Response to Draft Senior Mathematics Curriculum (ACARA)
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ACARA has recently released draft curricula for two Senior Secondary mathematics subjects: *Mathematical Methods* (*MM*) and *Specialist Mathematics* (SM), which are aimed at students in Years 11 and 12 who intend to go on to use mathematics at the tertiary level.

The introduction of a national high school mathematics curriculum across Australia offers the opportunity for us to improve mathematics education in all states, encourage new generations of students to aim for careers involving mathematics, such as engineering, science and finance, and strengthen our role as an education leader in the Asia region. The current drafts fail to do this.

A national curriculum should reflect the needs of students and best practice across the states. In NSW, the proposed curriculum is far inferior in key ways to what we already have; in terms of flexibility, balance, interest and usefulness to students, and alignment with the international first-year mathematics curriculum which underpins tertiary study in engineering and science. Forcing students to choose between only two levels of HSC maths will steer many NSW students away from careers in engineering and science disciplines.

Specifically, our main concerns are that the current draft has

a) *unbalanced syllabus content out of line with top international standards*

b) emphasis on rote learning rather than mathematical understanding, with a notable absence of engaging applications

c) unclear design methodology, mathematical inconsistencies, and lack of justification

d) excessive implementation costs for teachers and State Boards of Education.

The draft national curriculum replaces core material such as algebra, geometry and applications of calculus with a lot of advanced statistics and, for the higher strand, tertiary level topics. The subject MM is now largely calculus and statistics, while the subject SM is an ad-hoc and overfull compilation of advanced topics, which will only appeal to a small cohort of students, and will be very difficult for any teacher to effectively teach.

We are particularly concerned that future engineering and sciences students will fall through the gap between these subjects, avoiding SM because of its difficulty, and not getting enough crucial knowledge and applications from MM.

The combination of both subjects will likely lead to a reduction of students studying mathematics at high school. We need to be encouraging more students to engage with mathematics, not fewer. Another inevitable consequence will be a weakening of standards, and programs, for tertiary level engineering and science, at least here in NSW. The whole exercise will be accompanied by a significant cost and stress to the teaching profession, which will have to undergo massive retraining---in a direction, ironically, not in the national interest.

This response consists of

A) A SUMMARY OF CONCERNS, which can be read by a general audience,

B) SUGGESTIONS FOR CHANGE for both subjects MM and SM, meant for an audience familiar with details of high school mathematics, and

C) DETAILED COMMENTS AND CORRECTIONS meant for ACARA developers and educators.

A) SUMMARY OF CONCERNS

The task of writing a national curriculum is a challenging one. Despite the dedicated work of ACARA, the drafts suffer from serious shortcomings. We now elaborate on what we consider to be the main weaknesses in the proposed drafts.

Unbalanced curriculum content, out of line with top international standards

Advanced level high school syllabuses around the world include the traditional broad topics of *algebra, geometry* and *calculus*, together with the relatively newer but also important topic of *statistics*. A good syllabus ought to have a balance of these four subjects and bring out the connections between them to reinforce ideas and strengthen applications.

In the current draft MM course, *algebra is minimised* and *geometry is omitted altogether.* Of the four basic topics, these are historically the most important, and the most fundamental. Algebraic skill is fundamental for almost all mathematical computations. Geometric thinking, including both Euclidean, coordinate and three dimensional geometry, is essential not only in mathematics, but in many of its applications; especially in engineering, physics, chemistry, architecture, industry and technology. It also provides an ideal topic to develop logical reasoning and problem solving skills.

Asian countries are quickly establishing the international benchmarks for quality school mathematics programs. In the latest PISA rankings, Singapore, Hong Kong, China (Shanghai), Korea, Taipei and Japan took out 6 of the 9 top places, while Australia was placed 15th. The syllabi for these countries strongly respect the importance of algebraic competence and geometric reasoning and understanding; for example China has considerable material on spacial geometry, transformations and vectors. If we pursue a national agenda strongly out of line with these countries, our attractiveness as a regional educational provider in the sciences and engineering professions will be placed in jeopardy.

The current draft has replaced algebra and geometry with a formulaic approach to calculus, and an inordinate amount of statistics. This is a drastic proposal which, if adopted, will isolate Australia's mathematics education internationally, and will impact directly on the capabilities of our future engineering and science students.

While an increase in statistics from the current NSW levels can certainly be justified, the proposed syllabus swings the pendulum too far in this direction. The amount of statistics in the MM subject, which is much beyond anything our Asian colleagues see, swamps any attempt to provide a balanced view of mathematics. This new direction will magnify the existing weaknesses of students, and will very likely turn off many mathematically minded students. Although engineers certainly can benefit from learning some statistics, it is more important that they are skilled in algebraic and geometrical reasoning and computation.

Despite half the MM syllabus being devoted to statistics, the treatment is undermined by the lack of proper grounding in calculus and, to a lesser extent, algebra. The approach in MM is a standard one, adapted for students without the requisite foundations in calculus, requiring so many corners to be cut that the mathematical underpinnings of the subject become lost. This risks creating the impression that statistics is a “black-box” discipline, where magical results are pulled out of a hat, without proper justification.

It is optimistic to expect that students will be attracted to a mathematics subject if the logical structure is not accessible to them. Unfortunately there is a real risk that the proposed curriculum will turn off many young people from statistics, and hence also from mathematics. In addition, many teachers here in NSW will find the statistics *very* hard to teach, and students will perceive this subject as being more challenging *than our current Mathematics (2 unit) course*, so they will drop down. It means we risk losing students studying advanced level mathematics.

