \quad (23)$$

$$\widehat{M}_5\psi(\mathbf{p}(\theta, \phi)) := \frac{2\pi G_5(\theta_s^5, \phi_s^5) e^{ikf(\theta_s^5, \phi_s^5)}}{k \left[ \frac{3}{2} \cos\left(\frac{\theta+\pi}{3}\right) \sin\theta \sin\left(\frac{\pi-2\theta}{3}\right) \right]^{1/2}}. \quad (24)$$

We follow [13, Chapter VIII] to derive the power of approximating  $M_j\psi$  by  $\widehat{M}_j\psi$ , for  $j = 1, \dots, N(\theta)$ .

**Theorem 4.1** For  $j = 1, \dots, N(\theta)$ , and for large  $k$ ,

$$M_j\psi(\mathbf{p}(\theta, \phi)) - \widehat{M}_j\psi(\mathbf{p}(\theta, \phi)) = O\left(\frac{1}{k^2}\right).$$

**Proof:** First we note that, with  $f := f(\theta', \phi')$ ,

$$\left. \begin{array}{l} \frac{\partial^2 f}{\partial \theta'^2} \frac{\partial^2 f}{\partial \phi'^2} - \frac{\partial^2 f}{\partial \theta' \partial \phi'} \left\{ \begin{array}{l} > 0 \text{ if } (\theta', \phi') = (\theta_s^i, \phi_s^i) \quad i = 1, 4, \\ < 0 \text{ if } (\theta', \phi') = (\theta_s^j, \phi_s^j) \quad j = 2, 3, 5. \end{array} \right. \end{array} \right.$$

Thus  $(\theta_s^1, \phi_s^1)$  is a local maximum,  $(\theta_s^4, \phi_s^4)$  is a local minimum, and  $(\theta_s^j, \phi_s^j)$ ,  $j = 2, 3, 5$ , are each saddle points. Following [13, Chapter VIII] we can write an expansion for  $M_j$ ,  $j = 1, \dots, N(\theta)$ , in increasing powers of  $1/k$ , with in each case the leading order term given by  $\widehat{M}_j$ . Noting that  $\cos \frac{\theta_s^2}{2} = \cos \frac{\theta_s^3}{2} = 0$ , the result follows.  $\square$

We approximate  $M_{N(\theta)+1}\psi(\mathbf{p}(\theta, \phi))$  by  $\widehat{M}_{N(\theta)+1}\psi(\mathbf{p}(\theta, \phi))$ , defined by

$$\begin{aligned} \widehat{M}_{N(\theta)+1}\psi(\mathbf{p}(\theta, \phi)) := & 2\pi\psi(\mathbf{p}(\theta, \phi)) \left( \frac{i\tilde{H}(k, 0)}{k|\cos\theta|} - \frac{\tilde{H}(k, 0)}{k^2 \cos^4\theta} \left[ 1 + \frac{1}{2} \sin^2\theta \right] - \frac{\frac{\partial \tilde{H}}{\partial \theta'}(k, 0)}{k^2 |\cos^3\theta|} \right) \\ & - \frac{\tilde{H}(k, 0)}{k^2} \int_{\phi}^{2\pi+\phi} \frac{\partial \psi(\mathbf{p}(\alpha, \beta))}{\partial \theta'} \Big|_{\theta'=0} \frac{d\phi'}{(1 - \sin\theta \cos(\phi - \phi'))^2}, \quad (25) \end{aligned}$$

and we estimate the error in this approximation in Theorem 4.3. First we need the following result.

**Lemma 4.2** *For any constant  $c$ , and for  $m = 1, 2, \dots$ ,*

$$\begin{aligned} K_m(\theta) &:= \int_c^{2\pi+c} \frac{1}{(1 - \sin \theta \cos(y - c))^m} dy, \\ &= \frac{2\pi}{|\cos \theta|^{2m-1}} \sum_{j=0}^{m-1} \frac{(2j-1)!!}{j!} \binom{m-1}{j} \sin^j \theta (1 - \sin \theta)^{m-1-j}, \end{aligned}$$

where  $(2j-1)!! = 1$  if  $j = 0$ , and  $(2j-1)!! := 1.3.5 \dots (2j-3)(2j-1)$ ,  $j \geq 1$ .

