Simple proofs of Ramanujan’s partition congruences

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Introduction

Let $p(n)$ be the number of partitions of $n$. For example, $p(4) = 5$, since we can write

$$4 = 4$$
$$= 3 + 1$$
$$= 2 + 2$$
$$= 2 + 1 + 1$$
$$= 1 + 1 + 1 + 1$$

and there are 5 such representations.

It was known to Euler that

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{E(q)}$$
where
\[ E(q) = (1 - q)(1 - q^2)(1 - q^3) \cdots , \]
and by convention, \( p(0) = 1 \).
Euler also knew that
\[ E(q) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - + + \]
\[ = \sum_{-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}. \]
It follows that
\[ (1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - + + \cdots ) \times \sum_{n \geq 0} p(n)q^n = 1, \]
from which it follows that for \( n > 0 \),
\[ p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) \]
\[ + p(n - 12) + p(n - 15) + + - - \cdots . \]
Thus,

\[ p(1) = p(0) = 1, \]
\[ p(2) = p(1) + p(0) = 2, \]
\[ p(3) = p(2) + p(1) = 3, \]
\[ p(4) = p(3) + p(2) = 5, \]

and so on.

P. MacMahon, in 1917, calculated \( p(n) \) for \( n \) up to 200, and presented his results thus:

\begin{tabular}{cccccccc}
1 & 7 & 42 & 176 & 627 & 1958 & 5604 & \\
1 & 11 & 56 & 231 & 792 & 2436 & 6842 & \\
2 & 15 & 77 & 297 & 1002 & 3010 & 8349 & \\
3 & 22 & 101 & 385 & 1255 & 3718 & 10143 & \\
5 & 30 & 135 & 190 & 1575 & 4565 & 12310 & \\
\end{tabular}

Look at the bottom row. Ramanujan did, and made the exciting discovery that

\[ p(5n + 4) \equiv 0 \pmod{5}. \]
He also discovered that

\[ p(7n + 5) \equiv 0 \pmod{7} \quad \text{and} \quad p(11n + 6) \equiv 0 \pmod{11}. \]

And these were just the simplest of his conjectures. (There is nothing quite so simple for other primes.) Ramanujan proved these three congruences, but his proof of the mod 11 congruence is much deeper than his proofs of the mod 5 and mod 7 congruences. The purpose of this talk is to give simple proofs of all three congruences. The proofs of the mod 5 and mod 7 congruences are simpler versions of Ramanujan’s proofs, and the proof of the mod 11 congruence is new.
2 Two proofs of the mod 5 congruence

We start with two well–known identities, proofs of which can be found in Hardy & Wright, Introduction to the Theory of Numbers, for example.

Euler’s identity

\[ E = (1 - q)(1 - q^2)(1 - q^3) \cdots \]
\[ = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots \]
\[ = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} \]

and Jacobi’s identity

\[ J = E^3 = 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + 13q^{21} \]
\[ - 15q^{28} + 17q^{36} - 19q^{45} + \cdots \]
\[ = \sum_{n \geq 0} (-1)^n (2n + 1) q^{(n^2+n)/2}. \]
First we notice that the exponents in the series for $E$, namely $0, 1, 2, 5, 7, 12, 15$ and so on are all congruent to 0, 1 or 2 modulo 5.
So we can write

$$E \equiv E_0 + E_1 + E_2,$$

where for $i = 0, 1, 2$, $E_i$ comprises those terms in $E$ in which the exponent is congruent to $i \pmod{5}$.

Similarly, we notice that, modulo 5,

$$J \equiv J_0 + J_1,$$

where $J_i$ comprises terms in which the exponent is congruent to $i \pmod{5}$, since $(n^2 + n)/2 \equiv 0, 1$ or 3 $(\pmod{5})$, but $2n + 1 \equiv 0 \pmod{5}$ precisely when $(n^2 + n)/2 \equiv 3 \pmod{5}$. 

