# Positive weight quadrature on the sphere and monotonicities of Jacobi polynomials

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**Abstract**

In 2000, Reimer proved that a positive weight quadrature rule on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ has the property of quadrature regularity. Hesse and Sloan used a related property, called Property (R) in their work on estimates of quadrature error on $\mathbb{S}^d$. Using a variation on Reimer's bounds on the sum of the quadrature weight within a spherical cap, and using Jacobi polynomials of the form $P_t^{(1+d/2,d/2)}$, as well as the Sturm comparison theorem, the constants related Property (R) can be estimated. A recent conjecture on monotonicities of Jacobi polynomials would, if true, provide improved estimates for these constants.

## 1 Introduction

This paper addresses some inequalities related to the Jacobi polynomials and their largest zeros, and then applies these inequalities to estimates related to positive weight quadrature on the unit sphere.

The work originated with the deeper investigation of Reimer's approach to estimating the local density of positive weight quadrature rules on the unit sphere [8] [9, Section 6.5].

In 1995, Sloan [10] introduced the hyperinterpolation approximation, essentially a truncated Fourier expansion where integration is replaced by quadrature. Sloan and Womersley [11] found a bound on the uniform norm of the hyperinterpolation operator on the unit sphere in $\mathbb{R}^3$, subject to a condition on the quadrature rule, which they called *quadrature regularity.* Le Gia and Sloan [5] generalized this bound to the arbitrary higher dimensional case, subject to a quadrature regularity assumption appropriate to the dimension. Reimer [8] [9] independently found this generalized bound,

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and proved that the quadrature regularity assumption is not needed, ie. he proved that any quadrature rule which is admissible for hyperinterpolation is automatically quadrature regular. Hesse and Sloan, in their study of quadrature on the unit sphere in $\mathbb{R}^3$ [3] [4], introduced a property related to quadrature regularity, which they called *Property (R).* Reimer's proofs also apply to Property (R).

This paper concentrates on the estimation of the constants related to Property (R). We use Jacobi polynomials of the form $P_t^{(d/2+1,d/2)}$ to produce a variation on Reimer's ([8, Lemma 1, p. 272] [9, Lemma 6.23, p. 220]) bounds on the sum of the quadrature weight within a spherical cap. We then use the Sturm comparison theorem to estimate the constants related Property (R). We also conjecture an improvement on these estimates, based on a recent conjecture on monotonicities of Jacobi polynomials [2].

## 2 Preliminaries

For dimension $d$, the unit sphere $\mathbb{S}^d$ embedded in $\mathbb{R}^{d+1}$ is

$$\mathbb{S}^d := \left\{ \mathbf{x} \in \mathbb{R}^{d+1} \;\middle|\; \sum_{k=1}^{d+1} x_k^2 = 1 \right\}.$$

The spherical distance $s(\mathbf{a}, \mathbf{b})$ between two points $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$ is defined as

$$s(\mathbf{a}, \mathbf{b}) := \cos^{-1}(\mathbf{a} \cdot \mathbf{b}),$$

where the inner product is that of $\mathbb{R}^{d+1}$.

For $d \geqslant 1$, for any point $\mathbf{a} \in \mathbb{S}^d$ and any angle $\theta \in [0, \pi]$, the closed *spherical cap* $S(\mathbf{a}, \theta)$ is

$$S(\mathbf{a}, \theta) := \{\mathbf{b} \in \mathbb{S}^d \;\mid\; s(\mathbf{a}, \mathbf{b}) \leqslant \theta\},$$

that is the set of points of $\mathbb{S}^d$ whose spherical distance to $\mathbf{a}$ is at most $\theta$. The angle $\theta$ is called the *spherical radius* of the cap.

We use $\sigma$ to denote the Lebesgue area measure on $\mathbb{S}^d$. It is well known [7] that the area of a spherical cap $S(\mathbf{x}, \theta)$ of spherical radius $\theta$ and centre $\mathbf{x}$ is

$$\mathcal{V}_d(\theta) := \sigma\left(S(\mathbf{x}, \theta)\right) = \omega_{d-1} \int_0^\theta \sin^{d-1} \xi \; d\xi, \tag{2.1}$$

independent of $\mathbf{x}$.