**Proof:** Making the substitution  $t = \tan((y-c)/2)$ , and defining  $a^2 := \frac{1-\sin \theta}{1+\sin \theta}$ ,

$$K_m(\theta) = \frac{4}{(1 + \sin \theta)^m} \sum_{j=0}^{m-1} \binom{m-1}{j} (1 - a^2)^j \int_0^\infty \frac{1}{(a^2 + t^2)^{j+1}} dt.$$

Noting that  $\int_0^\infty \frac{1}{(a^2+t^2)^{j+1}} dt = \frac{1}{a^{2j+1}} \frac{(2j-1)!!\pi}{2^{j+1}j!}$ , the result follows.  $\square$

**Theorem 4.3** *For fixed  $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$  and  $\phi \in [0, 2\pi]$ , there exist constants  $C_1 > 0$ ,  $C_2(\theta, \phi) > 0$ , each bounded independently of  $k$ , such that for  $k$  sufficiently large*

$$\left| M_{N(\theta)+1} \psi(\mathbf{p}(\theta, \phi)) - \widehat{M}_{N(\theta)+1} \psi(\mathbf{p}(\theta, \phi)) \right| \leq \frac{1}{k} \left( \frac{C_1}{\sin^2 \theta} + \frac{C_2(\theta, \phi)}{k} \right). \quad (26)$$

**Proof:** Following [13, p.425], since  $\nabla f \neq 0$  for  $(\theta', \phi') \in \text{supp}(g)$  it follows from the divergence theorem and the identity  $\nabla \cdot (\mathbf{u} e^{ikf}) = (\nabla \cdot \mathbf{u}) e^{ikf} + ik g e^{ikf}$ , where  $\mathbf{u} = \mathbf{u}_0 := \frac{\nabla f}{|\nabla f|^2} g$ ,  $g := g(\theta', \phi')$ , that for  $n = 1, 2, \dots$ ,

$$M_{N(\theta)+1} \psi(\mathbf{p}(\theta, \phi)) = -J(n) + \left( \frac{i}{k} \right)^n \int \int_{\text{supp}(g)} g_n e^{ikf} d\theta' d\phi', \quad (27)$$

where with  $\Gamma$  the positively oriented (anticlockwise) boundary of  $\text{supp}(g)$ ,  $\sigma$  the arc length of  $\Gamma$ , and  $\mathbf{n} := (n_1, n_2)$  the unit outward normal vector to  $\Gamma$ ,

$$J(n) := \sum_{s=0}^{n-1} \left( \frac{i}{k} \right)^{s+1} \int_{\Gamma} (\mathbf{u}_s \cdot \mathbf{n}) e^{ikf} d\sigma, \quad g_{s+1} := (\nabla \cdot \mathbf{u}_s), \quad \mathbf{u}_{s+1} := \frac{\nabla f}{|\nabla f|^2} g_{s+1}. \quad (28)$$

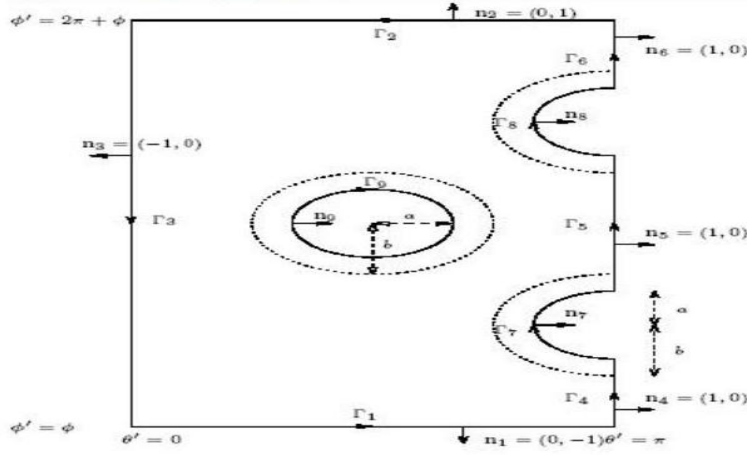
We immediately deduce that for  $n = 1, 2, \dots$ ,

$$\left| M_{N(\theta)+1} \psi(\mathbf{p}(\theta, \phi)) + J(n) \right| \leq \frac{1}{k^n} \left| \int \int_{\text{supp}(g)} g_n e^{ikf} d\theta' d\phi' \right| \leq \frac{C(\theta, \phi)}{k^{n+1}} \|g_n\|_\infty.$$