Also, it is easy to check that

$$(1 - q)^5 \equiv 1 - q^5 \pmod{5}$$

so

$$E^5 = (1 - q)^5 (1 - q^2)^5 (1 - q^3)^5 \cdots$$

$$\equiv (1 - q^5)(1 - q^{10})(1 - q^{15}) \cdots$$

$$= E(q^5).$$

So now we have, modulo 5,

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{E} = \frac{E^4}{E^5} \equiv \frac{E^4}{E(q^5)} \equiv \frac{EJ}{E(q^5)}$$

$$\equiv \frac{(E_0 + E_1 + E_2)(J_0 + J_1)}{E(q^5)}$$

$$\equiv \frac{E_0 J_0 + E_0 J_1 + E_1 J_0 + E_1 J_1 + E_2 J_0 + E_2 J_1}{E(q^5)},$$
and if we look for terms in which the exponent is congruent to 4 (mod 5), we find

\[ \sum_{n \geq 0} p(5n + 4)q^{5n+4} \equiv 0 \]

and the mod 5 congruence is proved!
(This is a simplified version of Ramanujan’s proof.)
An alternative proof using only $J$ goes as follows. Modulo 5,

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{E} = \frac{E^9}{E^{10}} = \frac{(E^3)^3}{(E^5)^2} \equiv \frac{J^3}{E(q^5)^2}$$

$$\equiv \frac{(J_0 + J_1)^3}{E(q^5)^2}$$

$$\equiv \frac{J_0^3 + 3J_0^2J_1 + 3J_0J_1^2 + J_1^3}{E(q^5)^2}$$

so

$$\sum_{n \geq 0} p(5n + 4)q^{5n+4} \equiv 0.$$
3 Proof of the mod 7 congruence

We have

\[ E = E_0 + E_1 + E_2 + E_5, \]

where \( E_i \) comprises those terms of \( E \) in which the exponent is congruent to \( i \pmod{7} \), and, modulo 7,

\[ J \equiv J_0 + J_1 + J_3, \]

where \( J_i \) comprises those terms of \( J \) in which the exponent is congruent to \( i \pmod{7} \).

Note that \( 2n + 1 \equiv 0 \pmod{7} \) whenever \( (n^2 + n)/2 \equiv 6 \pmod{7} \).
It follows that, modulo 7,

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{E} = \frac{E^6}{E^7} = \frac{(E^3)^2}{E^7} = \frac{J^2}{E^7}$$

$$\equiv \frac{(J_0 + J_1 + J_3)^2}{E(q^7)}$$

$$= \frac{J_0^2 + 2J_0J_1 + J_1^2 + 2J_0J_3 + 2J_1J_3 + J_3^2}{E(q^7)},$$

So if we look for those terms in which the exponent is 5 (mod 7), we find

$$\sum_{n \geq 0} p(7n + 5)q^{7n+5} \equiv 0.$$
4 Proof of the mod 11 congruence

We have

\[ E = E_0 + E_1 + E_2 + E_5 + E_7 + E_{15} \]

and, modulo 11,

\[ J \equiv J_0 + J_1 + J_3 + J_6 + J_{10}, \]

since \( 2n + 1 \equiv 0 \pmod{11} \) whenever \( (n^2 + n)/2 \equiv 4 \pmod{11} \).

So we have

\[
\sum_{n \geq 0} p(n)q^n = \frac{1}{E} = \frac{E^{21}}{E^{22}} = \frac{(E^3)^7}{(E^{11})^2}
\]

\[
= \frac{J^7}{(E^{11})^2} \equiv \frac{J^7}{E(q^{11})^2}
\]

\[
\equiv \frac{(J_0 + J_1 + J_3 + J_6 + J_{10})^7}{E(q^{11})^2}.
\]
If we now extract those terms in which the exponent is congruent to 6 (mod 11), we find

\[ \sum_{n \geq 0} p(11n + 6)q^{11n+6} \equiv \frac{P}{E(q^{11})^2}, \]

where \( P \) is a polynomial in \( J_0, J_1, J_3, J_6 \) and \( J_{10} \) of degree 7.
Indeed, modulo 11,

\[ P \equiv 7 J_6^6 J_6 + 10 J_0^5 J_3^2 + J_0^4 J_1 J_6 J_{10} + 8 J_0^3 J_3^3 J_3 + 2 J_0^3 J_1 J_3^2 J_{10} \\
+ 8 J_0^3 J_6^3 J_{10} + 3 J_0^2 J_1^2 J_3 J_6^2 + 3 J_0^2 J_1 J_6 J_{10}^2 + 3 J_0 J_3 J_6^2 J_{10} \\
+ 2 J_0^2 J_3 J_6 J_{10}^3 + 10 J_0^2 J_{10}^5 + 7 J_0 J_1^6 + J_0 J_1^4 J_3 J_{10} + 2 J_0 J_1^2 J_3 J_6 \]

