

Pseudodifferential equations on the sphere with spherical splines: Error analysis

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Abstract

Spherical splines are used to define approximate solutions to strongly elliptic pseudodifferential equations on the unit sphere. These equations arise from geodesy. The approximate solutions are found by using Galerkin method. We prove optimal convergence (in Sobolev norms) of the approximate solution by spherical splines to the exact solution. Our numerical results underlie the theoretical result.

Keywords: pseudodifferential equation, sphere, spherical spline

1 Introduction

Pseudodifferential operators have long been used [9, 10] as a modern and powerful tool to tackle linear boundary-value problems. Svensson [15] introduces this approach to geodesists who study [7, 8] these problems on the sphere which is taken as a model of the earth. Efficient solutions to pseudodifferential equations on the sphere become more demanding when given data are collected by satellites. In this paper, we study the use of spherical splines to find approximate solutions to these equations.

For spherical data interpolation and approximation, tensor products of univariate splines are not a good choice since data locations are not usually spaced over a regular grid. Spherical radial basis functions seem to be a better choice [16]; however, the resulting matrix systems from this approximation are very ill-conditioned.

Spherical splines [1, 3, 2] appear to have many properties in common with classical polynomial splines over planar triangulations and are well suited for scattered data interpolation and approximation problems. Baramidze and Lai [5] use these functions to solve the Laplace–Beltrami equation on the unit sphere in \mathbb{R}^3 .

In this paper, we solve strongly elliptic pseudodifferential equations on the sphere by the Galerkin method using spherical splines. This class of equations include the Laplace–Beltrami equation, Stokes equation, weakly singular integral equations, and

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many others [7, 8, 15]. Our main result is an optimal convergence rate of the approximation. The key of the analysis is proving the approximation property of spherical splines as a subset of Sobolev spaces. Since the pseudodifferential operators to be studied can be of any real order, it is necessary to obtain an approximation property in Sobolev norms of real orders, negative and positive.

The paper is organised as follows. In Section 2 we define Sobolev spaces and the pseudodifferential equations to be study. The definition of spherical splines is recalled in Section 3. In Section 4 we prove the approximation property of spherical splines as a subset of Sobolev spaces. Section 5 presents the main result of the Galerkin scheme. Our numerical results are reported in Section 6.

Throughout this paper, C is a generic constant which may take different value at different occurrence.

2 Preliminaries

2.1 Sobolev spaces

Throughout this paper, we denote by \mathbb{S} the unit sphere in \mathbb{R}^3 , i.e., $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ where $|\cdot|$ is the Euclidean norm in \mathbb{R}^3 . A spherical harmonic of order ℓ on \mathbb{S} is the restriction to \mathbb{S} of a homogeneous harmonic polynomial of degree ℓ in \mathbb{R}^3 . The space of all spherical harmonics of order ℓ is the eigenspace of the Laplace–Beltrami operator $\Delta_{\mathbb{S}}$ corresponding to the eigenvalue $\lambda_{\ell} = -\ell(\ell + 1)$. The dimension of this space being $2\ell + 1$, see e.g. [12, page 4], one may choose for it an orthonormal basis $\{Y_{\ell,m}\}_{m=-\ell}^{\ell}$. The collection of all the spherical harmonics $Y_{\ell,m}$, $m = -\ell, \dots, \ell$ and $\ell = 0, 1, \dots$, forms an orthonormal basis for $L_2(\mathbb{S})$.

For $s \in \mathbb{R}$, the Sobolev space H^s is defined as usual by

$$H^s := \left\{ v \in \mathcal{D}'(\mathbb{S}) : \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2s} |\widehat{v}_{\ell,m}|^2 < \infty \right\},$$

where $\mathcal{D}'(\mathbb{S})$ is the space of distributions on \mathbb{S} and $\widehat{v}_{\ell,m}$ are the Fourier coefficients of v ,

$$\widehat{v}_{\ell,m} = \int_{\mathbb{S}} v(\mathbf{x}) Y_{\ell,m}(\mathbf{x}) d\sigma_{\mathbf{x}}.$$

Here $d\sigma_{\mathbf{x}}$ is the element of surface area. The space H^s is equipped with the following norm and inner product:

$$\|v\|_s := \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2s} |\widehat{v}_{\ell,m}|^2 \right)^{1/2} \quad (2.1)$$

and

$$\langle v, w \rangle_s := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)^{2s} \widehat{v}_{\ell,m} \widehat{w}_{\ell,m}.$$

When $s = 0$ we write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_0$; this is in fact the L_2 -inner product. We note that

$$|\langle v, w \rangle_s| \leq \|v\|_s \|w\|_s \quad \forall v, w \in H^s, \quad \forall s \in \mathbb{R}, \quad (2.2)$$

and

$$\|v\|_{s_1} = \sup_{\substack{w \in H^{s_2} \\ w \neq 0}} \frac{\langle v, w \rangle_{\frac{s_1+s_2}{2}}}{\|w\|_{s_2}} \quad \forall v \in H^{s_1}, \quad \forall s_1, s_2 \in \mathbb{R}. \quad (2.3)$$

In the case $k \in \mathbb{Z}^+$, the Sobolev space H^k can also be defined by using atlas for the unit sphere \mathbb{S} [13]. This construction uses a concept of homogeneous extensions.