Next we evaluate (where  $f_{\theta'} := df/d\theta'$ ,  $f_{\phi'} := df/d\phi'$ )

$$\int_{\Gamma} (\mathbf{u}_s \cdot \mathbf{n}) e^{ikf} d\sigma = \int_{\Gamma} \frac{n_1 f_{\theta'} + n_2 f_{\phi'}}{f_{\theta'}^2 + f_{\phi'}^2} e^{ikf} g_s d\sigma, \quad \text{for } s = 0, 1. \quad (29)$$

As shown in Figure 1 (for  $\theta \in (\pi/2, \pi)$ ),  $\text{supp}(g)$  is bounded by the lines  $\phi' = \phi$ ,  $\phi' = 2\pi + \phi$ ,  $\theta' = 0$ ,  $\theta' = \pi$  and the supports of  $1 - \chi_j(\theta', \phi')$ ,  $j = 1, \dots, N(\theta)$ .



**Figure 1: Domain of integration  $\text{supp}(g)$ , for  $\theta \in (\pi/2, \pi)$ .**

The contributions to (29) from the sections of  $\Gamma$  corresponding to  $\phi' = \phi$  and  $\phi' = 2\pi + \phi$  ( $\Gamma_1$  and  $\Gamma_2$  in Figure 1) are both zero, since for  $\phi' = \phi$  and  $\phi' = 2\pi + \phi$  we have  $n_1 = 0$  and  $f_{\phi'} = 0$ . On the sections of  $\Gamma$  corresponding to  $\theta' = 0$  and  $\theta' = \pi$  ( $\Gamma_3$ ,  $\Gamma_4$ ,  $\Gamma_5$  and  $\Gamma_6$  in Figure 1) we have

$$\frac{n_1 f_{\theta'} + n_2 f_{\phi'}}{f_{\theta'}^2 + f_{\phi'}^2} e^{ikf} = \begin{cases} -1/(1 - \sin \theta \cos(\phi - \phi')), & \text{on } \theta' = 0, \\ e^{ik(2-2\cos\theta)}/\sin \theta \cos(\phi - \phi'), & \text{on } \theta' = \pi. \end{cases}$$

Recalling (28),

$$g_{s+1} = \left[ \frac{f_{\theta'\theta'} + f_{\phi'\phi'}}{f_{\theta'}^2 + f_{\phi'}^2} - 2 \frac{f_{\theta'}^2 f_{\theta'\theta'} + 2f_{\theta'} f_{\phi'} + f_{\phi'}^2 f_{\phi'\phi'}}{(f_{\theta'}^2 + f_{\phi'}^2)^2} \right] g_s + \frac{f_{\theta'}}{f_{\theta'}^2 + f_{\phi'}^2} \frac{\partial g_s}{\partial \theta'} + \frac{f_{\phi'}}{f_{\theta'}^2 + f_{\phi'}^2} \frac{\partial g_s}{\partial \phi'}. \quad (30)$$

From the definition of  $\chi_j$ ,  $j = 1, \dots, N(\theta)$ ,  $g$  and all its derivatives, and hence  $g_s$ ,  $s = 0, 1, \dots$ , then vanish on all other sections of  $\Gamma$  ( $\Gamma_7$ ,  $\Gamma_8$  and  $\Gamma_9$  in Figure 1, plus four other semicircles in the case  $\theta \in (0, \pi/2)$ ). Thus

$$\int_{\Gamma} (\mathbf{u}_s \cdot \mathbf{n}) e^{ikf} d\sigma = \int_{\phi}^{2\pi+\phi} \frac{g_s(0, \phi')}{1 - \sin \theta \cos(\phi - \phi')} d\phi' + \frac{e^{ik(2-2\cos\theta)}}{\sin \theta} \int_{\phi}^{2\pi+\phi} \frac{g_s(\pi, \phi')}{\cos(\phi - \phi')} d\phi'. \quad (31)$$

