EXACT VALUES OF TRIGONOMETRIC FUNCTIONS

For certain values of $\theta$, the trigonometric functions $\cos \theta$, $\sin \theta$ and $\tan \theta$ have values which are easily expressed, for example, as fractions or surds. You need to know all of the following, without the assistance of a calculator.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos \theta$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\sin \theta$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\sqrt{3}/2$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>0</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>$\sqrt{3}$</td>
<td>undefined</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Comments.

- The value of $\tan \theta$ is undefined when $\theta = \frac{\pi}{2}$. Please **do not** write “$\tan \frac{\pi}{2} = \infty$”: this is nonsense, because $\infty$ is not a number.
- In university level mathematics, the only sensible way to measure angles is in radians. If you are in the habit of saying “$\cos 60^\circ = \frac{1}{2}$”, you need to learn the radian version, otherwise the time you take for trig problems will be hugely increased. And don’t forget that “$\cos 60 = \frac{1}{2}$” is not just inferior, it is **wrong**.

The above table gives (mostly) the cases when $\theta$ is an angle in the first quadrant. To evaluate trigonometric functions for angles in other quadrants you need the following formulae – for more details see the “Trigonometric identities” worksheet:

\[
\begin{align*}
\cos(\theta \pm 2\pi) &= \cos \theta, \quad \cos(\theta \pm \pi) = -\cos \theta, \quad \cos(-\theta) = \cos \theta, \\
\sin(\theta \pm 2\pi) &= \sin \theta, \quad \sin(\theta \pm \pi) = -\sin \theta, \quad \sin(-\theta) = -\sin \theta, \\
\tan(\theta \pm \pi) &= \tan \theta, \quad \tan(-\theta) = -\tan \theta.
\end{align*}
\]

Also frequently useful are

\[
\cos(\pi - \theta) = -\cos \theta, \quad \sin(\pi - \theta) = \sin \theta.
\]

Examples.

- $\cos(-\frac{\pi}{4}) = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.
- $\tan(-\frac{\pi}{3}) = -\tan \frac{\pi}{3} = -\sqrt{3}$.
- $\sin(\frac{5}{6}\pi) = \sin(\pi - \frac{5}{6}\pi) = \sin(\frac{\pi}{6}) = \frac{1}{2}$.
- $\tan(\frac{7}{4}\pi) = \tan(\frac{7}{4}\pi - \pi) = \tan(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.
- To calculate $\sin(\frac{100}{3}\pi)$, we begin by writing the coefficient as a mixed number, \(\frac{100}{3} = 33 + \frac{1}{3}\).

Then we have

\[
\begin{align*}
\sin(\frac{100}{3}\pi) &= \sin(33 + \frac{1}{3})\pi = \sin[(33 + \frac{1}{3})\pi - 34\pi] \\
&= \sin(-\frac{2}{3}\pi) = -\sin(-\frac{2}{3}\pi + \pi) \\
&= -\sin(\frac{1}{3}\pi) = -\frac{\sqrt{3}}{2}.
\end{align*}
\]

Note carefully that in the second step we subtracted an even multiple of $\pi$ from the angle. You will often need to do this kind of calculation when studying complex numbers.
EXERCISES.

Please try to complete the following exercises. Remember that you cannot expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop-in Centre.

1. Write out the table of exact values on page 1 from memory.

2. Calculate exact values of the following:
   
   (a) \( \cos \left( \frac{3}{4} \pi \right) \);
   
   (b) \( \cos \left( -\frac{\pi}{6} \right) \);
   
   (c) \( \tan \left( -\frac{\pi}{4} \right) \);
   
   (d) \( \cos \left( \frac{5}{6} \pi \right) \);
   
   (e) \( \sin \left( -\frac{\pi}{3} \right) \);
   
   (f) \( \cos \left( -\frac{7}{4} \pi \right) \);
   
   (g) \( \tan \left( \frac{5}{6} \pi \right) \);
   
   (h) \( \cos \left( \frac{5}{3} \pi \right) \);
   
   (i) \( \tan \left( \frac{11}{4} \pi \right) \).

3. Other exact values can be computed by using the addition formulae for cos, sin and tan (see the “Trigonometric identities” revision worksheet if you don’t remember them).

   (a) By substituting \( \alpha = \frac{\pi}{3} \), \( \beta = \frac{\pi}{4} \) in the formula

   \[
   \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta ,
   \]

   find the exact value of \( \cos \left( \frac{\pi}{12} \right) \).

   (b) Use a similar method to find \( \sin \left( \frac{\pi}{12} \right) \). Hence find \( \tan \left( \frac{\pi}{12} \right) \).

ANSWERS.

2. (a) \( -\frac{1}{\sqrt{2}} \);
   
   (b) \( -\frac{\sqrt{3}}{2} \);
   
   (c) \( -1 \);
   
   (d) \( \frac{1}{2} \);
   
   (e) \( -\frac{\sqrt{3}}{2} \);
   
   (f) \( \frac{1}{\sqrt{2}} \);
   
   (g) \( -\frac{1}{\sqrt{3}} \);
   
   (h) \( -1 \);
   
   (i) \( 1 \).

3. The exact values are

   \[
   \cos \left( \frac{\pi}{12} \right) = \frac{\sqrt{3} + 1}{2\sqrt{2}} , \quad \sin \left( \frac{\pi}{12} \right) = \frac{\sqrt{3} - 1}{2\sqrt{2}} ,
   \]

   and so

   \[
   \tan \left( \frac{\pi}{12} \right) = \frac{\sin \left( \frac{\pi}{12} \right)}{\cos \left( \frac{\pi}{12} \right)} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} .
   \]

   The last can be simplified to give

   \[
   \tan \left( \frac{\pi}{12} \right) = 2 - \sqrt{3} ;
   \]

   if you don’t know how to do this, please see the “Surds” worksheet.