MATHEMATICS ENRICHMENT CLUB.¹
Solution Sheet 10, July 30, 2013

1.

\[
\begin{align*}
(x^{-1} + y^{-1})^{-1} &= \frac{1}{x^{-1} + y^{-1}} \\
&= \frac{1}{\frac{1}{x} + \frac{1}{y}} \\
&= \frac{xy}{x + y}.
\end{align*}
\]

2. For a number to be a cube it’s prime factorisation must contain only cubes. The prime factorisation of 60 = $2^2 \times 3 \times 5$ so $n = 2 \times 3^2 \times 5^2 = 450$.

3. Using long division we can see that 12 950 264 876 is divisible by 3 but that $4 650 088 292 = \frac{12 950 264 876}{3}$ is not. Thus the prime factorisation of 12 950 264 876 contains a 3 which is not squared, so cannot be a square.

4. There is a multiplier $\alpha$ such that the angles of our triangle are $2\alpha$, $3\alpha$ and $4\alpha$. Using the angle sum $2\alpha + 3\alpha + 4\alpha = 180 \implies \alpha = 20$. So the angles are 40, 60 and 80.

5. Let the median from $C$ to $AB$ meet $AB$ at $D$. Since $DC$ has length $\frac{1}{2}AB$, both triangles $ADC$ and $BDC$ are isosceles with $AD = DC$ and $DC = DB$. So $\angle DCA = \angle DAC = \alpha$, $\angle DCB = \angle DBC = \beta$ and $\angle DAC + \angle DBC + \angle ACB = \alpha + \beta + (\alpha + \beta) = 180 \implies \alpha + \beta = 90$. Since $\angle ACB = \alpha + \beta = 90$.

6. (a) Recall that $\text{gcd}(a + mb, b) = \text{gcd}(a, b)$. So if we have $\text{gcd}(m, n)$ with $m > n$ and we divide $m$ by $n$ to get a remainder $r$, then $\text{gcd}(m, n) = \text{gcd}(n, r)$. So divide $2^{50} + 1$ by $2^{20} + 1$ and we get

\[
2^{50} + 1 = (2^{20} + 1)(2^{30} - 2^{10}) + 2^{10} + 1
\]

so

\[
\text{gcd}(2^{50} + 1, 2^{20} + 1) = \text{gcd}(2^{20} + 1, 2^{10} + 1).
\]

¹Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.
Now we divide \(2^{20} + 1\) by \(2^{10} + 1\), and continue dividing the larger by the smaller
and replacing the larger with the remainder, so it goes
\[
gcd(2^{20} + 1, 2^{10} + 1) = gcd(2^{10} + 1, 2) \\
= gcd(2, 1) \\
= gcd(1, 0) \\
= 1.
\]

(b) Note that for odd powers \(n\), the remainder when dividing \(2^n\) by 3 is 2, whereas
for even powers \(n\) the remainder is 1. The remainder when dividing 1 by 3 is 1.
Since the sum of the remainders of \(2^n/3\) for odd \(n\), and 1/3 is 3, then 3 divides
\(2^n + 1\) for odd \(n\). So the greatest common divisor of \(2^n + 1\) and \(2^m + 1\) for odd \(n\)
and \(m\) must be at least 3.

**Senior Questions**

1. (a) The surface area of a surface of revolution constructed by rotating the graph
\(y = f(x)\) about the \(x\)-axis for \(a \leq x \leq b\) is given by
\[
A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2} \, dx.
\]
Since Gabriel’s horn is infinitely long we actually mean
\[
A = \lim_{n \to \infty} 2\pi \int_1^n \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx.
\]
Note that \(\frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \geq \frac{1}{2} > 0\) for \(x \geq 1\) so
\[
\int_1^n \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx > \int_1^n \frac{1}{x} \, dx = \ln n.
\]
Since \(\ln n \to \infty\) as \(n \to \infty\) so does \(\int_1^n \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx\) and \(A\) is infinitely large.

(b) The volume of a solid of revolution constructed by rotating \(y = f(x)\), \(a \leq x \leq b\)
about the \(x\)-axis is given by
\[
V = \pi \int_a^b f(x)^2 \, dx.
\]
For Gabriel’s horn then
\[
V = \pi \lim_{n \to \infty} \int_1^n \frac{1}{x^2} \, dx \\
= \pi \lim_{n \to \infty} \left[-\frac{1}{x}\right]_1^n \\
= \pi \lim_{n \to \infty} \left(1 - \frac{1}{n}\right) \\
= \pi.
\]
So if Gabriel wanted to paint the infinite surface area of the inside of his infinitely
long horn he’d need at most \(\pi\) units of paint. Wait, what?
2. Using various angle expansions obtain

\[
\cos((n + 2)\theta) = \cos((n + 1)\theta) \cos \theta - \sin((n + 1)\theta) \sin \theta \\
= \cos((n + 1)\theta) \cos \theta - \sin(\theta) (\cos(n\theta) \cos \theta + \cos(n\theta) \sin \theta) \\
= \cos((n + 1)\theta) \cos \theta - \sin(n\theta) \sin \theta \cos \theta - \cos(n\theta) \sin^2(\theta) \\
= \cos((n + 1)\theta) \cos \theta - \frac{1}{2} \sin(n\theta) \sin(2\theta) - \cos(n\theta) \sin^2 \theta.
\]

Now note

\[
\sin(n\theta) \sin(2\theta) = \cos(n\theta) \cos(2\theta) - \cos(n\theta + 2\theta) \\
= \cos(n\theta)(2 \cos^2 \theta - 1) - \cos((n + 2)\theta).
\]

So

\[
\cos((n + 2)\theta) = \cos((n + 1)\theta) \cos \theta - \frac{1}{2} (\cos(n\theta)(2 \cos^2 \theta - 1) - \cos((n + 2)\theta)) - \cos(n\theta) \sin^2 \theta \\
= \cos((n + 1)\theta) \cos \theta - \cos(n\theta) \cos^2 \theta + \frac{1}{2} \cos(n\theta) + \frac{1}{2} \cos((n + 2)\theta) - \cos(n\theta) \sin^2 \theta
\]

\[
\frac{1}{2} \cos((n + 2)\theta) = \cos((n + 1)\theta) \cos \theta + \frac{1}{2} \cos(n\theta) - \cos(n\theta)(\cos^2 \theta + \sin^2 \theta) \\
= \cos((n + 1)\theta) \cos \theta - \frac{1}{2} \cos(n\theta) \\
\cos((n + 2)\theta) = 2 \cos((n + 1)\theta) \cos \theta - \cos(n\theta).
\]

Iteratively expanding \(\cos 5\theta\) we obtain

\[
\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.
\]