Using (30),

$$g_{s+1}(0, \phi') = \frac{\cos \theta}{(1 - \sin \theta \cos(\phi - \phi'))^2} g_s(0, \phi') + \frac{\frac{\partial g_s}{\partial \theta'}(0, \phi')}{1 - \sin \theta \cos(\phi - \phi')}, \quad (32)$$

$$g_{s+1}(\pi, \phi') = \left( \frac{1/2 - \cos \theta}{\sin^2 \theta \cos^2(\phi - \phi')} \right) g_s(\pi, \phi') + \frac{\frac{\partial g_s}{\partial \theta'}(\pi, \phi')}{\sin \theta \cos(\phi - \phi')},$$

and since  $\mathbf{p}(\alpha, \beta)|_{\theta'=0} = \mathbf{p}(\theta, \phi)$  and  $\mathbf{p}(\alpha, \beta)|_{\theta'=\pi} = \mathbf{p}(\pi - \theta, \phi)$ ,

$$g(0, \phi') = \tilde{H}(k, 0) \psi(\mathbf{p}(\theta, \phi)), \quad g(\pi, \phi') = 0,$$

$$g_1(0, \phi') = \left[ \frac{\tilde{H}(k, 0) \cos \theta}{(1 - \sin \theta \cos(\phi - \phi'))^2} + \frac{\tilde{H}'(k, 0)}{(1 - \sin \theta \cos(\phi - \phi'))} \right] \psi(\mathbf{p}(\theta, \phi))$$

$$+ \left[ \frac{\tilde{H}(k, 0)}{(1 - \sin \theta \cos(\phi - \phi'))} \right] \frac{\partial \psi(\mathbf{p}(\alpha, \beta))}{\partial \theta'} \Big|_{\theta'=0},$$

$$g_1(\pi, \phi') = \frac{- \left[ 1 - \sum_{j=1}^N \chi_j(\theta', \phi') \right] \tilde{H}(k, \pi) \psi(\mathbf{p}(\pi - \theta, \phi))}{2 \sin \theta \cos(\phi - \phi')}.$$

Using these in (31), for  $s = 0, 1$ ,

$$\int_{\Gamma} (\mathbf{u}_0 \cdot \mathbf{n}) e^{ikf} d\sigma = \tilde{H}(k, 0) \psi(\mathbf{p}(\theta, \phi)) \int_{\phi}^{2\pi+\phi} \frac{1}{(1 - \sin \theta \cos(\phi - \phi'))} d\phi', \quad (33)$$

$$\int_{\Gamma} (\mathbf{u}_1 \cdot \mathbf{n}) e^{ikf} d\sigma = \int_{\phi}^{2\pi+\phi} \frac{\tilde{H}(k, 0) \psi(\mathbf{p}(\theta, \phi)) \cos \theta}{(1 - \sin \theta \cos(\phi - \phi'))^3} d\phi' + \int_{\phi}^{2\pi+\phi} \frac{\tilde{H}'(0) \psi(\mathbf{p}(\theta, \phi))}{(1 - \sin \theta \cos(\phi - \phi'))^2} d\phi'$$

$$+ \int_{\phi}^{2\pi+\phi} \frac{\partial \psi(\mathbf{p}(\alpha, \beta))}{\partial \theta'} \Big|_{\theta'=0} \frac{\tilde{H}(k, 0)}{(1 - \sin \theta \cos(\phi - \phi'))^2} d\phi'$$

$$+ \frac{e^{ik(2-2\cos\theta)} \tilde{H}(k, \pi) \psi(\mathbf{p}(\pi - \theta, \phi))}{2 \sin^2 \theta} \int_{\phi}^{2\pi+\phi} \frac{[1 - \sum_{j=1}^N \chi_j(\theta', \phi')]}{\cos^2(\phi - \phi')} d\phi'. \quad (34)$$

Applying Lemma 4.2 and the fact that  $\tilde{H}(k, \pi)$  is of order  $k$  in (33) and (34), the result (26) follows from (28) (with  $n = 2$ ), (27) and (25).  $\square$