We use Pochhammer's shifted factorial [1, p. 2],

$$(x)_n := \prod_{k=0}^{n-1} (x+k) = \frac{\Gamma(x+n)}{\Gamma(x)} \quad (-x \notin \mathbb{N}_0).$$

It is well known that for $\alpha, \beta > -1$ the $n$ zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ are real and distinct and are contained in the open interval $(-1,1)$ [12, Theorem 3.3.1, p. 44], [1, Theorem 5.4.1, p. 253]. We can therefore define

$$\Theta_n^{(\alpha,\beta)} := \cos^{-1} x_n^{(\alpha,\beta)}, \tag{2.2}$$

where is the largest zero of $P_n^{(\alpha,\beta)}$.

In this paper we use the normalized Jacobi polynomials $\widetilde{P}_n^{(\alpha,\beta)}$ defined by

$$\widetilde{P}_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}.$$

We use $\mathbb{P}_t(\mathbb{S}^d)$ to denote the real polynomials on $\mathbb{R}^{d+1}$, of maximum total degree $t$, restricted to $\mathbb{S}^d$, and This space is known [9, (4.4)] to have dimension

$$\mathcal{D}(d,t) := \dim \mathbb{P}_t(\mathbb{S}^d) = \frac{2t+d}{d!}\ (t+1)_{d-1} \tag{2.3}$$

and is known [9, (4.31)] to be a reproducing kernel Hilbert space with inner product

$$\langle f, g \rangle := \int_{\mathbb{S}^d} f(y) g(y)\ d\sigma(y) \tag{2.4}$$

and reproducing kernel $\Phi_t^{(d+1)}(x,y) := \Phi_t^{(d+1)}(x \cdot y)$, where

$$\Phi_t^{(d+1)} := \frac{\mathcal{D}(d,t)}{\omega_d}\ \widetilde{P}_t^{(\frac{d}{2},\frac{d}{2}-1)}. \tag{2.5}$$

A *quadrature rule* $Q := (X, W)$ on $\mathbb{S}^d$ of *strength* $t$ and *cardinality* $\mathcal{N}$ is a linear functional on the set of real-valued functions on $\mathbb{S}^d$, $(\mathbb{S}^d \to \mathbb{R})$ which is defined by a sequence $X$ of $\mathcal{N}$ quadrature points $(\mathbf{x}_1, \ldots, \mathbf{x}_{\mathcal{N}})$ on $\mathbb{S}^d$ and a sequence $W$ of $\mathcal{N}$ corresponding real quadrature weights $(w_1, \ldots, w_{\mathcal{N}})$,

$$Q\ f := \sum_{k=1}^{\mathcal{N}} w_k\ f(\mathbf{x}_k),$$

such that, for all $p \in \mathbb{P}_t(\mathbb{S}^d)$,

$$Q\ p = \int_{\mathbb{S}^d} p(\mathbf{y})\ d\sigma(\mathbf{y}).$$

We consider sequences of quadrature rules $(Q_1, \ldots)$ on $S^d$, where each $Q_t$ has strength $t$ and cardinality $\mathcal{N}_t$, with all weights positive. Such a sequence is said to be *admissible.*

An admissible sequence of quadrature rules is said to have *Property (R)* [3, (20)] [4, Definition 4] if and only if, given $\delta \in (0, \frac{\pi}{2}]$, there exists positive constants $\gamma$ and $T$ such that for all $\mathbf{y} \in \mathbb{S}^d$ and each $Q_t := (X_t, W_t)$ in the sequence, if $t \geqslant T$ then

$$\sum_{\mathbf{x}_{t,k} \in S\left(\mathbf{y}, \frac{\delta}{t}\right)} w_{t,k} \leqslant \gamma \; \sigma \left( S \left( \mathbf{x}_{t,k}, \frac{\delta}{t} \right) \right).$$

For $\theta \in (0, \xi]$, $\xi \in (0, \pi/2]$, we have the well-known estimate

$$\sin \theta \in [\operatorname{sinc} \xi, 1] \; \theta. \tag{2.6}$$

where $\operatorname{sinc} \theta := \frac{\sin \theta}{\theta}$. From (2.1) we have $\frac{\partial}{\partial \theta} \mathcal{V}_d(\theta) = \omega_{d-1} \sin^{d-1} \theta$. Using the estimate (2.6) therefore gives us

$$\frac{\partial}{\partial \theta} \mathcal{V}_d(\theta) \in [(\operatorname{sinc} \xi)^{d-1}, 1] \; \omega_{d-1} \theta^{d-1},$$

so

$$\mathcal{V}_d(\theta) \in [(\operatorname{sinc} \xi)^{d-1}, 1] \; \frac{\omega_{d-1}}{d} \theta^d. \tag{2.7}$$

## 3 Positive weight quadrature and Property (R)

Reimer's full method of proof that any admissible sequence of positive weight quadrature rules is quadrature regular [8, Section 2] [9, Section 6.5, pp. 220–222] makes it difficult to estimate the constants involved. The 2000 version of Reimer's proof [8, Section 2] uses $\Phi_\nu^{(d+1)}$, the reproducing kernel for $\mathbb{P}_\nu(\mathbb{S}^d)$. The first part of Reimer's proof is adapted for use here.