\[ + 3 J_0 J_1^2 J_3^2 J_{10} + J_0 J_1 J_3 J_6^4 + 2 J_0 J_1 J_3^3 J_{10}^2 + J_0 J_3^4 J_6 J_{10} \\
+ 8 J_0 J_3^3 J_{10}^3 + 10 J_1 J_6^5 + 2 J_1^3 J_3 J_6^2 J_{10} + 8 J_1 J_6 J_3^3 + 10 J_1 J_3^5 \\
+ 8 J_1 J_3^3 J_6^3 + 3 J_1 J_3^2 J_6 J_{10}^2 + J_1 J_3 J_6 J_{10}^4 + 7 J_1 J_{10}^6 + 7 J_6^6 J_{10} \\
+ 7 J_3 J_6^6 + 10 J_6^5 J_{10}^2 \]

We want to show that \( P \equiv 0 \).
The crucial idea

We have

\[ J^4 = E^{12} = E^{11}E \equiv E(q^{11})(E_0 + E_1 + E_2 + E_5 + E_7 + E_{15}). \]

That is,

\[ (J_0+J_1+J_3+J_6+J_{10})^4 \equiv E(q^{11})(E_0+E_1+E_2+E_5+E_7+E_{15}). \]

If we extract those terms in which the exponents are 3, 6, 8, 9 and 10 (mod 11), we find

\[
\begin{align*}
J^3_0J_3 &+ J_0J^3_1 + 6J_0J_1J_3J_6 + 7J^2_1J^2_6 + 3J^2_3J^2_6J_{10} + J_6J^3_{10} \equiv 0, \\
J^3_0J_6 &+ 7J^2_0J^2_3 + 6J_0J_1J_6J_{10} + J^3_1J_3 + 3J^2_1J^2_3J_{10} + J^3_6J_{10} \equiv 0, \\
3J^2_0J^2_6 &+ 6J_0J_3J_6J_{10} + J_0J^3_{10} + 7J^2_1J^2_3 + J_1J^3_6 + J^3_3J_{10} \equiv 0, \\
3J^2_0J^2_3 &+ 7J^2_0J^2_{10} + J_0J^3_3 + 6J_1J_3J_6J_{10} + J^3_1J_6 + J_1J^3_{10} \equiv 0, \\
J^3_0J_{10} &+ 6J_0J_1J_3J_6 + 3J_0J_1J^2_{10} + J_1J^3_3 + J^3_3J_6 + 7J^2_6J^2_{10} \equiv 0.
\end{align*}
\]
Now we make the observation that

\[ P \equiv (7J_1J_3J_{10} + 5J_0^2J_3 + 7J_1^3) \]
\[ \times (J_0^3J_3 + J_0^2J_1 + 6J_0J_1J_3J_6 + 7J_1^2J_6^2 + 3J_3J_6^2J_{10} + J_6J_{10}^3) \]
\[ + (7J_0^3 + 7J_0J_1J_{10} + 5J_6^2J_{10}) \]
\[ \times (J_0^3J_6 + 7J_0^2J_3^2 + 6J_0J_1J_6J_{10} + J_1^3J_3 + 3J_1J_3^2J_{10} + J_6J_{10}) \]
\[ + (7J_0J_3J_6 + 5J_0J_{10}^2 + 7J_3^3) \]
\[ \times (3J_0J_1^2J_6 + 6J_0J_3J_6J_{10} + J_0J_{10}^3 + 7J_1^2J_3^2 + J_1J_6^3 + J_3J_{10}) \]
\[ + (5J_1^2J_6 + 7J_3J_6J_{10} + 7J_{10}^3) \]
\[ \times (3J_0^2J_3J_6 + 7J_0^2J_{10} + J_0J_{10}^3 + 6J_1J_3J_6J_{10} + J_1J_6^3 + J_1J_{10}^3) \]
\[ + (7J_0J_1J_6 + 5J_1J_{10}^2 + 7J_6^3) \]
\[ \times (J_0^3J_{10} + 6J_0J_1J_3J_6 + 3J_0J_1J_{10}^2 + J_1J_3^3 + J_3J_6^3 + 7J_6^2J_{10}^2) \]

and so

\[ P \equiv 0. \]