Emphasis on rote learning rather than mathematical understanding, with a notable absence of engaging applications

The MM syllabus appears to be heavily symbolic and rule-driven, with a lack of clear unifying ideas and themes. It focuses more on accumulating methodological tools rather than on developing mathematical understanding; teaching students how to compute answers without teaching why methods work.

Applications of calculus, which are usually of considerable interest and importance in the physical sciences, are in this draft largely limited to statistics. Applications not only reinforce understanding by giving concrete realisations of abstract mathematical concepts, but they also lead to a better appreciation of the discipline. Indeed, except for the abstractly inclined, we find many students turned off from mathematics because they “can't see what it's good for”. Consequently, teaching applications is much more important in the MM syllabus than in the SM syllabus. Historically applications of calculus include prominently *volumes*, *motion* and *mechanics;* topics which appeal to students, yet receive insufficient attention in these syllabi.

The Singapore Secondary Maths Syllabus states that “Students should develop and explore the mathematics ideas in depth, and see that mathematics is an integrated whole, not merely isolated pieces of knowledge.” Our fear is that the rather routine methodological approach of MM, with very few gems and little development of abstract thinking, will dull the interest of students who otherwise would be drawn to the mathematical sciences.

The syllabus for SM discourages in-depth understanding in a different way: it covers an enormous amount of material and is a major leap up in difficulty from the MM subject. Such breadth comes at the expense of depth, as students will have fewer opportunities to develop deep understanding, problem solving skills, and an appreciation for logical structure and the role of proofs.

Several topics, such as graph theory, are debatable additions to the high school syllabus and generally not found elsewhere in the world; at UNSW we teach graph theory in our first year Discrete Mathematics course, amongst several other equally important topics. *It is neither necessary nor desirable that students tackle university level mathematics in high school---we would much rather that they learn the basics well.*

Potential science and engineering students are an important “clientele” group which the National Curriculum must cater for. It is alarming to see that their needs have not been met. For the very strong students amongst them, the SM subject is a good option, but it is unsuitable for the vast majority of such students who will find the excessive material overwhelming.

The MM syllabus on the other hand, is seriously inadequate, as they get only a minimum of algebra, with hardly any review, miss out entirely on any kind of geometry beyond that in the year 10 syllabus, and get hardly any important applications of calculus. Geometric thinking is absolutely vital to much of science and most of engineering, and many students appreciate the strongly visual aspect of this subject, as opposed to the formula-driven approach to calculus currently in the draft. Geometric intuition is best built up steadily and continuously over time; it is naive to think that science and engineering students will pick this up easily at university after a lapse of two years.

The undue emphasis on methods at the expense of ideas appears in the SM syllabus as well as MM, conspicuously in the treatment of statistics. The two new inferential ideas of using sign tests and using permutation tests are presented separately as a “cookbook” of specialised solutions to specialised problems. By applying either technique, or both, across a range of useful applications (including the important problem of comparing two independent samples, currently ignored), students could develop a broader intuition for how to approach analysis of the relationship between variables.

Unclear design methodology, mathematical inconsistencies and lack of justification

The current proposal appears to be the result of a compromise between vested interest groups, rather than the result of a dedicated investigation into the principles that inform effective syllabi, and a study of those ideas that are effective elsewhere in the world. Even the names of *Mathematics Methods* and *Specialist Mathematics* come from the Victorian system, suggesting a pre-conceived bias rather than a clean fresh start.

In any case, the various design decisions implicit in the proposed curricula ought to be *justified and explained*. How is the proposed system of two subjects (MM and SM) superior to the more flexible arrangement of 2/3/4 unit courses (Mathematics/Extension 1/Extension2) that we have here in NSW, which give students a better opportunity to find a level suitable to them? Why abandon geometry and scale down algebraic competence in MM? Why the large amount of material in the SM syllabus, and in particular why graph theory, as opposed to any number of other potential topics, for example: transformations and symmetry, elementary number theory, projective geometry, or logic and boolean algebra? Do prior studies show that such a high level of statistics encourages students to pursue the subject at the tertiary level?

The case for a dramatic reorientation, at least to educators here in NSW, needs to be made much more convincingly. In addition, a contentious draft syllabus such as this ought to be accompanied by a schedule of extensive trialling in one or several districts before any serious move is made to roll it out nationally. Surely some serious feedback from teachers is needed before one can properly assess the likely effectiveness of the new curriculum. In fact we are pretty confident that the results will be notably underwhelming, at least here in NSW.

The draft presents ample evidence that the project has not been thought out carefully; with a depressing number of both minor and major mathematical misunderstandings, omissions and inconsistencies throughout, especially in the MM course. Even the topics that are treated at length in this course, namely calculus and statistics, are very poorly handled. In the Detailed Comments section we make a list of many such problems; their prevalence and seriousness weakens our confidence in this project considerably.

For example: the Calculus section of MM has serious pedagogical issues: with the traditional geometric underpinnings minimised or removed entirely, trigonometry introduced in an unmotivated fashion (again the lack of geometry), anti-differentiation preceding any motivating discussion of integrals as areas, a bizarre definition of the exponential function, the quotient rule and implicit differentiation used before they are taught, the main techniques of integration (integration by substitution, partial fractions and parts) completely missing, and applications largely bereft of interest. In fact Unit 4 of MM has no integration applications at all.

The Learning Outcomes that precede the description of each unit are written in a uniform and perfunctory fashion, and are all of the following form: “understand the concepts and techniques in Topic X”, “solve problems in Topic X”, “apply reasoning skills in Topic X”, “communicate argument and strategies when solving problems and interpret and evaluate mathematical information and ascertain the reasonableness of solutions to problems”. This generic approach shows that the goals of the syllabi have not been properly thought out: is this the best we can do to enunciate the national needs for hundreds of thousands of students?