Definition 2.1. Given any spherical function v , its *homogeneous extension* of degree $n \in \mathbb{Z}^+$ to $\mathbb{R}^3 \setminus \{0\}$ is a function v_n defined by

$$v_n(\mathbf{x}) := |\mathbf{x}|^n v\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right).$$

Let $\{(\Gamma_j, \phi_j)\}_{j=1}^J$ be an atlas for \mathbb{S} , i.e, a finite collection of charts (Γ_j, ϕ_j) , where Γ_j are open subsets of \mathbb{S} , covering \mathbb{S} , and where $\phi_j : \Gamma_j \rightarrow B_j$ are infinitely differentiable mappings whose inverses ϕ_j^{-1} are also infinitely differentiable. Here B_j , $j = 1, \dots, J$, are open subsets in \mathbb{R}^2 . Also, let $\{\alpha_j\}_{j=1}^J$ be a partition of unity subordinate to the atlas $\{(\Gamma_j, \phi_j)\}_{j=1}^J$, i.e., a set of infinitely differentiable functions α_j on \mathbb{S} vanishing outside the sets Γ_j , such that $\sum_{j=1}^J \alpha_j = 1$ on \mathbb{S} . For any $k \in \mathbb{Z}^+$, the Sobolev space H^k on the unit sphere is defined as follows

$$H^k := \{v : (\alpha_j v) \circ \phi_j^{-1} \in H^k(B_j), \quad j = 1, \dots, J\}, \quad (2.4)$$

which is equipped with a norm defined by

$$\|v\|_k^* := \sum_{j=1}^J \|(\alpha_j v) \circ \phi_j^{-1}\|_{H^k(B_j)}. \quad (2.5)$$

This norm is equivalent to the norm defined in (2.1); see [11].

Let $v \in H^k$, $k \geq 1$. For each $l = 1, \dots, k$, let v_{l-1} denote the unique homogeneous extension of v of degree $l - 1$. Then

$$|v|_l := \sum_{|\alpha|=l} \|D^\alpha v_{l-1}\|_{L_2(\mathbb{S})} \quad (2.6)$$

is a Sobolev-type seminorm of v in H^k . Here $\|D^\alpha v_{l-1}\|_{L_2(\mathbb{S})}$ is understood as the L_2 -norm of the restriction of the trivariate function $D^\alpha v_{l-1}$ to \mathbb{S} . When $l = 0$ we define

$$|v|_0 := \|v\|_{L_2(\mathbb{S})},$$

which can now be used together with (2.6) to define another norm in H^k :

$$\|v\|'_k := \sum_{l=0}^k |v|_l. \quad (2.7)$$

This norm turns out to be equivalent to the Sobolev norm defined in (2.5); see [13].

2.2 Pseudodifferential operators

Let $\{\widehat{L}(\ell)\}_{\ell \geq 0}$ be a sequence of real numbers. A pseudodifferential operator L is a linear operator that assigns to any $v \in \mathcal{D}'(\mathbb{S})$ a distribution

$$Lv := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{L}(\ell) \widehat{v}_{\ell,m} Y_{\ell,m}. \quad (2.8)$$

The sequence $\{\widehat{L}(\ell)\}_{\ell \geq 0}$ is referred to as the *spherical symbol* of L . In this paper, we assume that L is a *strongly elliptic* pseudodifferential operator of order 2α , i.e.,

$$C_1(\ell+1)^{2\alpha} \leq \widehat{L}(\ell) \leq C_2(\ell+1)^{2\alpha} \quad \forall \ell = 0, 1, 2, \dots, \quad (2.9)$$

for some positive constants C_1 and C_2 . It can be easily seen that $L : H^{s+\alpha} \rightarrow H^{s-\alpha}$ is a homeomorphism for all $s \in \mathbb{R}$.

The following pseudodifferential operators are commonly seen; see [15].

- (i) The Laplace–Beltrami operator is an operator of order 2 and has as symbol $\widehat{L}(\ell) = \ell(\ell+1)$. This operator is the restriction of the Laplacian on the sphere.
- (ii) The weakly-singular integral operator is an operator of order -1 and has as symbol $\widehat{L}(\ell) = 1/(2\ell+1)$. This operator arises from the boundary-integral reformulation of the Dirichlet problem with the Laplacian interior or exterior to the sphere.

The problem we solve in this paper reads as: *Given $g \in H^{-\alpha}$, find $u \in H^{\alpha}$ satisfying*

$$Lu = g. \quad (2.10)$$

Under the assumption (2.9), a unique solution exists and can be expressed as

$$u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{L}(\ell)^{-1} \widehat{g}_{\ell,m} Y_{\ell,m}.$$

We define a bilinear form $a(\cdot, \cdot) : H^{\alpha+s} \times H^{\alpha-s} \rightarrow \mathbb{R}$, for any $s \in \mathbb{R}$, by

$$a(w, v) := \langle Lw, v \rangle \quad \forall w \in H^{\alpha+s}, v \in H^{\alpha-s}. \quad (2.11)$$

We note that

$$a(u, v) = \langle g, v \rangle \quad \forall v \in H^{\alpha}. \quad (2.12)$$

In the sequel, for any $x, y \in \mathbb{R}$, $x \simeq y$ means that there exist positive constants C_1 and C_2 satisfying

$$C_1 x \leq y \leq C_2 x.$$

The following simple results are often used in the next sections.