**Lemma 3.1** *Let $Q := (x, w)$ be a positive weight quadrature rule on $\mathbb{S}^d$ of strength $2\nu$. Let $K := \Phi_\nu^{(d+1)}$ denote the reproducing kernel of $\mathbb{P}_\nu(\mathbb{S}^d)$. Then for $\theta \in (0, \Theta_\nu^{(\frac{d}{2}, \frac{d}{2}-1)})$, for any $\mathbf{y} \in \mathbb{S}^d$,*

$$\sum_{x_k \in S(\mathbf{y}, \theta)} w_k \leqslant \frac{K(1)}{K^2(\cos \theta)} = \left( \widetilde{P}_\nu^{(\frac{d}{2}, \frac{d}{2}-1)}(\cos \theta) \right)^{-2} \; \frac{\omega_d}{\mathcal{D}(d, \nu)}, \tag{3.1}$$

*where $\cos \Theta_\nu^{(\frac{d}{2}, \frac{d}{2}-1)}$ is the largest zero in $\widetilde{P}_\nu^{(\frac{d}{2}, \frac{d}{2}-1)}$, as per (2.2).*

## 4 Monotonicity of functions related to Jacobi polynomials

The expression on the right hand side of the equality in (3.1) contains two factors which depend on both the dimension $d$ and the degree $\nu$. This makes it difficult to use in estimating the constants related to Property (R). We can replace the factor involving $\widetilde{P}_\nu^{(\frac{d}{2},\frac{d}{2}-1)}$ with a factor which does not depend on $\nu$, making the resulting expression easier to use.

**Theorem 4.1** *For $n > 1$, $\alpha \geqslant \beta > -\frac{1}{2}$, $\theta \in (0, \frac{\pi}{2}]$, we have*

$$\widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right) > (\operatorname{sinc}\theta)^{\alpha+\frac{1}{2}}\ \widetilde{P}_1^{(\alpha,\beta)}(\cos\theta).$$

We prove Theorem 4.1 by using Sturm's comparison theorem with a variation on one of Szegö's transformations of the differential equation for Jacobi polynomials [12, (4.24.2), p. 67].

**Theorem 4.2** *Let $\alpha, \beta > -1$, $n \in \mathbb{N}$. The function*

$$V_n^{(\alpha,\beta)}(\theta) := \left(2n\ \sin\frac{\theta}{2n}\right)^{\alpha+\frac{1}{2}} \left(\cos\frac{\theta}{2n}\right)^{\beta+\frac{1}{2}}\ \widetilde{P}_n^{(\alpha,\beta)}\left(\cos\frac{\theta}{n}\right) \tag{4.1}$$

*satisfies the linear differential equation*

$$\frac{\partial^2}{\partial\theta^2}V_n^{(\alpha,\beta)}(\theta) + F_n^{(\alpha,\beta)}(\theta)\ V_n^{(\alpha,\beta)}(\theta) = 0, \tag{4.2}$$

*where*

$$F_n^{(\alpha,\beta)}(\theta) := \frac{1}{n^2}\left(\frac{\frac{1}{4}-\alpha^2}{4\sin^2\frac{\theta}{2n}} + \frac{\frac{1}{4}-\beta^2}{4\cos^2\frac{\theta}{2n}}\right) + \left(1+\frac{\alpha+\beta+1}{2n}\right)^2. \tag{4.3}$$

The proof is elementary and is omitted.

We use a reformulation of Makai's special case of Sturm's comparison theorem [6, pp. 165–166].