**Remark 4.4** Using (31), (32) and Lemma 4.2, we see that for  $\theta = \pi/2 \pm \delta$ , the leading order term of  $\int_{\Gamma}(\mathbf{u}_s \cdot \mathbf{n})e^{ikf} d\sigma$  for fixed  $k$  as  $\delta \rightarrow 0$  is of order

$$\int_{\phi}^{2\pi+\phi} \frac{\cos^s \theta}{(1 - \sin \theta \cos(\phi - \phi'))^{2s+1}} d\phi' \sim \frac{\cos^s \theta}{|\cos \theta|^{4s+1}} \sim \frac{1}{\delta^{3s+1}}.$$

Thus as  $\delta \rightarrow 0$ , with  $k$  fixed, each term in  $J(n)$  (see (28)) is of order  $\frac{1}{k^{s+1}\delta^{3s+1}} = \frac{\delta^2}{(k\delta^3)^{s+1}}$ . Hence for fixed  $k$ , we require  $\delta \geq Ck^{-1/3}$ , for some constant  $C > 0$ .

## 5 Numerical results

We demonstrate our approach by computing efficient approximations to the highly oscillatory weakly singular integral  $M\psi(\mathbf{x})$  in (1), with  $m$  given by the acoustic scattering kernel (5) (with  $\eta = k$ ), for a spherical scatterer of radius 1 at 1000 observed directions  $\mathbf{x} = \mathbf{p}(\theta, 0)$ , and for various values of  $k$ . Analytical formulae for these integrals are not known even for the constant density  $\psi \equiv 1$ . In order to facilitate computation of “exact” values with which to compare our results, we take  $\psi \equiv 1$ , in which case recalling (6), (15), (16), we have

$$M\psi(\mathbf{x}) = \int_0^{\pi} \tilde{H}(k, \theta') \cos \frac{\theta'}{2} e^{ik(2 \sin \frac{\theta'}{2} + \cos \theta (\cos \theta' - 1))} \int_0^{2\pi} e^{ik \sin \theta \sin \theta' \cos \phi'} d\phi' d\theta'. \quad (35)$$

The inner integral can be evaluated exactly using the Bessel functions of order zero:  $\int_0^{2\pi} e^{ika \cos y} dy = 2\pi J_0(ka)$ . For comparison purposes we then evaluated the  $\theta'$  integral in (35) to a very high accuracy, using Gaussian quadrature with 30 nodes per half wavelength, taking the resulting computed number to be the exact value of  $M\psi(\mathbf{p}(\theta, 0))$  with  $\psi \equiv 1$ .

Using (20), Theorem 4.1 and Theorem 4.3, our approximation  $M_{app}\psi(\mathbf{p}(\theta, 0))$  to  $M\psi(\mathbf{p}(\theta, 0))$  for  $\theta \in (0, \pi)$ ,  $\theta \neq \pi/2$ , is defined by