Lemma 2.2. *Let s be any real number.*

(i) *The bilinear form $a(\cdot, \cdot) : H^{\alpha+s} \times H^{\alpha-s} \rightarrow \mathbb{R}$ is bounded, i.e.,*

$$|a(w, v)| \leq C \|w\|_{\alpha+s} \|v\|_{\alpha-s} \quad \forall w \in H^{\alpha+s}, v \in H^{\alpha-s}. \quad (2.13)$$

(ii) *If $w, v \in H^s$, then*

$$|\langle Lw, v \rangle_{s-\alpha}| \leq C \|w\|_s \|v\|_s. \quad (2.14)$$

(iii) *For any $v \in H^s$, there holds*

$$\langle Lv, v \rangle_{s-\alpha} \simeq \|v\|_s^2. \quad (2.15)$$

In particular, there holds $a(v, v) \simeq \|v\|_\alpha^2$ for all $v \in H^\alpha$.

Here C is a constant independent of v and w .

Proof. Let $w \in H^{\alpha+s}$ and $v \in H^{\alpha-s}$. Noting (2.9) and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |a(w, v)| &\leq \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |\widehat{L}(\ell)| |\widehat{w}_{\ell,m}| |\widehat{v}_{\ell,m}| \\ &\leq C \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2\alpha} |\widehat{w}_{\ell,m}| |\widehat{v}_{\ell,m}| \\ &= C \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{\alpha+s} |\widehat{w}_{\ell,m}| (\ell+1)^{\alpha-s} |\widehat{v}_{\ell,m}| \\ &\leq C \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2(\alpha+s)} |\widehat{w}_{\ell,m}|^2 \right)^{1/2} \\ &\quad \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2(\alpha-s)} |\widehat{v}_{\ell,m}|^2 \right)^{1/2} \\ &= C \|w\|_{\alpha+s} \|v\|_{\alpha-s}, \end{aligned}$$

proving (2.13). The proof for (2.14) and (2.15) can be done similarly, noting the definition (2.9) of strongly elliptic operators. \square

3 Approximation subspaces

The finite-dimensional subspace to be used in the approximation will be defined from spherical splines. In this section we review several basic concepts, present some properties of spherical splines. More details can be found in [1], [2] and [3].

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be linearly independent vectors in \mathbb{R}^3 . The *triheron* T generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is defined by

$$T := \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 \text{ with } b_i \geq 0, i = 1, 2, 3\}.$$

The intersection $\tau := T \cap \mathbb{S}$ is called a *spherical triangle*. Let $\Delta = \{\tau_i : i = 1, \dots, \mathcal{T}\}$ be a set of spherical triangles. If Δ satisfies

- (i) $\bigcup_{i=1}^{\mathcal{T}} \tau_i = \mathbb{S}$,
- (ii) each pair of triangles in Δ are either disjoint or share a common vertex or an edge,

then Δ is called a *spherical triangulation* of the sphere \mathbb{S} .

Given $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ a set of points on \mathbb{S} , we can form a spherical triangulation Δ . For any nonnegative integers r and d , the set of spherical splines of degree d and smoothness r associated with Δ is defined by

$$S_d^r(\Delta) := \{s \in C^r(\mathbb{S}) : s|_{\tau} \in \mathcal{P}_d, \tau \in \Delta\},$$

where \mathcal{P}_d denotes the set of homogeneous polynomials of degree d .

In this paper, we assume that the integers d and r defining $S_d^r(\Delta)$ satisfy

$$\begin{cases} d \geq 3r + 2 & \text{if } r \geq 1, \\ d \geq 1 & \text{if } r = 0. \end{cases} \quad (3.1)$$

We now use the definition of Sobolev spaces by local charts, see (2.4), to show the following necessary result.

Proposition 3.1. *Let Δ be a spherical triangulation on the unit sphere \mathbb{S} . Assume that (3.1) holds. There holds*

$$S_d^r(\Delta) \subset H^{r+1}.$$

Proof. We prove the result for $r = 0$ and use induction to obtain the result for $r > 0$. Here we use a specific atlas of charts to define the Sobolev space H^1 . We denote by $\mathbf{n} = (0, 0, 1)$ and $\mathbf{s} = (0, 0, -1)$ the north and south poles of \mathbb{S} , respectively. For any point $\mathbf{x} \in \mathbb{S}$ and $R > 0$, we denote the spherical cap centred at \mathbf{x} and having radius R by $C(\mathbf{x}, R)$, i.e.,

$$C(\mathbf{x}, R) := \{\mathbf{y} \in \mathbb{S} : \cos^{-1}(\mathbf{x} \cdot \mathbf{y}) \leq R\}. \quad (3.2)$$

The interior of $C(\mathbf{x}, R)$ is denoted by $C^o(\mathbf{x}, R)$. Let

$$U_1 = C^o(\mathbf{n}, \theta_0) \quad \text{and} \quad U_2 = C^o(\mathbf{s}, \theta_0),$$

with $\theta_0 \in (\pi/2, 2\pi/3)$. Assume that Δ is fine enough such that \mathbb{S} admits a simple cover $\mathbb{S} = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_j = \bigcup_{\tau \subset U_j} \tau, \quad j = 1, 2.$$

The stereographic projection ψ_1 from the punctured sphere $\mathbb{S} \setminus \{\mathbf{n}\}$ onto \mathbb{R}^2 is defined as a mapping that maps $\mathbf{x} \in \mathbb{S} \setminus \{\mathbf{n}\}$ to the intersection of the equatorial hyperplane $\{z = 0\}$ and the extended line that passes through \mathbf{x} and \mathbf{n} . The stereographic projection ψ_2 based on \mathbf{s} can be defined similarly. The scaled projections

$$\phi_1 := \frac{1}{\tan(\theta_0/2)} \psi_2|_{\Gamma_1} \quad \text{and} \quad \phi_2 := \frac{1}{\tan(\theta_0/2)} \psi_1|_{\Gamma_2}$$

map Γ_1 and Γ_2 , respectively, onto the unit ball of \mathbb{R}^2 . It is clear that $\{\Gamma_j, \phi_j\}_{j=1}^2$ is a C^∞ atlas for \mathbb{S} .