**Theorem 4.3** *Let $f_1$ and $f_2$ be functions continuous on the interval $(x_1, x_2]$, with*

$$f_1(x) < f_2(x) \quad \text{for } x \in (x_1, x_2]. \tag{4.4}$$

*Let the functions $y_1$ and $y_2$, both not identically zero, satisfy the differential equations*

$$\frac{d^2}{dx^2}y_1(x) + f_1(x)y_1(x) = 0, \quad \frac{d^2}{dx^2}y_2(x) + f_2(x)y_2(x) = 0, \tag{4.5}$$

*respectively, for $x \in (x_1, x_2]$. If, in addition, $y_1$ and $y_2$ satisfy*

$$y_1(x) > 0, \quad y_2(x) > 0 \quad \text{for } x \in (x_1, x_2), \tag{4.6}$$

$$\lim_{x\to x_1^+}\left(y_2(x)\frac{d}{dx}y_1(x) - y_1(x)\frac{d}{dx}y_2(x)\right) = 0, \tag{4.7}$$

$$\lim_{x\to x_1^+}\frac{y_1(x)}{y_2(x)} \geqslant 1$$

*then for $x \in (x_1, x_2]$ we have $y_1(x) > y_2(x)$.*

The proof follows Makai's proof [6, pp. 165–166], using some details from the proofs of Watson [13, 15.83, pp. 518–519] and Andrews, Askey and Roy, [1, Theorem 4.14.2, p. 227-228]. Details are omitted.

The following theorem shows that the function $F_n^{(\alpha,\beta)}$ of Theorem 4.2 satisfies condition (4.4) of Theorem 4.3.

**Theorem 4.4** *For $n \geqslant 1$, $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $\theta \in (0, \frac{\pi}{2}]$, the function $F_n^{(\alpha,\beta)}$ defined by (4.3) satisfies*

$$F_n^{(\alpha,\beta)}(\theta) \geqslant F_{n+1}^{(\alpha,\beta)}(\theta),$$

*with equality only when $\alpha = \beta = -\frac{1}{2}$.*

We use the following well known result.

**Lemma 4.5** *For $\alpha \geqslant \beta > -\frac{1}{2}$, we have*

$$\frac{\pi}{2n} = \Theta_n^{(-1/2,-1/2)} < \Theta_n^{(\alpha,\beta)}.$$

We can now apply Theorem 4.3 to the function $F_n^{(\alpha,\beta)}$ of Theorem 4.2 to obtain the following result, which is used in the proof of Theorem 4.1.

**Theorem 4.6** *For $n \geqslant 1$, $\alpha \geqslant \beta > -\frac{1}{2}$, $\theta \in (0, \frac{\pi}{2}]$, for $V_n^{(\alpha,\beta)}$ defined by (4.1) we have*

$$V_n^{(\alpha,\beta)} < V_{n+1}^{(\alpha,\beta)}.$$

## 5 Estimation of constants related to quadrature rules

Using Theorem 4.1, we can now estimate the constants related to Property (R).

**Theorem 5.1** *For $t \geqslant T \geqslant 2$, let $Q = (X, W)$ be a positive weight quadrature rule on $\mathbb{S}^d$ which is exact on $\mathbb{P}_t(\mathbb{S}^d)$. Then for $\delta \in (0, \pi)$, for any $\mathbf{y} \in \mathbb{S}^d$,*

$$\sum_{\mathbf{x}_k \in S\left(\mathbf{y}, \frac{\delta}{t}\right)} w_k \leqslant c(d, \delta)\ t^{-d} \leqslant \gamma(d, \delta, T)\ \sigma\left(S\left(\mathbf{y}, \frac{\delta}{t}\right)\right), \tag{5.1}$$

*where*

$$c(d, \delta) := 2^{d-1}\ \omega_d\ d!\ \left(\operatorname{sinc}\frac{\delta}{2}\right)^{-d-1}\ \left(\widetilde{P}_1^{(\frac{d}{2}, \frac{d}{2}-1)}\left(\cos\frac{\delta}{2}\right)\right)^{-2} \tag{5.2}$$

$$\gamma(d, \delta, T) := \frac{d}{\omega_{d-1}}\ \left(\operatorname{sinc}\frac{\delta}{T}\right)^{-d+1}\ \delta^{-d}\ c(d, \delta) \tag{5.3}$$

Observation of the graphs of $\widetilde{P}_n^{(1,0)}$ for small values of $n$, suggests a conjecture on a montonicity property of these and similar Jacobi polynomials.