Cut and pastes are replicated inside the curriculum: the phrase “use Topic X to solve practical problems” is found repeatedly at the end of sections; but how should an educator interpret such unspecified instructions? The Achievement Standards, which summarise the four units, are all repeats of generic phrases along the lines of: “understands and applies concepts and techniques in Topics X,Y and Z”, “uses digital technologies appropriately to solve a range of problems”, etc. Compare this with the Singapore Syllabus, with clearly enunciated objectives and use of precise, meaningful language.

It is not clear that the project was preceded by asking some key questions; such as "*What makes a successful national mathematics curriculum*?", "*How do we get more students excited about studying mathematics*?" and "*How do we determine the effectiveness of a proposed curriculum*?"

Finally, at the level of mathematical details, we have had a good look at the curricula and found many inconsistencies, errors of judgement and omissions (see the DETAILED COMMENTS AND CORRECTIONS section). How many more will be found when an unprepared and reluctant teaching cohort takes on the massive job of implementing these changes?

Excessive implementation costs for teachers and State Boards of Education

The scale of the effort and cost required to implement the proposed curriculum is huge – particularly in terms of resources for developing lesson plans and teacher training. We do understand that implementation is not the main concern of ACARA, but we believe that it would be foolish to ignore this issue.

Maths teachers already struggle with the present syllabi, and they need a lot of guidance as to what to teach. Any major rewrite of the curriculum, as is being proposed here, ought to be accompanied by information and resources more complete than the proposed MM and SM syllabi-- to ensure that teachers and students see how the topics fit together in a logical fashion. Otherwise, inherent inconsistencies will be revealed when individual lesson plans are fleshed out, and each state will have to reinvent the wheel fleshing out the proposals. It is far more appropriate that the curriculum body ACARA itself prepare such secondary materials and documentation. This is also a good way of discovering mistakes, inconsistencies and omissions.

One of the key differences between Australia and Asian mathematics education is that Asian education has a culture of using textbooks, and we generally don't. In China or Singapore, the curriculum is supported by carefully produced textbooks that ensure that student work is focussed on particular topics and that teachers and parents have a consistent resource.

The ICEEM texts created by AMSI in the lower years are a great step forward in this direction, but at the year 11 and 12 level we don't see any supporting texts arising in the near future along the lines of the proposed MM and SM, which is a bottleneck to successful implementation.

The other very serious concern with implementation is *the issue of teacher training*. These syllabi differ significantly, even dramatically, from the current ones here in NSW, and also in other states. They incorporate new material that many of our high school teachers will never have seen before. This is true both of the MM course, with its influx of high level statistics, and the SM course, with new sections on graph theory, linear algebra, vectors and statistics. *It will be an enormous and costly task to re-train teachers in the new topics* and they will find this stressful. Unless this is done well, a generation of students will suffer.

SUMMARY: This draft is inferior to the current NSW syllabus; the intermediate mathematics course (MM) has too little geometry and algebra, too much statistics, and an inconsistent approach to calculus with not enough interesting applications. In addition we lose the flexibility we currently have with 2/3/4 unit mathematics. It also restricts the advanced subject SM to a small cohort of very gifted students, who will get too much material instead of a deeper understanding of key topics.

Since calculus will be primarily taught in MM, our stronger students will not develop a deeper understanding of this fundamental sub-discipline and its wide-ranging applications. This will weaken the mathematical abilities of our incoming tertiary students heading to be our future engineers and scientists.

The new national curriculum risks lowering the standards at many universities,

diminishing the quality of our educational export industry. Inevitably it will lead to a reduction of mathematics enrolments in high schools, and impose a very heavy cost on teachers, as they struggle to learn the new material, and on some state governments such as NSW, as they juggle their offerings to minimise the damage.

The proposed national curriculum will be, in our opinion, a setback for mathematics education in NSW, and also for other parts of the country, and we at UNSW would support the Federal government having a fresh look at the project.

B) SUGGESTIONS FOR CHANGE: Mathematical Methods (MM)

Most advanced mathematics students will be taking the MM subject, which is why it is particularly important to identify and address its weaknesses---many aspects of the curriculum properly need to be completely rethought. Probably it would be best if a strong and balanced group of mathematicians and statisticians were recruited to tackle the problem again from the beginning. This is politically perhaps unlikely, but it would be the best outcome for future advanced mathematics students in Australia. Making adjustments to a flawed design can be much more expensive than redesigning. Having said that, here are our general suggestions for improvements (detailed critical analysis follows in the Appendix).

The gravest problem with the MM syllabus is the lop-sided choice of material. We strongly recommend that the content be balanced across the subjects **1) Algebra ...2) Geometry ... 3) Calculus …** and **4) Probability and Statistics.**

The algebra content needs to be strengthened and earlier material reviewed. We suggest introducing arithmetic and geometric series, inequalities, more algebraic work with polynomials and rational functions, including the factor theorem. For example decomposition into partial fractions is an important precursor to integration, and a more thorough appreciation of the algebraic manipulations that correspond to translations, reflections and other transformations is very useful.

The combinatorial side of the binomial coefficients ought to be moved from the SM to MM-- it is fundamental. There could also be more trigonometry in the MM syllabus, preceded by some geometry of right triangles, and discussion of areas and arc lengths associated to circles. The sine and cosine laws are central and ought to be established carefully. This is a good environment to practice algebraic skills, and including some practical trigonometry with some clearly spelt out applications is valuable.

Geometry, Euclidean and vector, needs to be incorporated into the syllabus. One obvious possibility is to move the sub-unit "Vector geometry" currently in the SM syllabus to the MM syllabus. This would give a good basis for some Geometry that would be useful, not just to future science and engineering students, but to anyone who wishes to pursue mathematical studies.