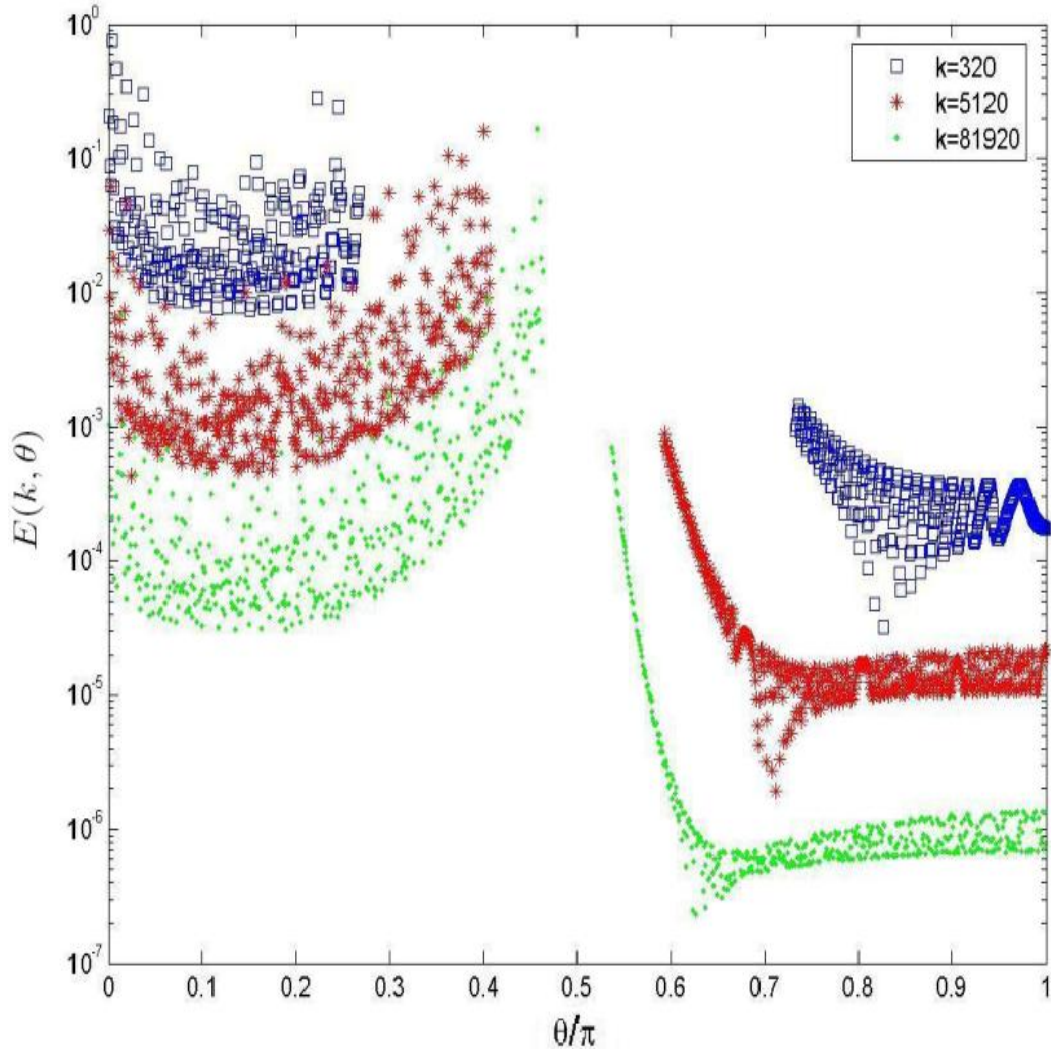
$$M_{app}\psi(\mathbf{p}(\theta, 0)) := \sum_{j=1}^{N(\theta)+1} \widehat{M}_j\psi(\mathbf{p}(\theta, 0)), \quad (36)$$

where  $\widehat{M}_j\psi$  for  $j = 1, \dots, N(\theta) + 1$  are given by (21)–(25), and  $N(\theta)$  is as defined in (19). From Theorems 4.1 and 4.3 and recalling Remark 4.4, we

would expect that for  $|\theta - \pi/2| > Ck^{-1/3}$  for some fixed constant  $C > 0$ ,

$$E(k, \theta) := \frac{|M\psi(\mathbf{p}(\theta, 0)) - M_{app}\psi(\mathbf{p}(\theta, 0))|}{|M\psi(\mathbf{p}(\theta, 0))|} \leq \frac{c(\theta)}{k}. \quad (37)$$

The relative errors  $E(k, \theta)$  evaluated for a thousand evenly spaced values of  $\theta$  with  $|\theta - \pi/2| > 5k^{-1/3}$  are shown for  $k = 320$ ,  $k = 5120$  and  $k = 81920$  in Figure 2. Evaluation of just the one dimensional  $\theta'$  integral in (35) for the exact solution of  $M\psi(\mathbf{p}(\theta, 0))$  took several hours on a AMD Opteron 2.0Ghz computer, with the computational time increasing for larger values of  $k$ . On the other hand our approximation  $M_{app}\psi(\mathbf{p}(\theta, 0))$  was computed for all values of  $\theta$  and  $k$  in less than a second, and the relative errors are clearly decreasing as  $k$  increases.



**Figure 2: Errors  $E(k, \theta)$  for  $|\theta - \pi/2| > 5k^{-1/3}$ .**  
 ( $\square$ ,  $k = 320$ ;  $*$ ,  $k = 5120$ ;  $\cdot$ ,  $k = 81920$ .)

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