MATHEMATICS ENRICHMENT CLUB.¹
Problem Sheet 3, May 14, 2012

1. The perimeter of a base of a rectangular brick with integer sides is 18 cm, whilst its
volume is 42 cm³. What is its height?

2. Calculate
\[
(1 - \frac{1}{2}) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \ldots \left(1 - \frac{1}{2008}\right).
\]

3. Find the smallest positive integer whose square ends in (a) 09 and (b) 9009.

4. Show that if \(a, b\) are positive numbers such that \(ab \leq 1\) then
\[
\frac{a}{b+1} + \frac{b}{a+1} + (1-a)(1-b) \leq 1.
\]

5. Suppose we have the numbers \(x_0 = 0, x_1 = 1\) and \(x_{n+1} = x_n + 2x_{n-1}\) for \(n \geq 2\).
   a. Write down the numbers \(x_n\) for \(n = 2, 3, 4, 5, 6\).
   b. Show that there is no \(n\) for which \(x_n = 1999\). (Hint: Use modulo 8 arithmetic).
   c. Show that \(x_n = \frac{2^n - (-1)^n}{3}\) satisfies the equation.

6. In \(\triangle ABC\), extend the sides \(AB\) and \(AC\) and draw a circle outside the triangle which
touches \(BC\) and these two produced sides. This circle is called the escribed circle of the
triangle.
   a. Show that \(r_1 = \frac{A}{s-a}\), where \(r_1\) is the radius of the escribed circle, \(A\) is the area of
   \(\triangle ABC\), \(a\) is the length of \(BC\) and \(s\) is half the perimeter of \(\triangle ABC\).
   b. Show that \(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}\), where \(r_2, r_3\) are the radii of the other two escribed
   circles and \(r\) is the radius of the incircle (recall last week’s result.)

7. \(ABCD\) is a parallelogram, \(Q\) a point inside it. Prove that the sum of the areas of
\(AQB\) and \(CQD\) is half the area of \(ABCD\).

¹Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.
Senior Questions.

1. Find \( \int_0^1 \frac{1}{1 + t + t^2} \).

2. Find the limit \( \lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3} \).

3. A hand of eight cards is dealt from a standard pack. How many hands contain exactly three cards of the same value and the remaining cards from the remaining suit?