There should also be more co-ordinate geometry in the syllabus, and it should include the connection between quadratic equations in x and y and conic sections in the plane, and some three-dimensional sketching, and equations of planes. In particular volumes and surface areas of spheres, cylinders and cones in three dimensional space should be covered, as well as some basic familiarity with translations, reflections and rotations, at least in the plane.

The calculus section needs to be completely re-organised to ensure a logical presentation, avoid important omissions and encourage greater understanding. The subject ought to be integrated with coordinate geometry, which motivates the subject, for example by a prior discussion of tangent lines. Fundamental techniques such as the product, quotient and chain rules should be covered earlier. Integration using algebraic manipulations such as partial fractions and substitution should be brought in from the SM syllabus. Polynomials and rational functions form the workhorse for calculations and here allow simultaneous development of important algebraic skills. The exponential function ought to be defined in a more standard way, and connected with logarithms.

It is traditional to develop the differential calculus first, including second derivatives, so that the applications in Unit 4 ought to be more of integration, and they should include some interesting gems, of which there are many. Newton's law F=ma ought to be mentioned, as this is a core reason for so much of the importance of the calculus. The use of Leibniz notation should more clearly distinguish between infinitesimals and differences.

We have to consider that future engineers will likely not be taking SM, so MM must provide them enough basic skills and classical applications to adequately engage with their tertiary courses. Applications of calculus should include some of those traditional ones found in textbooks; calculating maximums and minimums, area, volumes, masses, motion and forces. A small section on numerical integration, say the trapezoidal rule, should be included to strengthen students' understanding of the integral, and give them tools for approximating integrals they cannot compute.

Needless to say, the time spent on Statistics in the MM syllabus needs to be cut back to make way for other topics. Most international curricula have the Probability and Statistics component at *roughly a quarter of the course* and we recommend that the MM syllabus should roughly adopt this.

We believe that it is possible to both reduce the time spent on Statistics and improve the quality and interest of the material for this topic. The key aspect of Statistics which we should teach to high school students is an understanding of the fundamental idea of statistical inference: *When we use a sample of data to calculate a statistic, the process of sampling introduces uncertainty. We can use the rules of probability to quantify this uncertainty, then take it into account when drawing general conclusions ("inferences") from a sample.*

Logically, the key is the 2SE rule: approximately 95% of sample proportions are within two standard errors of their mean, which can be taught without reference to the Central Limit Theorem or continuous distributions. Given that these latter topics cannot be supported with the inadequate Calculus foundations in the MM syllabus, we recommend that continuous distributions and the Central Limit Theorem be omitted, or moved to the SM syllabus.

This ensures that students will better appreciate statistical inference without the distraction of the more sophisticated continuous random variables, and is in keeping with the maxim that material should be taught properly or not at all. Indeed, this is how Probability and Statistics is taught at first year at UNSW, even though the students have a much stronger foundation in Calculus. With similar judicious trimming of the Statistics strand, we believe that it can easily (and should be) cut back to a quarter of the syllabus, delivering a more concise and attractive introduction to the discipline which focusses more directly on the key elements.

The idea of statistical inference is a major conceptual advance for most students, and so it needs to be treated carefully and taught in a methodical way. Topics such as the exponential distribution are a diversion. Clearly probability, data and the notion of a discrete random variable are required, with the associated probability function, mean and variance. The Bernoulli distribution is an important special case. A statistic calculated from a random sample can be treated as a random variable, a key idea which can be studied by simulation; standard errors of statistics can be introduced via a study of how means and variances change when random variables are linearly transformed or summed; and then the 2SE rule mentioned above.

This sequence of ideas could be taught using a much smaller portion of the MM syllabus than has been assigned to statistics in the draft proposal. Some topics in the current proposal (in particular, the exponential distribution) are frankly counter-productive and should be eliminated.

A modified version of the above syllabus, where we primarily teach inference via hypothesis testing using the binomial distribution, is taught in our first year course at UNSW in 8 lectures. Hence we believe it is possible to teach fundamental ideas behind statistical inference without dedicating almost half of the MM course to the topic.

SUGGESTIONS FOR CHANGE: Specialist Mathematics (SM)

Currently the topics in this course seem ad-hoc and disjointed, with material skipped over lightly without encouraging depth of understanding. We suggest having a smaller number of topics, built around structured, interconnected themes, that are explored in some depth. There are certainly many options for such topics, but the choice should be guided by good pedagogical principles: teach a topic well or not at all, students should obtain good exposure to *mathematical proofs,* students should have ample opportunity to tackle *challenging problems* etc. We illustrate with some examples below.

The topics on vectors (in the plane and in 3 dimensions), matrices and linear systems of equations could form a nice introduction to a university linear algebra course. However, as written, they are poorly integrated, and many aspects are touched on all too briefly or lead nowhere. For example, the connection between the vectors and matrices topic is lost as one has not viewed nx1-matrices as vectors. Other missed connections are the dot product as a matrix product, and bizarrely enough, although the (orthogonal) projection is touched on in the vectors topic, it is not included in the transformations in the plane topic. Matrices are introduced without motivation, and though the definitions of sums and scalar multiples of matrices are not unnatural, their utility is completely hidden.

The material on vector and Cartesian forms of lines and planes is too brief and misses out on many challenging problems. For example, we suggest students also learn vector equations for line segments and regions in a plane, to promote their understanding of parametric forms and the meaning of parameters. Also, they should look at the intersections of lines and/or planes, where the lines or planes can be given in parametric or Cartesian form. There are many nice problems here which not only promote their understanding of the subject, but can challenge good students. The notions of convexity and barycentric co-ordinates could also be introduced and connected with the physical concept of center or mass. The section on row reduction methods is rather pointless as back-substitution is not taught. We suggest either Gaussian elimination is taught properly and allotted the requisite amount of time, or better still, left off until students reach university.