**Conjecture 5.2** *For* $\alpha \geqslant \frac{1}{2}$, $n \geqslant 1$, $\theta \in (0, \Theta_1^{(\alpha,\alpha-1)})$, *we have*

$$\widetilde{P}_n^{(\alpha,\alpha-1)}\left(\cos\frac{\theta}{n}\right) < \widetilde{P}_{n+1}^{(\alpha,\alpha-1)}\left(\cos\frac{\theta}{n+1}\right),$$

*where* $\Theta_1^{(\alpha,\beta)}$ *is defined by (2.2).*

In the context of this paper, the restriction to $\alpha \geqslant \frac{1}{2}$, $\beta = \alpha - 1$ is natural. Gautschi [2] has recently generalized this conjecture to deal with $\alpha, \beta > -1$.

We can improve the estimates given in Theorem 5.1 if we assume that Conjecture 5.2 is true.

**Corollary 5.3** *For* $t \geqslant T \geqslant 2$, *let* $Q = (X, W)$ *be a positive weight quadrature rule on* $\mathbb{S}^d$ *which is exact on* $\mathbb{P}_t(\mathbb{S}^d)$. *If Conjecture 5.2 is true for* $\alpha = \frac{d}{2}$, *then for* $\delta \in (0, \pi)$, *for any* $\mathbf{y} \in \mathbb{S}^d$, *we have*

$$\sum_{\mathbf{x}_k \in S\left(\mathbf{y}, \frac{\delta}{t}\right)} w_k \leqslant c'(d,\delta)\; t^{-d} \leqslant \gamma'(d,\delta,T)\; \sigma\left(S\left(\mathbf{y}, \frac{\delta}{t}\right)\right),$$

*where*

$$c'(d,\delta) := \left(\operatorname{sinc}\frac{\delta}{2}\right)^{d+1} c(d,\delta), \quad \gamma'(d,\delta,T) := \left(\operatorname{sinc}\frac{\delta}{2}\right)^{d+1} \gamma(d,\delta,T).$$

# 6 Proofs

*Proof of Lemma 3.1.* This proof follows Reimer [8, Lemma 1, p. 272] [9, Lemma 6.23, p. 220]. Assume that $Q$ has cardinality $\mathcal{N}$. Because $Q$ is exact for degree $2\nu$ and because $K$ is a reproducing kernel of degree $\nu$, for any $\mathbf{y} \in \mathbb{S}^d$ we have

$$\sum_{k=1}^{\mathcal{N}} w_k K^2(\mathbf{x}_k \cdot \mathbf{y}) = \int_{\mathbb{S}^d} K^2(\mathbf{z} \cdot \mathbf{y})\; d\sigma(\mathbf{z}) = K(1).$$

Since $\theta \in (0, \Theta_\nu^{(\frac{d}{2},\frac{d}{2}-1)})$, $K(\cos\phi)$ is monotonically decreasing in $\phi$ for $\phi \in (0, \theta]$, and so for any $\mathbf{y} \in \mathbb{S}^d$,

$$0 \leqslant K^2(\cos\theta) \sum_{\mathbf{x}_k \in S(\mathbf{y},\theta)} w_k \leqslant \sum_{\mathbf{x}_k \in S(\mathbf{y},\theta)} w_k K^2(\mathbf{x}_k \cdot \mathbf{y}) \leqslant K(1).$$

This gives the inequality (3.1). The equality is a result of (2.5). □

*Proof of Theorem 4.4.* We first note that $F_n^{(-\frac{1}{2},-\frac{1}{2})}(\theta) = 1$ for $n \geqslant 1$.

Since we are concerned with $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, we define $b := \beta + \frac{1}{2}$, $c := \alpha - \beta$ so that $b \geqslant 0$, $c \geqslant 0$, $\alpha = b + c - \frac{1}{2}$, $\beta = b - \frac{1}{2}$, and introduce the function

$$g(n,b,c,\theta) := F_n^{(b+c-\frac{1}{2},b-\frac{1}{2})}(\theta) - F_{n+1}^{(b+c-\frac{1}{2},b-\frac{1}{2})}(\theta).$$

We are therefore required to prove that for $n >= 1$, $b \geqslant 0$, $c \geqslant 0$, $\theta \in (0, \frac{\pi}{2}]$ we have $g(n,b,c,\theta) \geqslant 0$, with equality only when $b = c = 0$.

