Problem Sheet 16, September 10, 2012

1. A box of apples costs $4, a box of oranges costs $3 and a box of lemons costs $2. A person buys 8 boxes of fruit at a cost of $23. If at least one box of each kind of fruit is bought, find the largest possible number of boxes of apples.

2. Suppose $a, b$ are positive real numbers. Use the diagram below to give a geometric proof that $a^2 = (a - b)^2 + 2ab - b^2$.

![Figure 1: Two squares of length $a$ and $b$](image)

3. The perimeter of a rectangle is 20 cm what is the least value of the diagonal?

4. Consider the two sequences $x_0 = 1, x_1 = 1, x_{n+1} = x_n + 2x_{n-1}$ and $y_n = 8n + 1$. Prove that for $n > 1$ these two sequences never have a common term.

5. Use the fact that $a^2 + b^2 \geq 2ab$ for any positive real numbers $a, b$ to show that, for $a, b, c$ positive real numbers, $\frac{a^2 + b^2 + c^2}{3} \geq \left(\frac{a + b + c}{3}\right)^2$.

6. (a) Paul measured all 6 edges of a tetrahedron $ABCD$ and found them to be 1, 3, 4, 5, 6, 8 cm. Can this be correct?

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1Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.
(b) Paul then measured the edges to be 2, 3, 4, 5, 6, 8. If $AB = 2$ what is the length of $CD$?

7. A circle is drawn which touches $BC$ in triangle $ABC$ and also touches the two sides $AB$ and $AC$ produced at $T$ and $S$ respectively. Let $O$ be the centre of this circle.

(a) Explain why $OB$ bisects the angle $TBC$.
(b) Prove that the length of $AT$ equals half the perimeter of the triangle $ABC$.

0.1 Senior Questions.

1. Prove that
\[ \cos((n + 2)\theta) = 2\cos((n + 1)\theta) \cos \theta - \cos(n\theta), \]
for each integer $n \geq 0$.
Hence express $\cos 5\theta$ in terms of powers of $\cos \theta$.

2. For every positive real number $n > 1$, prove that
\[ 2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}. \]

3. Use the result in Q2 to prove that
\[ 2(\sqrt{N+1} - 1) < \sum_{n=1}^{N} \frac{1}{\sqrt{n}} < 2\sqrt{N} \]
and deduce that the sum of the first million terms of
\[ \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots \]
is between 1998 and 2000.