MATHEMATICS ENRICHMENT CLUB.¹
Problem Sheet 6, June 4, 2012

1. A parallelogram \( ABCD \) has \( BC = 4 \) cm and \( CD = 8 \) cm. The point \( A \) is 3 cm above \( CD \). Find the length of the perpendicular from \( A \) to \( BC \).

2. If \( a, b, c \) are real numbers and \( a > b \), which of the following must be true?

   \[
   \begin{align*}
   (a) \quad \frac{1}{a} &> \frac{1}{b} \\
   (b) \quad ac &> bc \\
   (c) \quad a^2 &> b^2 \\
   (d) \quad a + c &> b + c \\
   (e) \quad \frac{1}{a} &< \frac{1}{b}.
   \end{align*}
   \]

3. (a) Verify that \( x = 170, y = 39 \) satisfy \( x^2 = 19y^2 + 1 \).

   (b) Hence find integers \( x \) and \( y \) such that \( x^2 = 171y^2 + 1 \) and \( x^2 = 3211y^2 + 1 \).

4. A rectangle has perimeter 20 cm. What is the least value of the diagonal?

5. From the point \( (x, y) \) we can move a counter to any one of the following points:

   \( (2x, y), (x, 2y) \)

   or

   \( (x - y, y) \) if \( x > y \), \( (x, y - x) \) if \( y > x \).

   Starting from \((1, 1)\) can you see a rule to determine which points in the plane can be reached using the rules above?

6. The line joining a vertex of a triangle to the midpoint of the opposite side is called a median. Let \( m_A \) denote the median in triangle \( ABC \) from \( A \) to \( BC \).

   (a) Show that \( AB + AC > 2m_A \). (Hint: Think about parallelograms)

   (b) Deduce that \( AB + AC + BC > m_A + m_B + m_C \).

7. Given a circle \( K \) with centre \( O \) and diameter \( AB \), let \( C \) be any point on \( K \).

   (a) Prove that \( \angle ACB = 90^\circ \).

   (b) Describe how to construct a right-angled triangle \( ACB \) if we are given its hypotenuse \( AB \) and the length of the perpendicular dropped from \( C \) to \( AB \).

¹Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.
Senior Questions.

1. Let \( S(x) = \frac{e^x - e^{-x}}{2} \) and \( C(x) = \frac{e^x + e^{-x}}{2} \).

   (a) Show that \( (C(x))^2 - (S(x))^2 = 1 \).

   (b) If \( S(x) = \tan \theta \), express \( C(x) \) in terms of \( \theta \).

2. Find the integral

   \[
   \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^4 \theta}{\sin^2 \theta} \, d\theta.
   \]

3. A die is thrown \( n \) times. Show that if the probability that a 6 appears at least once is greater than \( \frac{1}{2} \), then \( n > \frac{\log 2}{\log 6 - \log 5} \).