We first express $g(n,b,c,\theta)$ as a fraction over a common denominator, obtaining

$$g(n,b,c,\theta) = \frac{h(n,b,c,\theta)}{\left(2\ n\ \sin\frac{\theta}{n}\ (n+1)\ \sin\frac{\theta}{n+1}\right)^2}, \tag{6.1}$$

where $h(n,b,c,\theta) := h_+(n,b,c,\theta) - h_-(n,b,c,\theta)$, with

$$\begin{aligned}
h_+(n,b,c,\theta) \quad &:= \quad 2\left(c\left(1+\cos\frac{\theta}{n}\right)+2b\right)(n+1)^2\sin^2\frac{\theta}{n+1} \\
&\quad +\Bigg[\big((1+2n)(2b+c)+4n(n+1)\big)(2b+c)\sin^2\frac{\theta}{n+1} \\
&\quad +2\left((2b+c)c\left(1+\cos\frac{\theta}{n+1}\right)+2b^2\right)n^2\Bigg]\sin^2\frac{\theta}{n}, \\
h_-(n,b,c,\theta) \quad &:= \quad 2\left(c\left(1+\cos\frac{\theta}{n+1}\right)+2b\right)n^2\sin^2\frac{\theta}{n} \\
&\quad +2\left((2b+c)c\left(1+\cos\frac{\theta}{n}\right)+2b^2\right)(n+1)^2\sin^2\frac{\theta}{n+1}.
\end{aligned}$$

Since the denominator of $g(n,b,c,\theta)$ at (6.1) is positive for $n \geqslant 1$, $\theta \in (0, \frac{\pi}{2}]$, we can ignore it and concentrate on the numerator $h(n,b,c,\theta)$. We have expressed $h(n,b,c,\theta)$ in a way which exposes the parts which are obviously positive and obviously negative within our domain of interest.

We see that for $b \geqslant 0$, $c \geqslant 0$, $n \geqslant 1$, $\theta \in (0,\pi]$, we have $h_+(n,b,c,\theta) \geqslant 0$, $h_-(n,b,c,\theta) \geqslant 0$, with equality only when $b = c = 0$.

We now expand the trig functions in $h_+(n,b,c,\theta)$ and $h_-(n,b,c,\theta)$ in Taylor's series and truncate appropriately. This means, for the positive part $h_+(n,b,c,\theta)$ we end each alternating sum with a negative term, giving an underestimate; for the negative part, $h_-(n,b,c,\theta)$ we end each alternating sum with a positive term, giving an overestimate. We truncate in such a way as to obtain polynomials in $\theta$ of degree at most 12. We therefore define

$$\begin{aligned}
&j_+(n,b,c,\theta) := \\
&2\left(c\left(2-\frac{\theta^2}{2n^2}\right)+2b\right)\left(\theta^2-\frac{\theta^4}{3(n+1)^2}+\frac{2\theta^4}{45(n+1)^6}-\frac{\theta^6}{315(n+1)^8}\right) \\
&+\left[\big((1+2n)(2b+c)+4n(n+1)\big)(2b+c)\left(\frac{\theta^2}{(n+1)^2}-\frac{\theta^4}{3(n+1)^4}\right)\right. \\
&\left.+2\left((2b+c)c\left(2-\frac{\theta^2}{2(n+1)^2}\right)+2b^2\right)n^2\right]\left(\frac{\theta^2}{n^2}-\frac{\theta^4}{3n^4}+\frac{2\theta^6}{45n^6}-\frac{\theta^8}{315n^8}\right), \\
&j_-(n,b,c,\theta) := \\
&2\left(c\left(2-\frac{\theta^2}{2(n+1)^2}+\frac{\theta^4}{24(n+1)^4}\right)+2b\right)\left(\theta^2-\frac{\theta^4}{3n^2}+\frac{2\theta^4}{45n^6}\right) \\
&+2\left((2b+c)c\left(2-\frac{\theta^2}{2n^2}+\frac{\theta^4}{24n^4}\right)+2b^2\right)\left(\theta^2-\frac{\theta^4}{3(n+1)^2}+\frac{2\theta^4}{45(n+1)^6}\right).
\end{aligned}$$

We have ensured that each of the truncated Taylor's series expansions is positive for $n \geqslant 1$, $\theta \in (0, \frac{\pi}{2}]$ and also ensured that $h_+(n,b,c,\theta) \geqslant j_+(n,b,c,\theta)$ and $h_-(n,b,c,\theta) \leqslant j_-(n,b,c,\theta)$, so that $j := j_+ - j_- \leqslant h$. Thus if we can prove that $j(n,b,c,\theta) \geqslant 0$ then we have proved that $h(n,b,c,\theta) \geqslant 0$ and therefore $g(n,b,c,\theta) \geqslant 0$.

