MATHEMATICS ENRICHMENT CLUB.\textsuperscript{1}
Problem Sheet 7, June 18, 2013

1. (a) Let $M$ be the midpoint of the side $AB$ in the triangle $ABC$. If $CM$ has length $h$, prove that

$$2(a^2 + b^2) = c^2 + 4h^2.$$ 

This is known as Apollonius’ theorem.

(b) Show how to draw a triangle knowing only the lengths of the three medians $h$, $k$ and $\ell$. (You can either use (i), or find a better way.)

2. Two circles $C_1, C_2$ with centres $O_1, O_2$ are externally tangent at the point $P$. A straight line through $P$ meets $C_1, C_2$ respectively at $A$ and $B$. Show that the tangents to the circles at $A$ and $B$ are parallel.

3. Find the last two digits (and then the last three digits) of $1! + 2! + 3! + \ldots + 99!$.

4. Denote the top of a cube by $ABCD$ and the bottom by $A_1, B_1, C_1, D_1$, so that $A$ is directly above $A_1$ and so on. Take midpoints of the six edges $AB, BB_1, B_1C_1, C_1D_1, D_1D$ and $DA$. Show that a plane containing any three of these points contains them all and deduce that these points form the vertices of a regular hexagon.

5. A quadrilateral in which a circle can be drawn which touches each of the four faces is called a circumscribable quadrilateral. If $r$ is the radius of the circle and $s$ is half the perimeter of the quadrilateral, prove that the area of the quadrilateral is $rs$.

6. What is the smaller angle between the hands of the clock at 12:25pm?

Senior Questions

1. Solve the equation $\cot^{-1} x - \cot^{-1}(x + 2) = \frac{\pi}{12}$.

2. If $x$ is a number between 4 and 8 and $y$ is a number between 20 and 40, what are the smallest and largest possible values of $\frac{y}{x}$?

\textsuperscript{1}Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.