1. (a) \[ \begin{align*}
0 &\leq (a-b)^2 \quad \text{with equality only if } a = b \\
0 &\leq a^2 + b^2 - 2ab \\
ab &\leq \frac{a^2 + b^2}{2}
\end{align*} \]

so \( ab \) is largest when \( a = b \), and since \( a + b = k \) then at \( a = b = \frac{k}{2} \).

(b) From above, first note that \( xy \leq \frac{c^2}{2} \), then

\[ \begin{align*}
c^4 &= (x^2 + y^2)^2 \\
c^4 &= x^4 + y^4 + 2x^2y^2 \\
x^4 + y^4 &= c^4 - 2x^2y^2,
\end{align*} \]

which is minimum when \( x^2y^2 \) is maximum, which from above is when \( x = y \) and has a value of \( \left( \frac{c^2}{2} \right)^2 \). So the minimum value of \( x^4 + y^4 = c^4 - \frac{c^4}{2} = \frac{c^4}{2} \).

2. Construct the triangles \( APB \) and \( AQB \). Let \( P' \) be at the intersection of the circle and the line \( AP \), now since \( AB \) is a diameter and \( P' \) on the circle, triangle \( AP'B \) is right at \( P' \), which also means triangle \( PP'B \) is right at \( P' \), and so \( \angle APB = \angle P'PB < 90^\circ \).

Similarly extend \( AQ \) to the circle and call the intersection \( Q' \), then \( \angle AQ'B \) is right, which implies \( \angle Q'QB < 90^\circ \). Since \( \angle Q'QB \) is an external angle of triangle \( AQB \) we have \( \angle Q'QB = \angle QAB + \angle QBA \), and so \( \angle QAB + \angle QBA < 90^\circ \) and so \( \angle AQB > 90^\circ \).

3. (a) Suppose a quadratic is factorised with roots \( \alpha \) and \( \beta \), then it is

\[ (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta. \]

Since \( 2343643 \) is odd, one of \( \alpha \) or \( \beta \) must be odd and the other even, but the product of an odd and even number is even, and so cannot be \( 2382987 \). Hence there are no integer solutions.

\[ ^1 \text{Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.} \]
(b) By similar logic, both $\alpha$ and $\beta$ must be even (to have even sum and even product). If they are both even, then the product $\alpha \beta$ must be divisible by 4 (write $\alpha = 2m$ \linebreak $\beta = 2n$ then $\alpha \beta = 4mn$), but 2382982 is not, and hence there are no integer solutions.

4. To make $10 out of $n$ 50c coins and $m$ 20c coins we must satisfy
\[ 5n + 2m = 100, \quad n, m \in \mathbb{Z}, \; n, m > 0 \]

or
\[ m = 100 - \frac{5n}{2}, \quad m, n \in \mathbb{Z}, \; n, m > 0. \]

So we merely count the number of $n$ which are divisible by 2 and satisfy the above, of which there are 9.

5. (a) In general, if we prime factorise $x = p_1^{m_1}p_2^{m_2} \cdots p_k^{m_k}$ then every factor can be written as $p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$ where each $a_1 = 0, 1, \ldots, m_i$. So there are $m_1 + 1$ choices for $a_1$, $m_2 + 1$ choices for $a_2$ and so on, and hence the number of factors is $(m_1 + 1)(m_2 + 2) \cdots (m_k + 1)$. Then $20 = 2^2 \times 5$ and so has $3 \times 2 = 6$ factors, so $\tau(20) = 6$.

If $n = p_1^{m_1} \cdots p_k^{m_k}$, then $n^2 = p_1^{2m_1} \cdots p_k^{2m_k}$ and so $\tau(n^2) = (2m_1 + 1)(2m_2 + 1) \cdots (2m_k + 1)$ which is a product of odd numbers and hence odd and so cannot be equal to the even number $2\tau(n)$.

(b) The number $144^2 = (3 \times 2^2)^4 = 3^4 \times 2^8$ so $\tau(144^2) = 5 \times 9 = 45$ and $\tau(144) = \tau((3 \times 2^2)^2) = \tau(3^2 \times 2^4) = 3 \times 5 = 15$ and $3 \times 15 = 45$.

6. Let \[ \sqrt[3]{x + \sqrt{x^2 + 1}} + \sqrt[3]{x - \sqrt{x^2 + 1}} = a + b = y, \] and recall \[ y^3 = (a + b)^3 = a^3 + b^3 + 3ab(a + b). \]

Now
\[ a^3 + b^3 = 2x \]
\[ ab = \sqrt[3]{x^2 - (x^2 + 1)} \]
\[ = \sqrt[3]{1} = 1. \]

so
\[ y^3 = 2x + 3 \times 1 \times y \]
\[ x = \frac{3y - y^3}{2}, \quad \text{where } y \in \mathbb{Z}. \]

Senior Questions

1. Expand the right hand side:
\[ (n^2 - (3n + 1))^2 - 25n^2 = n^4 - 2n^2(3n + 1) + (3n + 1)^2 - 25n^2 \]
\[ = n^4 - 6n^3 - 2n^2 + 9n^2 + 6n + 1 - 25n^2 \]
\[ = n^4 - 6n^3 - 18n^2 + 6n + 1. \]
Now if \( n^4 - 6n^3 - 18n^2 + 6n + 1 \) is prime its only factors are itself and 1. Since
\[
n^4 - 6n^3 - 18n^2 + 6n + 1 = (n^2 - 3n - 1 - 5n)(n^2 - 3n - 1 + 5n) = (n^2 - 8n - 1)(n^2 + 2n - 1)
\]
we must have one of the latter factors equal to 1. So first consider
\[
n^2 - 8n - 2 = 0
\]
but \((-8)^2 - 4 \times 1 \times (-2) = 72\) which is not a square, so this one has no integer solutions.
Also
\[
n^2 + 2n - 2 = 0
\]
has no integer solutions. So there are no integer \( n \) for which \( n^4 - 6n^3 - 18n^2 + 6n + 1 \) is prime.

2. It doesn’t matter which order the players win their games in, just the total number of games won or lost. If one player wins \( n \) games, the probability they won them all by luck alone is
\[
\binom{100}{n} \left( \frac{1}{2} \right)^n \left( \frac{1}{2} \right)^{100-n} = \frac{1}{2^{100}} \binom{100}{n}.
\]
So we must find \( N \) such that \( \sum_{n=1}^{N} \frac{1}{2^{100}} \binom{100}{n} = 0.05 \). This is prohibitively hard, but we can approximate the binomial distribution with a normal distribution with mean 50 and variance 25 (or standard deviation of 5). In a normal distribution 95% of values lie within 2 standard deviations of the mean, which means our players must win more than 60 of the 100 games to demonstrate that it is unlikely they won by luck alone.