Examining $j_+$ and $j_-$, we see that $j(n,b,c,\theta)$ is a polynomial in $\theta^2$. We also recall that $n \geqslant 1$ and we therefore substitute $\phi := \theta^2$ and $m := n-1$ into $j$, and rearrange to find common factors. We thus obtain

$$j(m+1,b,c,\sqrt{\phi}) = \frac{q(m,b,c,\phi)\ \phi^2}{3780\ (m+1)^8\ (m+2)^6},$$

where the coefficient corresponding to each power of $m, b, c$ in $q(m,b,c,\phi)$ is a polynomial in $\phi$ with degree at most 4 and with a positive constant term. Also, for $q(0,b,c,0)$, the coefficient corresponding to each power of $b, c$ is a positive constant. Therefore if $n = m+1 \geqslant 1$, $b \geqslant 0$ and $c \geqslant 0$ and if none of these polynomials has a zero in $(0, \pi^2/4]$, then for $\theta \in (0, \frac{\pi}{2}]$ we have $q(m,b,c,\phi) \geqslant 0$, with equality only if $b = c = 0$. This, in turn implies that $g(n,b,c,\theta) \geqslant 0$ under the same conditions, with equality only if $b = c = 0$.

Since each polynomial in $q(m,b,c,\phi)$ has degree at most 4, we can use a computer package such as Maple to find the zeros analytically and verify that there are no zeros in $(0, \pi^2/4]$. □

*Proof of Lemma 4.5.* For $n \geqslant 1$, $\alpha \geqslant \beta > -\frac{1}{2}$, the monotonicities of the Jacobi zeros as given by [12, Theorem 6.21.1] imply that $\Theta_n^{(\alpha,\beta)} > \Theta_n^{(-\frac{1}{2},-\frac{1}{2})}$. For $n \geqslant 1$, [12, (6.3.5)] gives us $n\ \Theta_n^{(-\frac{1}{2},-\frac{1}{2})} = \frac{\pi}{2}$. □

*Proof of Theorem 4.6.* From Theorem 4.2, the differential equation (4.2), with $F_n^{(\alpha,\beta)}$ defined by (4.3) is satisfied by the function $V_n^{(\alpha,\beta)}$ defined by (4.1).

We see that the expression

$$\left(2n \sin \frac{\theta}{2n}\right)^{\alpha+\frac{1}{2}} \left(\cos \frac{\theta}{2n}\right)^{\beta+\frac{1}{2}}$$

is positive in the the closed interval $\theta \in (0, \frac{\pi}{2}]$. Therefore, in this interval, $V_n^{(\alpha,\beta)}(\theta)$ is positive if and only if $\widetilde{P}_n^{(\alpha,\beta)}\left(\cos \frac{\theta}{n}\right)$ is positive.

In the formulation of Theorem 4.3, we now put $f_1 := F_{n+1}^{(\alpha,\beta)}$, $f_2 := F_n^{(\alpha,\beta)}$, $y_1 := V_{n+1}^{(\alpha,\beta)}$, $y_2 := V_n^{(\alpha,\beta)}$, $x_1 := 0$, $x_2 := \frac{\pi}{2}$.

Theorem 4.4 shows that condition (4.4) is satisfied.

Theorem 4.2 shows that the conditions (4.5) are satisfied.

Lemma 4.5 implies that condition (4.6) is satisfied.

Verification of condition (4.7) is elementary and is omitted.

We have therefore satisfied all of the conditions of Theorem 4.3 and our result then follows. □

*Proof of Theorem 4.1.* For $n > 1$, $\alpha \geqslant \beta > -\frac{1}{2}$, $\theta \in (0, \frac{\pi}{2}]$, as a consequence of Theorem 4.6, we have

$$V_1^{(\alpha,\beta)}(\theta) < V_n^{(\alpha,\beta)}(\theta).$$

For $\alpha \geqslant \beta > -\frac{1}{2}$ we can express $V_n^{(\alpha,\beta)}(\theta)$ as

$$V_n^{(\alpha,\beta)}(\theta) = \left(2n \ \sin \frac{\theta}{2n}\right)^{\alpha-\beta} \left(n \ \sin \frac{\theta}{n}\right)^{\beta+\frac{1}{2}} \ \widetilde{P}_n^{(\alpha,\beta)}\left(\cos \frac{\theta}{n}\right),$$

using a well-known half-angle formula.