Geometry is traditionally regarded as the best environment for exposing students to mathematical proofs, due to the many rich theorems and problems that have been discovered over the millennia. While circle geometry is well and good, it is rather limiting. We suggest that Euclidean geometry by reviewed, and this section be expanded, perhaps including some basic triangle geometry.

Students must have a good grounding in Cartesian planar co-ordinate geometry, including lines as linear relations and conics as quadratic relations, and algebraic forms of transformations. Another historically important topic are the classical theorems of Pappus and Desargues in projective geometry, and it would also be good to include some more physical applications, such as to laws of optics. For example, the law of reflection is an illustration of a minimum principle.

The introduction of graph theory is mostly unprecedented across the states. The material here is somewhat heavy on definitions at this level, so that while the subject has many applications, the students do not see many. If the subject is to be retained, we suggest an emphasis on Euler's formula, which connects to the geometry of the Platonic solids and other polyhedra; a pleasant and interesting topic for high school students to investigate using models. Other aspects of discrete mathematics might connect more easily to the rest of the course: for example the theory of generating functions, which connects to the algebra of rational functions as well as to calculus and infinite series.

The goals in SM statistics should be to develop a deeper intuition for the mathematical underpinnings of MM statistics. If time permitted, additional goals could be to extend notions of statistical inference to the important setting of testing for relationships between variables, perhaps via permutation testing as currently proposed. (The advantages of permutation testing being that it is intuitive to students and it exemplifies the important role of computing in modern statistics.)

The following syllabus outline could illustrate these ideas: deriving properties of the binomial distribution, including its normal approximation via the Central Limit Theorem; the hypergeometric distribution; inference about relationships by comparing independent samples and a paired sample using the hypergeometric and binomial distributions for exact inference (with a Bernouilli response) or approximate inference (using sign tests), and simulation-based inference about relationships via permutation tests. The relation between permutation tests and the binomial/hypergeometric should be explored. Some material on continuous distributions would be needed, especially the normal distribution, with numerical integration used to approximate normal probabilities, and to show students where the 2SE rule comes from.

C) DETAILED COMMENTS AND CORRECTIONS: *Mathematical Methods*

Note that more important points are indicated with (**) or (***).

Unit 1:

Topic 1: Algebra, functions and graphs 1

- It is important that students get lots of practice with the arithmetic of general polynomials and rational functions, and have some experience with conics. This is the heart of algebra, and links naturally to geometry. So general linear and quadratic polynomials and their graphs ought be included, not just specific examples. (***)
- In particular these students should appreciate that a linear relation between x and y geometrically means a line, and a quadratic relation geometrically means a conic section, and they should have seen ellipses, hyperbolas and parabolas as quadratic relations (not necessarily functions). This is historically and conceptually a key point. (***)
- The repeated use of phrases such as : “features of the graph of X...” is ambiguous. Which features exactly?
- The phrase “examples of inverse proportion” sounds stilted.
- When discussing the absolute value function, the phrase “absence of a tangent at the 'corner'” is ambiguous, and its accuracy debatable.
- The two dot points under *Graphs of relations* belong to the earlier section *Review of quadratic relationships*, and the second point is a direct repeat of an earlier one. (**)
- The relation between the graph of a function and its inverse function, when it exists, should probably be spelt out.
- When discussing the Binomial theorem, students should see the combinatorial role of the binomial coefficients, as in SM. This is historically a key point, as evidenced by its early introduction by Levi ben Gershon in the 1300's. (**)
- The last dot point under *Binomial theorem* is grammatically awkward and unclear.

Topic 2: Calculus 1

- The connection of the notion of derivative as the average rate of change of a function (a hard idea for students at this level to get their head around) and as the gradient of a tangent (a far more geometric approach) should be clarified. We recommend the geometrical approach. (***)
- There is inconsistent use of h and δx, presumably meaning the same thing. It is more standard to use Δx for a non-infinitesimal change rather than δx. Calling the latter “Leibniz notation” is debatable, since he used infinitesimals, which is not what is meant here. (**)
- Variable rates of change ought to be preceded by at least a mention of constant rates of change, arising from linear functions.
- The last dot point under *Computation of derivatives* introduces yet another notation for a derivative.

- There are no clear applications of the calculus given in this section - indeed the first application of calculus appears to be the last dot point in Calculus 2 - and then this is obscured by restricting the problem to finite interval domains (sic). In Calculus 1 we are not even asked to find the equation of the tangent line to a curve! This is a further sign of the unfortunate downplaying of geometry. (***)

Topic 3: Probability

- How much of the material under Probability has been covered in K-10?
- Under *Conditioning and independence*, is not the third dot point the definition of P(A|B)?

