The University of New South Wales  
School of Mathematics and Statistics  
Student Support Scheme  

ALGEBRAIC IDENTITIES

There are a number of algebraic identities which you need to know in order to help you solve equations and simplify expressions (by an identity we mean an equation involving one or more variables, which is true for all values of those variables).

• Addition of fractions:
  \[ \frac{w}{x} + \frac{y}{z} = \frac{wz + xy}{xz} \, . \]

• Square of a sum:
  \[ (x + y)^2 = x^2 + 2xy + y^2 \, . \]

• The same for a sum of three terms:
  \[ (x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \, . \]

• Difference of two squares:
  \[ x^2 - y^2 = (x - y)(x + y) \, . \]

• Difference of two powers:
  \[ x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \, . \]

• Sum of two powers: if \( n \) is odd then
  \[ x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}) \, . \]

Note that there is no formula like this if \( n \) is even: for example, \( x^2 + y^2 \) cannot be factorised in any simple way.

• The Binomial Theorem to expand \((x + y)^n\): this is dealt with in a separate sheet.

• Power laws and logarithm laws such as \( a^x a^y = a^{x+y} \): these are dealt with in separate sheets.

Comments.

• You must be able to use these identities “in both directions”. For example, if you see \( x^2 - y^2 \) you should know that it can be factorised, and if you see \((x - y)(x + y)\) you should know that it can be expanded and simplified.

• You must be able to replace the variables in all these identities by different variables, constants or expressions. From the “difference of two squares” formula we can find, for example,
  \[ m^2 - n^2 = (m - n)(m + n) \, . \]
  \[ x^2 - 5 = (x - \sqrt{5})(x + \sqrt{5}) \, . \]
  \[ 9a^2 - 100b^2 = (3a - 10b)(3a + 10b) \, . \]
  \[ x^6 - y^6 = (x^3 - y^3)(x^3 + y^3) \, . \]

• Sometimes we can use a number of these identities successively in order to give a detailed factorisation of a certain expression. For instance,
  \[ x^6 - y^6 = (x^3 - y^3)(x^3 + y^3) \quad \text{(difference of two squares)} \]
  \[ = (x - y)(x^2 + xy + y^2)(x^3 + y^3) \quad \text{(difference of 3rd powers)} \]
  \[ = (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2) \quad \text{(sum of 3rd powers)} \, . \]
EXERCISES.

Please try to complete the following exercises. Remember that you cannot expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions yourself. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the SSS.

1. Expand
   (a) \((4x + 5y)(4x - 5y)\);    (b) \((s + t)^2\);
   (c) \((x - 3y + 5z)^2\);    (d) \((z^3 - 4)(z^3 + 4)\);
   (e) \((ab - 2cd)^2\);    (f) \(((a + b) - c)((a + b) + c)\);
   (g) \((x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)\);
   (h) \((a + b + c + d)^2\)    (i) \((x + y)^2 - (x - y)^2\).

2. Factorise
   (a) \(x^2 + 10xy + 25y^2\);    (b) \(x^6 - y^4\);
   (c) \(x^8 - y^8\);    (d) \(4a^2 - 5b^2\);
   (e) \(x^6 - 2x^3 + 1\);    (f) \(z^7 - 128\) (hint: 128 = \(2^7\)).

3. (a) Factorise \(x^4 - x^2 + 1\) by writing it as \((x^4 + 2x^2 + 1) - 3x^2\) and using the above identities.
   (b) Hence factorise \(x^{12} - 1\) into linear and quadratic factors.

4. The sum and difference of fractions
   \[
   \frac{1}{x - 1} - \frac{2}{x^2 - 1} + \frac{1}{x^3 - 1}
   \]
   has a common denominator \((x - 1)(x^2 - 1)(x^3 - 1)\), but this is not the smallest common denominator. By factorising the three denominators, find the smallest common denominator and hence simplify the expression.

ANSWERS.

1. (a) \(16x^2 - 25y^2\);
   (b) \(s^2 + 2st + t^2\);
   (c) \(x^2 + 9y^2 + 25z^2 - 6xy + 10xz - 30yz\);
   (d) \(x^6 - 16\);
   (e) \(a^2b^2 - 4abcd + 4c^2d^2\);
   (f) \(a^2 + 2ab + b^2 - c^2\);
   (g) \(x^5 + y^5\);
   (h) \(a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd\);
   (i) \(4xy\).

2. (a) \((x + 5y)^2\);
   (b) \((x^3 - y^2)(x^3 + y^2)\);
   (c) \((x^4 - y^4)(x^4 + y^4)\) for a start, but hopefully you can continue and get \((x - y)(x + y)(x^2 + y^2)(x^4 + y^4)\);
   (d) \((2a - \sqrt{5}b)(2a + \sqrt{5}b)\);
   (e) \((x^3 - 1)^2\);
   (f) \((z - 2)(z^6 + 2z^5 + 4z^4 + 8z^3 + 16z^2 + 32z + 64)\).

3. (a) \((x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)\);
   (b) \((x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)(x^2 + 1)\);
   \((x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)\).

4. The smallest common denominator is \((x - 1)(x + 1)(x^2 + x + 1)\) and the expression is
   \[
   \frac{x^3 + x}{(x - 1)(x + 1)(x^2 + x + 1)}.
   \]