Let $B_j = \phi_j(\Gamma_j)$, $j = 1, 2$, and let $\{\alpha_j : \mathbb{S} \rightarrow \mathbb{R}\}_{j=1}^2$ be the partition of unity subordinate to $\{(\Gamma_j, \phi_j)\}_{j=1}^2$; see page 3. For any $v \in S_d^0(\Delta)$, it suffices to show that, see (2.4),

$$w_j := (\alpha_j v) \circ \phi_j^{-1} \in H^1(B_j) \quad \text{for } j = 1, 2.$$

This holds because w_j is continuous on $\overline{B_j}$ and $w_j|_\sigma$ belongs to $H^1(\sigma)$, where $\{\sigma := \phi_j(\tau) : \tau \subset \Gamma_j\}$ forms a partition of B_j ; see [6, page 38]. □

4 Approximation property

Before studying the approximation property of spherical splines, we introduce the concept of quasiuniform triangulation on \mathbb{S} .

For any spherical triangle τ , we denote by $|\tau|$ the diameter of the smallest spherical cap containing τ , and by ρ_τ the diameter of the largest spherical cap inside τ . Here the diameter of a cap is, as usual, twice its radius; see (3.2). We define

$$|\Delta| := \max\{|\tau|, \tau \in \Delta\}, \quad \rho_\Delta := \min\{\rho_\tau, \tau \in \Delta\} \quad \text{and} \quad h_\Delta := \tan \frac{|\Delta|}{2}. \quad (4.1)$$

Definition 4.1. Let β be a positive real number. A triangulation Δ is said to be β -quasiuniform provided that

$$\frac{|\Delta|}{\rho_\Delta} \leq \beta.$$

The following proposition is a special case of a result established in [4].

Proposition 4.2. [4, Theorem 2] Assume that Δ is a β -quasiuniform spherical triangulation with $|\Delta| \leq 1$, and that (3.1) holds. Then for any $v \in H^m$ there exists $Qv \in S_d^r(\Delta)$ satisfying

$$|v - Qv|_k \leq Ch_\Delta^{m-k} |v|_m,$$

where $k = 0, \dots, \min\{m-1, r+1\}$, and $m \in \{1, \dots, d+1\}$ satisfies $(d-m) \bmod 2 = 1$. Here C is a positive constant depending only on d and the smallest angle in Δ .

Remark 4.3. (i) In [13], Qv is defined as the quasi-interpolation of v .

(ii) The condition $k \leq r + 1$ is to ensure that $Qv \in H^k$; see Proposition 3.1.

(iii) Theorem 2 in [4] proves the result for $r \geq 0$ and $d \geq 3r + 2$. In fact, the result is also true when $r = 0$ and $d = 1$.

The following proposition extends the above result to Sobolev norms and to a larger range of values of m .

Proposition 4.4. *Assume that Δ is a β -quasiuniform spherical triangulation with $|\Delta| \leq 1$, and that (3.1) holds. Then for any $v \in H^m$ there exists $\eta \in S_d^r(\Delta)$ satisfying*

$$\|v - \eta\|_n \leq Ch_\Delta^{m-n} \|v\|_m,$$

where

$$\begin{cases} m \in \{1, 2, \dots, d + 1\} & \text{if } d \text{ is even,} \\ m \in \{2, 3, \dots, d + 1\} & \text{if } d \text{ is odd,} \end{cases}$$

and where $n = 0, 1, \dots, \min\{m - 1, r + 1\}$. Here the positive constant C depends only on d and the smallest angle in Δ .

Proof. Since the three norms defined in (2.1), (2.5), and (2.7) are equivalent, it suffices to prove the result with the norm $\|\cdot\|'$.

Case 1: Consider first the case when m and d are of different parity, i.e.,

$$m = \begin{cases} 1, 3, \dots, d + 1 & \text{if } d \text{ is even,} \\ 2, 4, \dots, d + 1 & \text{if } d \text{ is odd.} \end{cases}$$

It follows from Proposition 4.2 that for any $v \in H^m$ there exists $Qv \in S_d^r(\Delta)$ satisfying

$$|v - Qv|_k \leq Ch_\Delta^{m-k} |v|_m, \quad k = 0, \dots, n.$$

By summing over all $k = 0, \dots, n$ and noting that $h_\Delta < 1$, we obtain

$$\|v - Qv\|'_n \leq Ch_\Delta^{m-n} |v|_m.$$

Since $|v|_m \leq \|v\|'_m$, we deduce the required inequality.

Case 2: Next, we consider the case when m and d have the same parity, i.e.,

$$m = \begin{cases} 2, 4, \dots, d & \text{if } d \text{ is even,} \\ 3, 5, \dots, d & \text{if } d \text{ is odd.} \end{cases}$$

Note that here we can use the result proved in Case 1 for $m - 1$ and $m + 1$, but not for m , due to the parity of m when compared to d . Hence, we further consider different sub-cases.

Case 2.1: When $n \leq m - 2$, we have by using the result in Case 1 with n and $m - 1$

$$\|v - Qv\|'_n \leq Ch_\Delta^{m-1-n} \|v\|'_{m-1} \quad \forall v \in H^{m-1},$$

and then with n and $m + 1$

$$\|v - Qv\|'_n \leq Ch_\Delta^{m+1-n} \|v\|'_{m+1} \quad \forall v \in H^{m+1}.$$

By using interpolation we obtain the required inequality for $v \in H^m$.