Unit 2:

Topic 1: Algebra, functions & graphs 2

- Any reasonable treatment of classical trigonometry ought to include a prior discussion of the formulas for the area and arc-length of circular arcs, especially the definition of pi. (**)
- This section is again very theoretical, symbol-dominant and lacks any meaningful applications. Its purpose seems to be to introduce the trigonometric, logarithmic and exponential functions in one hit.
- Apart from the `revision' of right-angled trigonometry, the geometric aspects of the subject are suppressed---the focus of the section seems to be purely analytic. So students are given a very one-sided view of trigonometry, divorced from any geometrical discussion of Pythagoras' theorem, congruent triangles, properties of a circle, rotations etc. (***)
- The important sine and cosine rules are mentioned, but the approach to them is not specified. Curiously these rules are introduced *before* the definitions of the circular functions around the circle, so it is unclear how they extend to obtuse triangles. This important topic for scientists and engineers must be given a fuller treatment, with a wide variety of applications. (***)
- What are “extended domains”?
- The use of the term “indices” as synonymous with “exponents” might be reconsidered.
- This topic has many generic statements whose meaning is unclear. What do the vague instructions: “solve equations involving exponential functions etc”, “use exponential functions to solve practical problems” etc mean? Similarly for the logarithmic functions. Presumably compound interest is aimed at here? (**)
- The first and third dot points under *Logarithmic functions* appear to be identical.

Topic 2: Calculus 2

- We finally find some of the applications of calculus, mainly in questions from physics. However at this stage, we still do not have the product, chain or quotient rules, nor can we differentiate x^(-2) or sqrt(x), let alone (2x + 1)^7. (***)
- A notable omission here is Newton's method of finding zeroes of a function, which is an important application. (***)
- What is the difference between a polynomial and a linear combination of power functions?
- What is a “position such as time graph” ??
- In the fourth dot point under *Applications of derivatives*, there is no need to mention p(x).
- In the fifth dot point under *Applications of derivatives*, “interval domains” can be replaced with just “intervals”.

Discrete random variables

- A difficulty here is that the students have not seen geometric (or even arithmetic) series, and since there seems to be nothing in the syllabus on summing series of any sort, it is unclear how the mean and variance of the binomial distribution can be derived. Will summation notation be used? (**)

- What examples of non-uniform discrete random variables do you have in mind?
- Because of the lack of geometric series, the geometric distribution is missing. This is a shame because it misses an opportunity to better connect the different units in the course by linking statistics with an algebraic result that could otherwise have been introduced elsewhere. (**)
- Again, the instruction to “use binomial distributions to solve practical problems” is weak.

Unit 3:

This is a very daunting and problematic unit containing all the theory of the calculus of the circular, exponential and the logarithmic functions, as well as much of the integration theory, in addition to the difficult topic of continuous random variables. This will be very difficult to teach in a two-semester block, and the presentation of calculus here is highly dubious in places, with notable inconsistencies and vagueness.

Topic 1: Calculus 3

- The definition of the number e is awkward, not often used, and difficult to motivate to students. Please consider how you would go about computing e from your definition to say ten decimal places. (***)
- Once again, it is unclear what “use exponential functions and their derivatives to solve practical problems” means. The usual application is to growth and decay problems - is this what is meant? These kind of throw away lines allow you to say there are `applications' without these applications ever being explained. Ditto for the trigonometric functions. (***)
- The quotient rule is used in dot point 6; but the quotient rule is introduced in dot point 8. (!)
- Dot point 6 refers to the derivative of tan(x); this is then repeated in dot point 10.
- The wording of the phrase “the notion of composite functions and the use of the chain rule...” suggests that the rule need not be derived. Is this the intention? Hopefully not. (**)
- The logic of the unit is also of concern. The very important product, chain and quotient rules, which ought to have been introduced much earlier, now appear near the end of *Calculus 3*. This means that before this, students would not be able to differentiate expressions such as (x^2 + 1)^5 (without expanding), let alone x e^x. (***)

Topic 2: Calculus 4

- Anti-differentiation is completely unmotivated. The classical reasons for integrating---finding areas, volumes, centres of mass, and solving Newton's laws of motion, are nowhere to be found. (***)
- Students are expected to learn three different words for an anti-derivative, without motivation.
- The integration section contains no mention of numerical integration. How then can one make sense of the normal distribution in the following section - where the integral cannot be analytically evaluated and needs some numerical integration technique? (***)
- The meaning of the seventh dot point in *Anti-differentiation* is unclear.
- Some typos appear in the formulas for derivatives in dot point 9 of *Anti-differentiation,* and again in the second and third dot points of *Fundamental theorem.*
- The connection between the earlier section on anti-differentiation and definite integrals is not clearly spelt out.

Topic 3: Continuous Random Variables

- The notion of a *continuous random variable* is quite subtle. You have not made it clear what approach teachers are expected to take. A function on a probability space? This is a recipe for confusing the better students, who will struggle to try to pin down what the vague terminology really amounts to. (**)

- Under *Continuous random variables* most continuous random variables have density functions that are defined on the whole of the real line. Hence the probability notion leads to *improper integrals* -a topic which is beyond MM students in high school. (***)
- Without suitable techniques of integration, it is difficult to find the mean and variance of anything which is not `contrived' (***)
- Given the absence of the geometric distribution in Unit 3, the lack of any real integration tools, etc. it seems bizarre to have a whole section on the exponential distribution. You cannot even tell that it is a distribution without improper integrals! (**)
- Curiously the mean and variance of the exponential distribution are not mentioned, but the median, which we have not encountered before, is!
- Under the normal distribution the usual problems arise. Students cannot see that the formula even *defines* a density function; the presence of pi in the density function is inexplicable, since students have so little machinery at this stage. We cannot evaluate - or even see how one might evaluate numerically - the normal distribution, since no numerical integration was done previously. Students presumably will simply be told that `it can be done somehow' and it all becomes a black box. (***)
- The problem with all of this section is that it needs/assumes complicated mathematical ideas and techniques that are beyond the syllabus at that point and sweeps them under the carpet. This simply gives students (and perhaps teachers) the idea that statistics is a bag of tricks and calculations that they can never properly understand. (***)

Unit 4:

Once again this appears a very daunting and highly problematic unit. The statistics section contains a large amount of sophisticated material that is usually covered in second year university courses! Most advanced high school teachers in NSW will look at this material with trepidation, and likely contemplate an early retirement.

