A **geometric series** (or geometric progression, GP for short) is a sum of terms in which each and every number is a fixed ratio times the previous number. For example, in the sum

\[ 4 + 12 + 36 + 108 + 324 + 972 + 2916 \]

every term is 3 times the previous term. However a series like

\[ 5 + 10 + 20 + 30 + 90 \]

is not a geometric series because the second term is 2 times the first and the third is 2 times the second, but the fourth is \( \frac{1}{2} \) times the third (and the fifth is 3 times the fourth). A geometric series may have more, or fewer, terms than example (\( \ast \)). The general geometric series is

\[ a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}, \]

where every term is \( r \) times the previous one. We call \( a \) the **first term**, \( r \) the **common ratio** and \( n \) the **number of terms**. For instance, the geometric series (\( \ast \)) has \( a = 4 \), \( r = 3 \) and \( n = 7 \). The sum of a geometric series can be found from the formula

\[ a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a \frac{1 - r^n}{1 - r}, \]

as long as \( r \neq 1 \). For example, we can work out (\( \ast \)) as

\[ 4 + 12 + 36 + 108 + 324 + 972 + 2916 = 4 \frac{1 - 3^7}{1 - 3} = 4372. \]

You will often need to calculate the sum of a GP where the number of terms is unspecified; this can be done using exactly the same formula. For example, \( 4 + 4 \times 3 + 4 \times 3^2 + \cdots + 4 \times 3^{n-1} \) is a GP and its sum is

\[ 4 \frac{1 - 3^n}{1 - 3} = 2(3^n - 1). \]

**Warning.** You will need to take care with the number of terms. For example,

\[ 3 + 15 + 75 + 375 + \cdots + 3 \times 5^n \]

is a geometric series with \( n + 1 \) terms, not \( n \), and its sum is

\[ 3 \frac{1 - 5^{n+1}}{1 - 5} = \frac{3}{4} (5^{n+1} - 1). \]

The ratio for a GP may be a fraction. For example

\[ 2 + \frac{2}{5} + \frac{2}{5^2} + \cdots + \frac{2}{5^{n-1}} = 2 \frac{1 - \left(\frac{1}{5}\right)^n}{1 - \frac{1}{5}} = \frac{5}{2} \left(1 - \left(\frac{1}{5}\right)^n\right). \]

It is also possible for a geometric progression to consist of an infinite number of terms. In this case it will still add up to a finite number as long as the ratio is less than 1 in absolute value. The formula in this case is

\[ a + ar + ar^2 + \cdots = \frac{a}{1 - r}. \]

For example,

\[ 2 + \frac{2}{5} + \frac{2}{5^2} + \cdots = \frac{2}{1 - \frac{1}{5}} = \frac{5}{2}; \]

however \( 4 + 4 \times 3 + 4 \times 3^2 + 4 \times 3^3 + \cdots \) does not add up to a finite value because the ratio \( r = 3 \) is bigger than 1.
EXERCISES.

Please try to complete the following exercises. Remember that you cannot expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Which of the following are geometric series? If they are not, explain why not; if they are, write down the values of $a$, $r$ and $n$ and find the sum of the series.
   (a) $7 + 14 + 28 + 56 + 112 + 224 + 448 + 896$;
   (b) $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45$;
   (c) $2 + 10 + 50 + 250 + 1250 + 12500$;
   (d) $3 + 3 \times 7 + 3 \times 7^2 + \cdots + 3 \times 7^{100}$.

2. Which of the following are geometric series? If they are not, explain why not; if they are, write down the values of $a$ and $r$, and find the sum of the series if it is finite.
   (a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$;
   (b) $3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$;
   (c) $\frac{1}{6} + \frac{1}{30} + \frac{1}{150} + \frac{1}{750} + \cdots$;
   (d) $1 + 2 + 4 + 8 + 16 + 32 + \cdots$.

3. Find the sum of the following geometric series.
   (a) $7 + 7 \times 6 + 7 \times 6^2 + \cdots + 7 \times 6^{n-1}$;
   (b) $5 + 5 \times 11 + 5 \times 11^2 + \cdots + 5 \times 11^n$;
   (c) $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$ (careful!).

4. The same formulae work if $r$ is negative. Sum the following.
   (a) $-6 + 18 - 54 + 162 - 486 + 1458$;
   (b) $3 - 3(\frac{2}{3}) + 3(\frac{2}{3})^2 + \cdots$;
   (c) $1 - 2 + 4 - 8 + 16 - 32 + \cdots$.

ANSWERS.

1. (a) $a = 7$, $r = 2$, $n = 8$, $S = 1785$;
   (b) not a GP since (e.g.) the first ratio is 3, the second is 2;
   (c) not a GP as the last ratio is 10 and all the others are 5;
   (d) $a = 3$, $r = 7$, $n = 101$, $S = \frac{1}{2}(7^{101} - 1)$.

2. (a) not a GP as the first ratio is $\frac{1}{2}$ and the second is $\frac{2}{3}$;
   (b) $a = 3$, $r = \frac{1}{3}$; $S = \frac{9}{2}$;
   (c) $a = \frac{1}{6}$, $r = \frac{1}{5}$; $S = \frac{5}{24}$;
   (d) $a = 1$, $r = 2$, sum is not finite.

3. (a) $\frac{7}{6}(6^n - 1)$;
   (b) $\frac{1}{2}(11^{n+1} - 1)$;
   (c) $8(1 - (\frac{1}{2})^{n+3})$.

4. (a) 1094;
   (b) $\frac{15}{7}$;
   (c) no finite sum because $r = -2$ and the absolute value of $r$ is 2 which is greater than 1.