MATHEMATICS ENRICHMENT CLUB.¹
Solution Sheet 17, September 17, 2013

1. There are 5 odd integer digits to choose from. Once one is chosen for the first digit, only 4 remain, then 3. So there are $5 \times 4 \times 3 = 60$ 3 digit numbers with distinct odd digits.

2. Squaring palindromic numbers is a good way to find new palindromic squares, provided that the digits are small enough that there’s no need to do any “carrying” when we do the multiplication. So $121^2 = 14641$, $22^2 = 484$. It turns out there are no 4-digit palindromic squares.

3. Let the isosceles triangle be $ABC$ with base $BC$. The square is bisected by the altitude of the triangle through $A$, which meets $BC$ at $D$. Let $E$ be the vertex of the square on $BC$ between $B$ and $D$ and let $F$ be the vertex of the square above $E$. Then triangles $ABD$ and $FBE$ are similar, so, letting the side length of the square be $x$, we get the relation

$$\frac{x}{\sqrt{10^2 - 6^2}} = \frac{6 - \frac{x}{2}}{6}$$

the solution of which is

$$x = 4.8.$$

4. We wish to find $n$, such that for some $q_1, q_2, q_3$ and $r$ we have

$$364 = nq_1 + r, \quad 414 = nq_2 + r \quad \text{and} \quad 539 = nq_3 + r.$$

Combining the first two means

$$(q_2 - q_1)n = 414 - 364 = 50.$$

Since $n$ and all the $q$’s are integers, $n$ must be a factor of 50, which are 50, 25, 10, 5, 2 or 1. Dividing 364 or 414 by 50 gives a remainder of 14, whilst dividing 539 by 50 gives a remainder of 39, so $n$ is not 50. Dividing 364 or 414 by 25 still gives a remainder of 14, and so does dividing 539 by 25. So $n = 25$ works, and since it is larger than the other factors of 50 it is our answer.

¹Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.
5. (a) \( a_6 = 6 + 5 + 4 + 3 + 2 = 20 \)

(b) \( a_n \) is simply the sum of integers from 2 to \( n \), which is an arithmetic series, so 
\[ a_n = \frac{n-1}{2}(n + 2). \]

(c) \( b_6 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 = 90 \)

(d) Some may recognize that \( b_n \) is the \( n \)th square pyramidal number (en.wikipedia.org/wiki/Square_pyramidal_number) minus 1. The formula for the \( n \)th square pyramidal number is 
\( \frac{n}{6}(n+1)(2n+1) \), so 
\[ b_n = \frac{n}{6}(n+1)(2n+1) - 1. \]

6. (a) \( ABCB_1 \) is a parallelogram since \( BC \) is parallel to \( AB_1 \) and \( CB_1 \) is parallel to \( AB \). Similarly \( CBC_1A \) is a parallelogram. So now we know that \( A \) is the midpoint of \( B_1C_1 \).

Now \( \angle B_1AC = \angle ACB \) because they are alternate. If \( D \) is the point at which the altitude from \( A \) meets \( BC \) then \( \angle DAC = 90 - \angle ACD = 90 - \angle ACB \) so 
\( \angle DAC + \angle B_1AC = 90 \), and \( AD \) is the perpendicular bisector of \( B_1C_1 \).

(b) Since all the altitudes are also perpendicular bisectors of a triangle, and perpendicular bisectors of a triangle are concurrent, these altitudes are also.

7. \( P \) must be on the opposite side of the chord \( AB \) from \( O \) otherwise the angle with be zero. Instead, let the angle at \( P \) be \( \theta \), then the angle at \( O \) is \( 180 - 2\theta \). Setting these equal gives \( \theta = 180/3 = 60^\circ \).