Topic 1: Interval estimates for proportions and means

- The corresponding syllabi descriptions are heavy handed. There are many formulas, but few, if any, of the results quoted can be meaningfully proven or even justified - more black box stuff. The idea that forcing students to memorise lots of formulas will somehow motivate them to pursue the subject at tertiary level is fanciful. (**)
- This section is very content-heavy. Some options for simplification include: focussing on inference about sample proportions alone, rather than including means as well; focussing on inference via tail probabilities (calculated exactly from the binomial distribution) rather than confidence intervals. (**)
- Simulation is strongly recommended to demonstrate key ideas about variation in statistics and interval estimates from one sample to the next, e.g. "simulate repeated sampling" and "simulate to illustrate variations in confidence intervals". While these are useful, a more concrete tool is to get class members to collect data and calculate statistics, and to observe the variation from one student's sample to the next, when calculating various quantities of interest. Replicate samples of real data provides a more concrete tool for demonstrating these key ideas and should be specifically recommended in the syllabus.

Topic 2: Calculus 5

- Calculus 5 finishes off the calculus of the logarithm function - without covering basic integrals, and finally getting to the second derivative, its geometric meaning and applications. In most courses in calculus, this would be done much earlier. Most students will be wondering why the final topic in this course has reverted to such elementary material. Shouldn't students be seeing real applications of calculus at this point?? (***)

- Implicit differentiation appears as a technique for calculating the derivative of the logarithm, but it has not been previously established. (***)
- The increments formula uses non-standard notation.
- Once again, in dot point 4 of *Logarithmic functions* it is very unclear what applying logarithmic functions and their derivatives means. Is this another throw away line?

SUMMARY OF MATHEMATICAL METHODS SUBJECT:

This syllabus is *seriously flawed* and *ought to be completely overhauled*. Spending two whole years studying only heavily symbolic calculus and statistics in which the theory cannot be properly done, seems entirely inappropriate.

The lack of emphasis on geometric thinking even in two dimensions, let alone three, is *historically bizarre and pedagogically dangerous*. It does not provide the sorts of skills or mind set that practitioners of mathematics---especially future engineers and scientists---need. Developing algebraic competence by working with coordinate geometry and polynomial relations is a time-tested method for motivating and preparing for calculus--- an understanding that seems to have gone completely missing from this course.

The amount and range of statistics, with what will necessarily be an emphasis on black box applications, may not encourage interest in the subject nor appreciation for its use, and we fear it will turn off many students otherwise interested in mathematics.

DETAILED COMMENTS AND CONCERNS: *SPECIALIST MATHEMATICS*

Unit 1:

Topic 1: Recurrence relations:

- Recurrence relations ought to be motivated in part by the square, triangular, pyramidal and possibly pentagonal numbers, which were of interest to the Greeks and come up often in practice. (**)
- The "direct formulas" for the associated series (partial sums) of an arithmetic sequence: which formula do you have in mind? The crucial formula here is the sum of 1+2+...+n.
- The divisibility example is not a primary application of mathematical induction, there are more interesting and important ones.

Topic 2: Combinatorics:

- The material on permutations and combinations is fundamental in algebra, and naturally goes with the Binomial theorem. This topic ought to be in the MM subject. It is a good example of actually useful mathematics that everyone ought to know, especially as it underpins quite a lot of probability. (***)
- It might be good to specify the simple identities for Pascal's triangle that are to be derived.

Topic 3: Geometry:

- These advanced students should see more proofs in geometry. The general nature of proofs discussion is largely unnecessary if students are not going to do a lot of proofs. Probably they should just assimilate these points indirectly.
- The classical geometry here is confined to circle geometry, and all the proofs come from this area. While certainly a pleasant topic, this is too one-sided. Historically there are other important subjects; *Pappus'* and *Desargues' theorems* are results that high school students should see; these are elementary and fundamental in Projective Geometry, and can be introduced without a lot of preliminaries. (**)

- In circle geometry, the students should probably also see the *rational parametrisation of a circle,* as it relates to elementary number theory (Pythagorean triples) and to the tan(theta/2) rationalising substitution for integration which they meet at university. The basic idea goes back to Pythagorean triples and was used by Diophantus and Fermat (if a topic is important historically, there is usually a good reason).

Topic 4: Vectors:

- This is an important topic and an excellent addition to the curriculum, and perhaps should be moved to MM. However it must be properly handled and motivated. In addition to displacement and velocity, force and acceleration should be mentioned as key examples.
- The scalar product is an *algebraic object* and must be introduced as such. The curriculum suggests that the definition might be in terms of a cosine of an angle, which is seriously wrong. One derives this, and indeed the scalar product largely replaces the angle in higher dimensions. (***)
- Geometric proofs involving vectors are good.
- there is another useful notion; the idea of a *convex linear combination of vectors*, with importance for physics and engineering applications. (**)

OVERALL SUMMARY OF SM Unit 1: This unit has the makings of an interesting course.