For $n > 1$, $\alpha \geqslant \beta > -\frac{1}{2}$, $\theta \in (0, \frac{\pi}{2}]$, since $\sin\theta < n \sin \frac{\theta}{n} < \theta$, we therefore have

$$\sin^{\alpha+\frac{1}{2}} \theta \ \widetilde{P}_1^{(\alpha,\beta)}(\cos\theta) < \theta^{\alpha+\frac{1}{2}} \ \widetilde{P}_n^{(\alpha,\beta)}\left(\cos \frac{\theta}{n}\right),$$

and the result follows. □

*Proof of Theorem 5.1.* Let $\nu := \lfloor \frac{t}{2} \rfloor$. We have $2\nu \leqslant t$, so $Q_t$ is exact on $\mathbb{P}_{2\nu}(\mathbb{S}^d)$. Also, $0 < \delta < \pi$, so Lemma 4.5 yields

$$0 < \frac{\delta}{t} \leqslant \frac{\delta}{2\nu} < \frac{\pi}{2\nu} = \Theta_\nu^{(-\frac{1}{2},-\frac{1}{2})} < \Theta_\nu^{(\frac{d}{2},\frac{d}{2}-1)}.$$

By Lemma 3.1, we have

$$\mathrm{LHS} := \sum_{\mathbf{x}_k \in S\left(\mathbf{y}, \frac{\delta}{t}\right)} w_k \leqslant \left(\widetilde{P}_\nu^{(\frac{d}{2},\frac{d}{2}-1)}\left(\cos \frac{\delta}{t}\right)\right)^{-2} \frac{\omega_d}{\mathcal{D}(d,\nu)}.$$

Since $\widetilde{P}_\nu^{(\frac{d}{2},\frac{d}{2}-1)}(\cos\theta)$ is monotonically decreasing in $\theta$ for $0 < \theta < \Theta_\nu^{(\frac{d}{2},\frac{d}{2}-1)}$, we therefore have

$$\mathrm{LHS} \leqslant \left(\widetilde{P}_\nu^{(\frac{d}{2},\frac{d}{2}-1)}\left(\cos\frac{\delta}{2\nu}\right)\right)^{-2} \frac{\omega_d}{\mathcal{D}(d,\nu)}.$$

Since $\delta \in (0,\pi)$, by Theorem 4.1 we therefore have

$$\mathrm{LHS} \leqslant \left(\operatorname{sinc}\frac{\delta}{2}\right)^{-d-1} \left(\widetilde{P}_1^{(\frac{d}{2},\frac{d}{2}-1)}\left(\cos\frac{\delta}{2}\right)\right)^{-2} \frac{\omega_d}{\mathcal{D}(d,\nu)}. \tag{6.2}$$

From (2.3) we have

$$\begin{aligned}\mathcal{D}(d,\nu) &= \frac{2\nu+d}{d!}\,(\nu+1)_{d-1} > \frac{2\nu+d}{d!}\left(\nu+\frac{1}{2}\right)^{d-1} \\ &\geqslant \frac{2}{d!}\left(\nu+\frac{1}{2}\right)^{d} \geqslant \frac{2}{d!}\left(\frac{t}{2}\right)^{d} = \frac{t^d}{2^{d-1}\,d!}.\end{aligned}$$

Therefore from (6.2),

$$\mathrm{LHS} \leqslant \left(\operatorname{sinc}\frac{\delta}{2}\right)^{-d-1} \left(\widetilde{P}_1^{(\frac{d}{2},\frac{d}{2}-1)}\left(\cos\frac{\delta}{2}\right)\right)^{-2} \omega_d\, 2^{d-1}\, d!\, t^{-d},$$

which proves the first inequality of (5.1) given (5.2). Since $\delta \in (0,\pi)$ and $t \geqslant T \geqslant 2$, we have $\frac{\delta}{t} \leqslant \frac{\delta}{T} \leqslant \frac{\pi}{2}$. Therefore (2.1) and the estimate (2.7) yield

$$\sigma\left(S\left(\mathbf{y},\frac{\delta}{t}\right)\right) = \mathcal{V}_d\left(\frac{\delta}{t}\right) \geqslant \left(\operatorname{sinc}\frac{\delta}{T}\right)^{d-1} \frac{\omega_{d-1}}{d}\,\delta^d\, t^{-d}$$

and we have

$$t^{-d} \leqslant \left(\operatorname{sinc}\frac{\delta}{T}\right)^{-d+1} \frac{d}{\omega_{d-1}}\,\delta^{-d}\,\mathcal{V}_d\left(\frac{\delta}{t}\right),$$

which proves the second inequality of (5.1) given (5.3). □

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