

MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 3, May 15, 2016

1. Since 3 and 5 are co-prime, there is a unique solution to $3x + 5y = 18$; $x = 1, y = 6$. If we consider the number of solutions to $3x + 5y = 18 + 15$, then there is only one more compare to the number of solutions to $3x + 5y = 18$, because there is only two ways to choose between 3's or 5's to make the 15 we have added. Since there are 66 blocks of 15's between $18 - 1008$, we conclude that the number of solution to $3x + 5y = 1008$ is 67.
2. This is similar to the stairs problem we did in week one (how many ways to get up 10 stairs if we can either take 1 step or 2 steps at a time); Taking 1 step is analogous buying a drink for \$1, and taking 2 steps is analogous to buying a drink for \$2. The main different for the present problem is the choice between orange or pineapple juice for \$2.

To demonstrate how we handle this, consider the case where John buys 3 milks and 2 orange/pineapple juices to spend the \$7 he had. Treat each drink as an object, then the number of ways arrange these objects is $5! := 5 \times 4 \times 3 \times 2 \times 1$. However, we can not distinguish between the $3!$ ways the \$1 items are arranged, or the $2!$ ways the \$2 items are arranged. Hence, we remove them to give $5! \div 3! \div 2! = 10$ different ways to arrange the $3 \times \$1$ and $2 \times \$2$ drinks. Finally, there are a 2^2 ways we can pick orange and pineapple juice to make up $2 \times \$2$ drinks. Hence, for this case the number of ways to do this is $10 \times 4 = 40$.

Repeat with other cases should give 85 as the total number of ways John can spent his \$7 on drinks.

3. Line up 40 lots of the number "1"; that is

$$\underbrace{111 \dots 111}_{40 \text{ lots}}$$

Then we can think of this problem as partitioning the list of number 1's into 4 blocks (allowing some block(s) to be empty); The sum of the numbers in each block will be the value of either x_1, x_2, x_3 or x_4 , and the sum of all blocks will always be 40 because we have a list of 40 lots of "1", i.e $x_1 + x_2 + x_3 + x_4 = 40$.

To find the number of ways we can partition the 40 objects into 4 blocks, one can think of it as the number of ways to arrange 43 objects (with 3 walls that separates the 40

objects into blocks). The total number of ways we can do this is $43!$, but we don't care about how the objects are arranged because they are all 1's, nor do we care about how the walls are arranged. Hence, the number of solutions is $43! \div 40! \div 3! = 12341$.

4. This is a straightforward application of integration by parts https://en.wikipedia.org/wiki/Integration_by_parts. The solution is $5! = 120$.
5. Let r_1, r_2 be the remainder when the integers x_1, x_2 is divided by some integer d respectively. One can show that the remainder of $x_1 + x_2$ divided by d is $r_1 + r_2$, and the remainder of $x_1 \times x_2$ is $r_1 \times r_2$ (write x_1, x_2 in quotient and remainder form, then compute $x_1 + x_2 / x_1 \times x_2$).

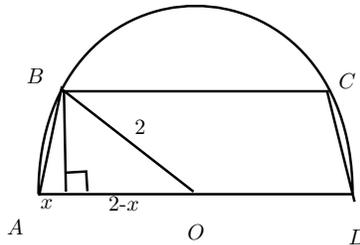
The remainder of 1^n divided by 7 is always 1.

The remainder of 2^1 divided by 7 is 2. Hence, the remainder of $2^2 = 2 \times 2$ divided by 7 is $2 \times 2 = 4$. Similarly, the remainder of $2^3 = 2^2 \times 2$ divided by 7 is $4 \times 2 = 8$, which is the same as 1. Finally, the remainder of $2^4 = 2^3 \times 2$ divided by 7 is $1 \times 2 = 2$. Since the remainder of 2^4 divided by 7 is the same as 2^1 , if we keep increasing the power n in 2^n we would expect the remainders to be 2, 4, 1 repeating, because we are multiplying the remainder by the same factor of 2 each time.

Analogously, the remainders of 4^n divided by 7 are 4, 2, 1 repeating.

Therefore, the remainder of $1^n + 2^n + 4^n$ divided by 7 is 1 + 2 + 4, 1 + 4 + 2, 1 + 1 + 1 repeating, which is the same as 0, 0, 3 repeating. Thus, we can argue that $2/3$ of 999 the numbers between $0 < n < 1000$ makes $1^n + 2^n + 4^n$ divisible by 7.

6. Draw the line BO , where O is the origin of the circle; see below. From the figure, we can easily find x using Pythagoras on the 2 right angled triangles. The length of BC is twice that of $x - 2$.



Senior Questions

- Using the supplied Fermat's theorem, one has a unique pair of integers x and y , such that $a^2 + b^2 = (x^2 + y^2)^2 = x^4 + y^4 + 2x^2y^2$. So $a = (x^2 - y^2)$ and $b = 2x^2y^2$. Since x, y are unique, so are a, b .
- We only need to prove that $a \geq b$ since we will then have $b \geq a$ by symmetry. Circle the largest number x_j in column j , $1 \leq j \leq n$, where n is the number of rows and therefore of columns. By relabelling if necessary, we may assume that x_1 is the smallest of these n numbers. We consider two cases.

Case 1. Two circled numbers x_j and x_k are on the same row. Then $a \geq x_j + x_k \geq 2x_1 \geq b$ since the sum of the largest two numbers in column 1 is b .

Case 2. Each circled number is in a different row. Let the second largest number y_1 in column 1 be in row j , and let the circled number in row j be x_k . Then $b = x_1 + y_1 \leq x_k + y_1 \leq a$ since the sum of the largest two numbers in row k is a .
- Let n be a positive integer and $n = dq + r$ where d, q, r are consecutive positive integer terms in a geometric sequence. Since $r < d$, we can always make the geometric sequence be increasing. We do this and look at three cases.

Case 1: If $q < r < d$, let $a > 1$ be the common ratio such that

$$r = aq, \quad d = ar = a^2q. \quad (1)$$

Then, substituting the above for n , we can write

$$n = a^2q^2 + aq = aq(aq + 1). \quad (2)$$

If a is an integer, we have the product of two integers greater than 1. Suppose a is a rational number which can be written in simplified form as $a = \frac{s}{t}$ for integers $s > 0, t > 1$. Since $d = a^2q = \frac{qs^2}{t^2}$ is an integer, we must have $q = t^2k$ for some positive integer k . In particular, $aq = stk$ and we have that $n = stk(stk + 1)$, the product of 2 (consecutive) integers greater than 1.

Case 2: If $r < q < d$, let $a > 1$ be the common ratio such that

$$q = ar, \quad d = aq = a^2r \quad (3)$$

Then substituting, we can write

$$n = a^3r^2 + r \quad (4)$$

If $r = 1$, then a must be a positive integer and $n = a^3 + 1 = (a + 1)(a^2 - a + 1)$, which cannot be prime. If a is an integer, then $n = r(a^3r + 1)$ which factors for $r > 1$. Now suppose a is a rational number. Let $a = \frac{s}{t}$ for integers $s > 0, t > 1$. Since $d = a^2r = \frac{rs^2}{t^2}$ is an integer, we must have that $r = t^2k$ for some positive integer k . Then we have

$$n = s^3tk^2 + t^2k = tk(s^3k + t). \quad (5)$$

which is the product of 2 positive integers greater than 1.

Case 3: If $r < d < q$, the proof is identical to case 2.