Unit 2:

Topic 1: Trigonometry:

- The compatibility with the Unit 2 of Methods is an issue. Students will be learning trigonometry from both courses simultaneously---has this been thought out? (**)
- The inclusion of some harmonic motion is good; however heights of tides is somewhat complicated; surely the classical example of an oscillating spring is fine. (**)

Topic 2: Matrices:

- The motivation of this subject is problematic. Matrices arise in practice when changing variables, solving systems of linear equations, or making linear transformations. At least one of these topics ought to be introduced first to justify the introduction of matrices. Solving 2x2 systems is hardly motivation enough for matrices, but three-dimensional planes and vector geometry are only introduced in Unit 3, so there is a difficulty here. (***)
- A solution is to study two-dimensional linear transformations first, and then introduce matrices as a convenient way of encoding these. A proper study of inverse matrices outside of the 2x2 case is beyond these students.
- The determinant should be linked to the change in area induced by a linear transformation of the plane, and/or the formula for the area of a triangle in the Cartesian plane. However the addition of matrices is then not so clearly motivated. So the order of this topic needs to be rethought. (**)
- The connection between vectors and nx1 matrices needs to be made explicit. (**)

Topic 3: Real and Complex numbers:

- Much of the discussion here is with integers and rational numbers, should this be omitted? As it is, the Topic perhaps ought to be called **Number systems**.
- Is not the conversion from rational numbers to recurring decimals done in a lower year?
- Any discussion of real numbers ought to include an attempt at defining pi properly. There should be some historical discussion about the problems in setting up real numbers. (**)
- With complex numbers, if we define Q(a+bi)=a^2+b^2 then the fundamental result Q(zw)=Q(z)Q(w) ought to be mentioned.

Topic 4: Graph theory:

- This section is problematic. Some of this material we teach to our first year students at UNSW in Discrete Mathematics—but we do not get as far as what is being proposed here. The implications seem not to have been thought out--- if this topic remains as proposed, high school teachers will struggle mightily to get on top of the material. Graph theory, while a lovely subject, is but one branch of combinatorics, and at this level relies overly on definitions, with the many interesting applications mostly beyond the reach of students. (***)
- The most important formula in the proposed graph theory section is Euler's polyhedral formula v-e+f=2 (this is a better way of writing it), and this Topic, if it must remain, could be cut down to just enough to prove that, and demonstrate it with the Platonic solids. (**)

OVERALL SUMMARY OF SM Unit 2: The graph theory section should be reduced and possibly eliminated altogether, and the introduction of matrices ought to follow planar linear transformations.

Unit 3:

Topic 1: Vectors in three dimensions

- A discussion of vectors in three dimensional space probably requires more time than is being allocated here. The students do not have a good three dimensional intuition, and this must be built up slowly and steadily. A good preliminary is some practice graphing coordinate axes and points in 3D space, along with lines, planes, spheres etc. (**)
- Vector equations should start out with lines and planes, not curves. This should be connected to the linear algebra topics of the course. (**)
- The cross (vector) product is an important topic which also takes some time to introduce properly.
- Vector equations of curves might include the circle, helix and a parabolic trajectory, but this should follow the linear examples.
- We suggest that the vector calculus material be confined (largely at least) to the plane. An interesting example is the cycloid, again of considerable historical and practical interest. (**)

Topic 2: Matrices and Systems of Equations:

- Students should first get plenty of practise with the nuts and bolts of making repeated substitutions in order to simplify a linear system before they move on to matrix inverses or augmented matrices and row-reduction. These other techniques are can wait the more serious context of a full course on linear algebra at the University level. (***)

Topic 3: Complex numbers:

- Getting to DeMoivre's theorem is not necessary. The need for polar forms is also debatable.

Topic 4: Functions and Calculus:

- What is meant by “find the inverse function of a one-to-one function”?? This is generally impossible in any other than a numerical way. Why are inverses introduced here after the inverse of matrices? (**)
- Sketching graphs of simple rational functions is very useful! (**)
- The primary integration techniques are 1) substitution 2) integration by parts. These should be introduced before trigonometric identities. (**)
- As for applications of the integral calculus, how about solving Newton's laws of motion? This was for several hundred years the main application! In particular simple harmonic motion and parabolic trajectories ought to be discussed. (***)
- A good preliminary to some of the Statistics material is some work on calculating the mass of a density distribution, or its center of mass (in one variable only). (**)

- Numerical integration using technology: does this include the trapezoidal rule? Hopefully it does. (**)

OVERALL SUMMARY OF SM Unit 3: The subject is too full, and parts of it are too advanced. More time is needed on three-dimensional intuition when developing vectors, reduce the amount of matrix material and reduce the complex numbers section. For Calculus, the applications should be foremost in mind.

Unit 4

Topic 1: Further Calculus and Applications of Calculus

- Integration by parts is out of order, it should come earlier. The applications here are mostly differential equations. It would be nice to see some area, volume or other geometric applications, mechanics or Newton's law of cooling. (**)
- The modelling motion section is presumably code for Newton's laws, in which case acceleration ought to figure prominently. (***)

Topic 2: Statistical Inference for Continuous Data

- Key results that were introduced in MM but not proved, should be proved in SM. Conspicuous examples are the normal approximation to the binomial, and the derivation of the mean and variance of the binomial. If CLT is to be taught in high school, our best students probably should see a proof of it, at least in the special case of the binomial. (***)
- The involved description of topics here suggest that this section needs more work. What are the main points? What kinds of problems will students be asked to solve? (**)
- This section seems easier than the corresponding Statistics section of the MM subject.
- There is undue emphasis on methods at the expense of ideas. Two new inferential ideas are introduced here: (1) hypothesis testing by reducing data to signs then using discrete probability results (2) using permutation tests. Either of these two tools can be used in a range of settings, however, they are presented separately as a “cookbook” of specialised solutions to specialised problems (sign test for paired data, permutation tests for regression) when in reality either method could be used for either problem. Meanwhile, the equally important problem of comparing two independent samples is ignored (which, incidentally, can be considered a special case of regression). Students would develop a deeper understanding of either of these approaches if applied across a range of different settings when studying the relationship between variables - using a paired sample, two independent samples, or regression. (**)

OVERALL SUMMARY OF SM Unit 4: This Unit needs rethinking.

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