MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 12, August 12, 2014[1]

1. (a) At the end of 30! there are 7 zeros. What is the digit to the left of these 7 zeros?

    (b) How many zeros does 1000! have at the end?

2. (a) Start with an equilateral triangle coloured red on a white background. Divide it into 4 congruent equilateral triangles and erase the red inside the centre triangle. How many equilateral triangles are there using only lines which separate red from white?

    (b) Do the same for each of the remaining red triangles. Again, how many equilateral triangles are there using only lines which separate the red from white?

    (c) Suppose you've done this process n times. How many equilateral triangles are there using only lines which separate red from white?

    (d) Suppose you've done this process n times. What is the ratio of the new red area to the original red area?

3. The positive integer d is not 2, 5 or 13. How many values of d are there such that, for *any* pair, (a, b), of numbers from $\{2, 5, 13, d\}$, $ab - 1$ is a perfect square?

4. How many ordered pairs, (a, b), are there of positive integers a and b between 1 and 999 (inclusive) such that

$$(a + 36b)(b + 36a)$$

is an integral power of 2?

5. Let A_1, A_2, ..., A_{2n} $(n \geq 2)$ be $2n$ ordered points equally spaced around a circle such that $A_1 A_2 \ldots A_{2n}$ is a regular polygon.

   Any 3 points form a triangle while only some sets of 4 points form rectangles. Given that there are 20 times more triangles than rectangles find n.

6. The segments that connect the midpoints of opposite sides of a convex quadrilateral $ABCD$ divide it into 4 quadrilaterals of the same perimeter. Prove that $ABCD$ is a parallelogram.

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[1]Some problems from UNSW's publication *Parabola*, others from www.brilliant.org

7. (**Just for fun**) Have a think about the following paradoxes involving infinities:

   (a) Hilbert has a Hotel with infinitely many rooms and all rooms are currently occupied. A weary traveller arrives and Hilbert, after hearing the tough tales of the traveller, would like to provide her with a room for the night. Can Hilbert move some customers around to accomodate one extra guest (without asking anyone to share a room)? What if infinitely many weary travellers arrive?

   (b) A non-negative integer is chosen at random, what is the probability that it is a perfect square?

   (c) Zeno steals your wallet and runs 10 metres away before you realise. At this point he turns and yells to you "you'll never catch me because by the time you get to where I am now, I'll be somewhere else!" Can you catch Zeno, get your wallet back and report him to the authorities?

   (d) A small town has one restaurant, *L'échec du Russell*, with one chef, Russell. When the townsfolk don't feel like making their own dinner they dine at *L'échec*. Thus, Chef Russell cooks dinner for anyone who doesn't cook dinner for themself. Who cooks Chef Russell's dinner?

Senior Questions

1. Let a_n be the Fibonacci sequence, i.e. $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$. Show that $a_n = \sum_{j=0}^{n} \binom{n-j}{j}$ where we let $\binom{n}{k} = 0$ if $k > n$.

2. It is known that for the Fibonacci sequence, a_n, as defined in question 1, that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2} \approx 1.62$. Show that it doesn't matter what we choose $0 < a_0 \leq a_1$ to be, the previous limit holds.