Case 2.2: Next, consider the case $n = m - 1 \leq r$. The result in Case 1 gives (noting that $m \geq 2$ in this case)

$$\|v - Qv\|'_{m-2} \leq Ch_\Delta \|v\|'_{m-1} \quad \forall v \in H^{m-1},$$

and

$$\|v - Qv\|'_m \leq Ch_\Delta \|v\|'_{m+1} \quad \forall v \in H^{m+1}.$$

By using interpolation again, we obtain

$$\|v - Qv\|'_{m-1} \leq Ch_\Delta \|v\|'_m \quad \forall v \in H^m.$$

Case 2.3: Finally, consider the case $n = m - 1 = r + 1$. Let P_{r+1} be the projection from H^{r+1} onto $S_d^r(\Delta)$ defined by

$$\langle P_{r+1}v, w \rangle_{r+1} = \langle v, w \rangle_{r+1} \quad \forall v \in H^{r+1} \text{ and } w \in S_d^r(\Delta).$$

It is well known that

$$\|P_{r+1}v - v\|'_{r+1} \leq \inf_{w \in S_d^r(\Delta)} \|w - v\|'_{r+1},$$

so that

$$\|P_{r+1}v - v\|'_{r+1} \leq \|v\|'_{r+1} \tag{4.2}$$

and

$$\|P_{r+1}v - v\|'_{r+1} \leq \|Qv - v\|'_{r+1}.$$

Using the result proved in Case 2.1 with $r + 1$ and $r + 3$ we have

$$\|Qv - v\|'_{r+1} \leq ch_\Delta^2 \|v\|'_{r+3},$$

implying

$$\|P_{r+1}v - v\|'_{r+1} \leq ch_\Delta^2 \|v\|'_{r+3}.$$

The above inequality and (4.2) yield

$$\|P_{r+1}v - v\|'_{r+1} \leq ch_\Delta \|v\|'_{r+2},$$

proving the proposition. □

Remark 4.5. *The approximant η to v in the above proposition is the quasi-interpolant Qv (Proposition 4.4), except when r and d have the same parity and $n = r + 1$ and $m = r + 2$. We believe that this restriction is only a technical difficulty.*

The following proposition extends Proposition 4.4 to the case when n and m are nonnegative real numbers. It will also relax the condition that $n \leq m - 1$.

Proposition 4.6. *Assume that Δ is a β -quasiuniform spherical triangulation with $|\Delta| \leq 1$, and that (3.1) holds. Then for any $v \in H^s$, there exists $\eta \in S_d^r(\Delta)$ satisfying*

$$\|v - \eta\|_t \leq Ch_\Delta^{s-t} \|v\|_s,$$

where $0 \leq t \leq r + 1$ and $t \leq s \leq d + 1$. Here C is a positive constant depending only on d and the smallest angle in Δ .

Proof. The proposition will be proved by considering different cases. The main idea of the proof is to use Proposition 4.4 and interpolation.

Case 1: $t \leq r + 1$ and $s = d + 1$.

By using Proposition 4.4 we have for any $v \in H^{d+1}$

$$\|v - Qv\|_{[t]} \leq Ch_\Delta^{d+1-[t]} \|v\|_{d+1},$$

and

$$\|v - Qv\|_{[t]} \leq Ch_\Delta^{d+1-[t]} \|v\|_{d+1}.$$

By using interpolation we derive

$$\|v - Qv\|_t \leq Ch_\Delta^{d+1-t} \|v\|_{d+1}. \quad (4.3)$$

We note that the only case that this argument fails is when $[t] = r + 1$ and $d + 1 = r + 2$, with d and r having the same parity; see Remark 4.5. This case never happens.

Case 2: $t \leq r + 1$ and $t \leq s < d + 1$.

Let $P_t : H^t \rightarrow S_d^r(\Delta)$ be the projection defined by

$$\langle P_t v, w \rangle_t = \langle v, w \rangle_t \quad \forall v \in H^t \text{ and } w \in S_d^r(\Delta).$$

Then

$$\|P_t v - v\|_t \leq \inf_{w \in S_d^r(\Delta)} \|w - v\|_t,$$

so that

$$\|P_t v - v\|_t \leq \|v\|_t, \quad (4.4)$$

and

$$\|P_t v - v\|_t \leq \|Qv - v\|_t.$$

Due to (4.3) there holds

$$\|P_t v - v\|_t \leq Ch_\Delta^{d+1-t} \|v\|_{d+1}.$$

This inequality and (4.4) yield

$$\|P_t v - v\|_t \leq Ch_\Delta^{s-t} \|v\|_s,$$

proving the proposition. \square

The following theorem is our most general result on the approximation property of $S_d^r(\Delta)$, which includes negative values of t and s .

Theorem 4.7. *Assume that Δ is a β -quasiuniform spherical triangulation with $|\Delta| \leq 1$, and that (3.1) holds. Then for any $v \in H^s$, there exists $\eta \in S_d^r(\Delta)$ satisfying*

$$\|v - \eta\|_t \leq Ch_\Delta^{s-t} \|v\|_s,$$

where $t \leq r + 1$ and $t \leq s \leq d + 1$. Here C is a positive constant depending only on d and the smallest angle in Δ .

