MATHEMATICS ENRICHMENT CLUB\(^1\)
Problem Sheet 10, July 30, 2013

1. Simplify \((x^{-1} + y^{-1})^{-1}\).

2. What is the least positive integer \(n\) such that \(60 \times n\) is a cube?

3. Show that the number 13950264876 is not a square by thinking about divisibility by 3.

4. The angles in a triangle are in the ratio 2 : 3 : 4. Find, in degrees, the size of the largest angle.

5. Suppose the median from the vertex \(C\) of a triangle \(ABC\) has length \(\frac{1}{2} AB\). Show that the triangle is right-angled at \(C\).

6. (a) Find the greatest common divisor of \(2^{50} + 1\) and \(2^{20} + 1\).

   (b) Explain why the greatest common divisor of \(2^m + 1\) and \(2^n + 1\) is at least 3 if \(m\) and \(n\) are both odd.

**Senior Questions**

1. Gabriel's Horn is constructed by rotating the graph \(y = \frac{1}{x}, ~ x \geq 1\) about the \(x\)-axis.

   (a) Prove that Gabriel’s Horn is infinite in surface area.

   (b) What is the volume of Gabriel’s Horn?

2. Prove that

   \[
   \cos((n + 2)\theta) = 2 \cos((n + 1)\theta) \cos \theta - \cos(n\theta),
   \]

   for each integer \(n \geq 0\). Hence express \(\cos 5\theta\) in terms of powers of \(\cos \theta\).

\(^1\)Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.