Proof. When both t and s are nonnegative, the result is proved in Proposition 4.6. The remaining cases are considered separately below.

Step 1: $-d - 1 \leq t \leq 0 \leq s \leq d + 1$.

We define $P_0 : H^0 \rightarrow S_d^r(\Delta)$ by

$$\langle P_0 v, w \rangle = \langle v, w \rangle \quad \forall v \in H^0 \text{ and } w \in S_d^r(\Delta).$$

It is well known that

$$\|P_0 v - v\|_0 \leq \inf_{w \in S_d^r(\Delta)} \|v - w\|_0,$$

which implies, together with Proposition 4.6, that

$$\|P_0 v - v\|_0 \leq h_\Delta^\sigma \|v\|_\sigma \quad \forall v \in H^\sigma, \quad 0 \leq \sigma \leq d + 1.$$

Noting (2.3) and using the above estimate with $\sigma = s$ and $\sigma = -t$ we obtain

$$\begin{aligned} \|P_0 v - v\|_t &= \sup_{\substack{w \in H^{-t} \\ w \neq 0}} \frac{\langle P_0 v - v, w \rangle}{\|w\|_{-t}} = \sup_{\substack{w \in H^{-t} \\ w \neq 0}} \frac{\langle P_0 v - v, w - P_0 w \rangle}{\|w\|_{-t}} \\ &\leq \|P_0 v - v\|_0 \sup_{\substack{w \in H^{-t} \\ w \neq 0}} \frac{\|w - P_0 w\|_0}{\|w\|_{-t}} \\ &\leq Ch_\Delta^{s-t} \|v\|_s. \end{aligned}$$

Step 2: $2s - d - 1 \leq t \leq s \leq 0$ and $-d - 1 \leq s$.

Let $P_s : H^s \rightarrow S_d^r(\Delta)$ be defined by

$$\langle P_s v, w \rangle_s = \langle v, w \rangle_s \quad \forall v \in H^s \text{ and } w \in S_d^r(\Delta).$$

Then it is obvious that

$$\|P_s v - v\|_s \leq \|v\|_s \quad \forall v \in H^s. \quad (4.5)$$

If $2s - d - 1 \leq t \leq 2s$, so that $0 \leq 2s - t \leq d + 1$, then the result proved in Step 1 yields

$$\|P_0 w - w\|_s \leq Ch_\Delta^{s-t} \|w\|_{2s-t} \quad \forall w \in H^{2s-t}. \quad (4.6)$$

Combining (2.3), (4.5) and (4.6), we have

$$\begin{aligned} \|P_s v - v\|_t &= \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\langle P_s v - v, w \rangle_s}{\|w\|_{2s-t}} = \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\langle P_s v - v, w - P_0 w \rangle_s}{\|w\|_{2s-t}} \\ &\leq \|P_s v - v\|_s \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\|w - P_0 w\|_s}{\|w\|_{2s-t}} \\ &\leq Ch_\Delta^{s-t} \|P_s v - v\|_s \\ &\leq Ch_\Delta^{s-t} \|v\|_s. \end{aligned}$$

In particular, we have

$$\|P_s v - v\|_{2s} \leq Ch_\Delta^{-s} \|v\|_s.$$

The result for $2s < t \leq s$ is obtained by interpolation, noting (4.5).

Step 3: $-2(d+1) \leq t \leq -d-1$ and $0 \leq s \leq d+1$. (This step extends the result of Step 1.)

Let $P_{-d-1} : H^{-d-1} \rightarrow S_d^r(\Delta)$ be the projection defined by

$$\langle P_{-d-1} v, w \rangle_{-d-1} = \langle v, w \rangle_{-d-1} \quad \forall w \in S_d^r(\Delta).$$

Then $P_{-d-1} v$ is the best approximation of v from $S_d^r(\Delta)$ in the H^{-d-1} -norm. The result in Step 1 gives

$$\|P_{-d-1} v - v\|_{-d-1} \leq Ch_\Delta^{s+d+1} \|v\|_s \quad \forall v \in H^s. \quad (4.7)$$

Since $\langle P_{-d-1} v - v, \eta \rangle_{-d-1} = 0$ for all $\eta \in S_d^r(\Delta)$, we have

$$\begin{aligned} \|P_{-d-1} v - v\|_t &= \sup_{\substack{w \in H^{-t-2(d+1)} \\ w \neq 0}} \frac{\langle P_{-d-1} v - v, w \rangle_{-d-1}}{\|w\|_{-t-2(d+1)}} \\ &= \sup_{\substack{w \in H^{-t-2(d+1)} \\ w \neq 0}} \frac{\langle P_{-d-1} v - v, w - \eta \rangle_{-d-1}}{\|w\|_{-t-2(d+1)}} \\ &\leq \|P_{-d-1} v - v\|_{-d-1} \sup_{\substack{w \in H^{-t-2(d+1)} \\ w \neq 0}} \frac{\|w - \eta\|_{-d-1}}{\|w\|_{-t-2(d+1)}}. \end{aligned} \quad (4.8)$$

Noting that $-d - 1 \leq -t - 2(d + 1) \leq 0$, we can use the result in Step 2 to choose, for any $w \in H^{-t-2(d+1)}$, a function $\eta \in S_d^r(\Delta)$ satisfying

$$\|w - \eta\|_{-d-1} \leq Ch_{\Delta}^{-t-d-1} \|w\|_{-t-2(d+1)}. \quad (4.9)$$

Substituting (4.7) and (4.9) into (4.8), we infer

$$\|P_{-d-1}v - v\|_t \leq Ch_{\Delta}^{s-t} \|v\|_s.$$

Step 4: $-d - 1 \leq s \leq 0$ and $4s - 2(d + 1) \leq t \leq 2s - d - 1$. (This step extends the result of Step 1.)

Defining the projection $P_{2s-d-1} : H^{2s-d-1} \rightarrow S_d^r(\Delta)$ in the same manner as P_{-d-1} , using the same argument as in Step 3, and invoking the result in Step 2, we obtain the required estimate.

The results for the cases $t < -2(d + 1)$ or $s < -d - 1$ can be obtained similarly by repeating the above arguments. The theorem is proved. \square

In the next section we will use the result developed in this section to estimate the error of the Galerkin approximation.

5 Galerkin method

5.1 Approximate solution

We shall find $\tilde{u} \in S_d^r(\Delta)$ by solving the Galerkin equation

$$a(\tilde{u}, v) = \langle g, v \rangle \quad \forall v \in S_d^r(\Delta). \quad (5.1)$$

Recalling Lemma 2.2, it can be seen that the bilinear form $a(\cdot, \cdot)$ is continuous and coercive in $S_d^r(\Delta)$. The well known Lax–Milgram Theorem confirms the unique existence of the approximate solution \tilde{u} . We now prove the main theorem of the paper.

Theorem 5.1. *Assume that Δ is a β -quasiuniform spherical triangulation with $|\Delta| \leq 1$ and that (3.1) hold. If the order 2α of the pseudodifferential operator L satisfies $\alpha \leq r + 1$, and if u and \tilde{u} satisfy, respectively, (2.12) and (5.1), then*

$$\|u - \tilde{u}\|_t \leq Ch_{\Delta}^{s-t} \|u\|_s,$$

where $s \leq d + 1$ and $2\alpha - d - 1 \leq t \leq \min\{s, \alpha\}$. Here C is a positive constant depending only on d and the smallest angle in Δ .

Proof. The proof is standard using Céa's Lemma and Nitsche's trick. We include it here for completeness. Céa's Lemma gives

$$\|u - \tilde{u}\|_{\alpha} \leq C \min_{v \in S_d^r(\Delta)} \|u - v\|_{\alpha},$$

which, together with Theorem 4.7, implies that there exists a constant C satisfying

$$\|u - \tilde{u}\|_\alpha \leq Ch_\Delta^{s-\alpha} \|u\|_s. \quad (5.2)$$

Now consider $t < \alpha$. It follows from (2.3) and (2.9) that

$$\|u - \tilde{u}\|_t \leq C \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{\langle u - \tilde{u}, v \rangle_\alpha}{\|v\|_{2\alpha-t}} \leq C \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{a(u - \tilde{u}, v)}{\|v\|_{2\alpha-t}}.$$

By using successively (2.12), (5.1) and Lemma 2.2, we deduce for $\eta \in S_d^r(\Delta)$

$$\|u - \tilde{u}\|_t \leq C \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{a(u - \tilde{u}, v - \eta)}{\|v\|_{2\alpha-t}} \leq C \|u - \tilde{u}\|_\alpha \sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{\|v - \eta\|_\alpha}{\|v\|_{2\alpha-t}}.$$

Since $2\alpha - d - 1 \leq t < \alpha$, there holds $\alpha < 2\alpha - t \leq d + 1$. By using Theorem 4.7 again, we can choose $\eta \in S_d^r(\Delta)$ satisfying

$$\|v - \eta\|_\alpha \leq Ch_\Delta^{\alpha-t} \|v\|_{2\alpha-t}.$$

Hence,

$$\|u - \tilde{u}\|_t \leq Ch_\Delta^{\alpha-t} \|u - \tilde{u}\|_\alpha \leq Ch_\Delta^{s-t} \|u\|_s,$$

proving the theorem. \square

6 Numerical experiments

In this section, we present our numerical results obtained from our experiments with different sets of points $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$, where $\mathbf{x}_i = (x_{i,1}, x_{i,2}, x_{i,3})$, $i = 1, \dots, N$, are points on the sphere generated by a simple algorithm [14] which partitions the sphere into equal areas. From these sets of data points, we use the FORTRAN package STRIPACK (<http://www.netlib.org/toms/77>), to obtain the spherical triangulations Δ .

We solve equation 5.1 by the space of spherical splines $S_1^0(\Delta)$, which is the space of continuous and piecewise homogenous polynomial of degree 1. A set of basis functions for $S_1^0(\Delta)$ is

$$\{B_i : i = 1, \dots, N\},$$

where, for each $i = 1, \dots, N$, B_i is the nodal basis function corresponding to the vertex \mathbf{x}_i defined as follows.

For any $\mathbf{x} \in \mathbb{S}$, there exists a spherical triangle $\tau \in \Delta$ containing \mathbf{x} . If \mathbf{x}_i is not a vertex of τ then $B_i(\mathbf{x}) = 0$. Otherwise, assume that τ is formed by three vertices \mathbf{x}_i , \mathbf{x}_j and \mathbf{x}_k . Then

$$B_i(\mathbf{x}) = \det \begin{bmatrix} x_1 & x_{j,1} & x_{k,1} \\ x_2 & x_{j,2} & x_{k,2} \\ x_3 & x_{j,3} & x_{k,3} \end{bmatrix} \cdot \det^{-1} \begin{bmatrix} x_{i,1} & x_{j,1} & x_{k,1} \\ x_{i,2} & x_{j,2} & x_{k,2} \\ x_{i,3} & x_{j,3} & x_{k,3} \end{bmatrix}. \quad (6.1)$$

Table 1: Errors in the L_2 -norm for the Laplace–Beltrami equation

N	h_Δ	$\ u_N - u\ _0$	EOC
101	0.2618	5.853E-2	
200	0.1819	2.713E-2	2.11
500	0.1136	1.093E-2	1.93
1001	0.0798	5.230E-3	2.09
2000	0.0570	2.675E-3	1.99
4000	0.0397	1.316E-3	1.96
8001	0.0283	6.627E-4	2.02

6.1 The Laplace–Beltrami equation

We consider the equation on the unit sphere of the form

$$Lu = g \quad \text{on } \mathbb{S}, \quad (6.2)$$

where $Lu := -\Delta_{\mathbb{S}}u + \omega^2u$ and where ω is some non-zero real constant. This equation arises, for example, when one discretizes in time the diffusion equation on the sphere.

We solve (6.2) with $\omega = 1$ and a right hand side function g given by

$$g(\mathbf{x}) = 7x_3^3 - 6x_3(x_1^2 + x_2^2),$$

so that the exact solution is

$$u(\mathbf{x}) = x_3^3.$$

We carry out the experiment with various values of N , the numbers of points, namely $N = 101, 200, 500, 1001, 2000, 4000$, and 8001 . The entry A_{ij} of the stiffness matrix A is computed by

$$\begin{aligned} A_{ij} &= \int_{\mathbb{S}} \left(-\Delta_{\mathbb{S}}B_i(\mathbf{x})B_j(\mathbf{y}) + B_i(\mathbf{x}) \cdot B_j(\mathbf{x}) \right) d\sigma_{\mathbf{x}} \\ &= \sum_{\tau \in \Delta} \int_{\tau} \left(-\Delta_{\mathbb{S}}B_i(\mathbf{x})B_j(\mathbf{y}) + B_i(\mathbf{x}) \cdot B_j(\mathbf{x}) \right) d\sigma_{\mathbf{x}} \end{aligned}$$

where $B_i(\mathbf{x})$ is computed by using (6.1). After solving the linear system, we find the L_2 -norm of the error $\|u_N - u\|_0$, where in this section we denote the approximate solution by u_N instead of \tilde{u} .

It is expected from our theoretical result (Theorem 5.1) that $\|u_N - u\|_0 = O(h_\Delta^2)$. The estimated orders of convergence (EOC) shown in Table 1 agree with our theoretical result.

6.2 Weakly singular integral equation

We also solve the weakly singular integral equation

$$Lu = g \text{ on } \mathbb{S}, \quad (6.3)$$

where L is the operator given by

$$Lv(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{S}} v(\mathbf{y}) \frac{1}{\|\mathbf{x} - \mathbf{y}\|} d\sigma_{\mathbf{y}},$$

for any $v \in \mathcal{D}'(\mathbb{S})$. This equation is the boundary integral reformulation of the Dirichlet problem for the Laplacian in the exterior or interior of the sphere [7]. Note that the weakly singular integral operator L is a pseudodifferential operator of order -1 with symbol being $1/(2\ell + 1)$.

We choose the right hand side g to be $g(\mathbf{x}) \equiv 1$ on the sphere \mathbb{S} . For this equation, we solve with various values of N , namely $N = 326, 350, 375, 400, 426, 450, 476, 500$, and 600 . Each entry A_{ij} of the stiffness matrix A is

$$A_{ij} = \frac{1}{4\pi} \int_{\mathbb{S}} \int_{\mathbb{S}} \frac{B_i(\mathbf{y}) \cdot B_j(\mathbf{x})}{\|\mathbf{x} - \mathbf{y}\|} d\sigma_{\mathbf{x}} d\sigma_{\mathbf{y}}, \quad i, j \in \{1, \dots, N\}.$$

Since the right hand side function $g = 1$ on \mathbb{S} , each entry of the right hand side vector is computed by

$$RHS(i) = \int_{\mathbb{S}} B_i(\mathbf{x}) d\sigma_{\mathbf{x}}, \quad i = 1, \dots, N.$$

To check the accuracy of the approximation, we compute the $H^{-1/2}$ -norm of the error $u - u_N$. This is calculated as follows (noting that in this case we do not know the exact solution):

$$\begin{aligned} \|u - u_N\|_{-1/2}^2 &\simeq \langle L(u - u_N), u - u_N \rangle = \langle L(u - u_N), u \rangle \\ &= \langle Lu, u \rangle - \langle u_N, Lu \rangle = \int_{\mathbb{S}} u d\sigma - \int_{\mathbb{S}} u_N d\sigma \\ &\simeq \int_{\mathbb{S}} u_{600} d\sigma - \int_{\mathbb{S}} u_N d\sigma. \end{aligned}$$

Here we have used the property that L is self-adjoint and $Lu = 1$. The estimated orders of convergence shown in Table 2 appear to agree with the theoretical result, which states that $\|u - u_N\|_{-1/2} = O(h_{\Delta}^{5/2})$.

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Table 2: Errors in the $H^{-1/2}$ -norm for the weakly singular integral equation

N	h_Δ	$\ u_{600} - u_N\ _{-1/2}$	EOC
326	0.1415	0.3295	
350	0.1360	0.2960	2.69
375	0.1315	0.2755	2.16
400	0.1289	0.2682	1.34
426	0.1227	0.2297	3.13
450	0.1206	0.2212	2.18
476	0.1160	0.1940	3.78
500	0.1136	0.1849	2.26

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