COMPLETENESS OF A GENERAL SEMIMARTINGALE MARKET UNDER CONSTRAINED TRADING

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Contents

1 Introduction

In this note, we provide a rather detailed and comprehensive study of the basic properties of self-financing trading strategies in a general security market model driven by discontinuous semimartingales. Our main goal is to analyze the issue of replication of a generic contingent claim using a self-financing trading strategy that is additionally subject to an algebraic constraint, referred to as the *balance condition*. Although such portfolios may seem to be artificial at the first glance, they appear in a natural way in the analysis of hedging strategies within the reduced-form approach to credit risk.

Let us mention in this regard that in a companion paper by Bielecki et al. (2004b) we also include defaultable assets in our portfolio, and we show how to use constrained portfolios to derive replicating strategies for defaultable contingent claims (e.g., credit derivatives). The reader is also referred to Bielecki et al. (2004a), where the case of continuous semimartingale markets was studied, for some background information regarding the probabilistic and financial set-up, as well as the terminology used in this note. The main emphasis is put here on the relationship between completeness of a security market model with unconstrained trading and completeness of an associated model in which only trading strategies satisfying the balance condition are allowed.

2 Trading in Primary Assets

Let $Y^1_t, Y^2_t, \ldots, Y^k_t$ represent cash values at time t of k primary assets. We postulate that the prices $Y^1, Y^2, \ldots, Y^k$ follow (possibly discontinuous) semimartingales on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F}$ satisfying the usual conditions. Thus, for example, general Lévy processes, as well as jump-diffusions are covered by our analysis. Note that obviously $\mathbb{F}^Y \subseteq \mathbb{F}$, where $\mathbb{F}^Y$ is the filtration generated by the prices $Y^1, Y^2, \ldots, Y^k$ of primary assets. As it is usually done, we set $X_{0-} = X_0$ for any stochastic process X, and we only consider semimartingales with càdlàg sample paths. We assume, in addition that at least one of the processes $Y^1, Y^2, \ldots, Y^k$, say Y^1, is strictly positive, so that it can be chosen as a numeraire asset. We consider trading within the time interval $[0,T]$ for some finite horizon date $T > 0$. We emphasize that we do not assume the existence of a risk-free asset (a savings account).

2.1 Unconstrained Trading Strategies

Let $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ be a trading strategy; in particular, each process ϕ^i is predictable with respect to the reference filtration $\mathbb{F}$. The component ϕ^i_t represents the number of units of the ith asset held in the portfolio at time t. Then the wealth $V_t(\phi)$ at time t of the trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ equals

$$V_t(\phi) = \sum_{i=1}^{k} \phi^i_t Y^i_t, \quad \forall\, t \in [0,T], \tag{1}$$

and ϕ is said to be a *self-financing strategy* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{k} \int_0^t \phi^i_u \, dY^i_u, \quad \forall\, t \in [0,T]. \tag{2}$$

Let Φ be the class of all self-financing trading strategies. By combining the last two formulae, we obtain the following expression for the dynamics of the wealth process of a strategy $\phi \in \Phi$

$$dV_t(\phi) = \Big(V_t(\phi) - \sum_{i=2}^{k} \phi^i_t Y^i_t \Big) (Y^1_t)^{-1} \, dY^1_t + \sum_{i=2}^{k} \phi^i_t \, dY^i_t.$$

The representation above shows that the wealth process $V(\phi)$ depends only on $k-1$ components of ϕ. Note also that, in our setting, the process $\Big(V_t(\phi) - \sum_{i=2}^k \phi_t^i Y_t^i\Big)(Y_t^1)^{-1}$ is predictable.

Remark. Let us note that Protter (2001) assumes that the component of a strategy ϕ that corresponds to the savings account (which is a continuous process) is merely optional. The interested reader is referred to Protter (2001) for a thorough discussion of other issues related to the regularity of sample paths of processes $\phi^1, \phi^2, \dots, \phi^k$ and $V(\phi)$.

Choosing Y^1 as a numeraire asset, and denoting $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$, $Y_t^{i,1} = Y_t^i(Y_t^1)^{-1}$, we get the following well-known result showing that the self-financing feature of a trading strategy is invariant with respect to the choice of a numeraire asset.

Lemma 2.1 (i) *For any $\phi \in \Phi$, we have*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^k \int_0^t \phi_u^i \, dY_u^{i,1}, \quad \forall\, t \in [0,T]. \tag{3}$$

(ii) *Conversely, let X be an $\mathcal{F}_T$-measurable random variable, and let us assume that there exists $x \in \mathbb{R}$ and $\mathbb{F}$-predictable processes ϕ^i, $i = 2, 3, \dots, k$ such that*

$$X = Y_T^1 \left(x + \sum_{i=2}^k \int_0^T \phi_t^i \, dY_t^{i,1} \right).$$

Then there exists an $\mathbb{F}$-predictable process ϕ^1 such that the strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ is self-financing and replicates X. Moreover, the wealth process of ϕ satisfies $V_t(\phi) = V_t^1 Y_t^1$, where the process V^1 is given by formula (4) *below.*

Proof. The proof of part (i) is given, for instance, in Protter (2001). We shall thus only prove part (ii). Let us set

$$V_t^1 = x + \sum_{i=2}^k \int_0^t \phi_u^i \, dY_u^{i,1}, \quad \forall\, t \in [0,T], \tag{4}$$

and let us define the process ϕ^1 as

$$\phi_t^1 = V_t^1 - \sum_{i=2}^k \phi_t^i Y_t^{i,1} = (Y_t^1)^{-1}\Big(V_t - \sum_{i=2}^k \phi_t^i Y_t^i\Big),$$

where $V_t = V_t^1 Y_t^1$. From (4), we have $dV_t^1 = \sum_{i=2}^k \phi_t^i \, dY_t^{i,1}$, and thus

$$\begin{aligned} dV_t &= d(V_t^1 Y_t^1) = V_{t-}^1 dY_t^1 + Y_{t-}^1 dV_t^1 + d[Y^1, V^1]_t \\ &= V_{t-}^1 dY_t^1 + \sum_{i=2}^k \phi_t^i \big(Y_{t-}^1 \, dY_t^{i,1} + d[Y^1, Y^{i,1}]_t\big). \end{aligned}$$

From the equality

$$dY_t^i = d(Y_t^{i,1} Y_t^1) = Y_{t-}^{i,1} dY_t^1 + Y_{t-}^1 dY_t^{i,1} + d[Y^1, Y^{i,1}]_t,$$

it follows that

$$\begin{aligned} dV_t &= V_{t-}^1 dY_t^1 + \sum_{i=2}^k \phi_t^i \big(dY_t^i - Y_{t-}^{i,1} dY_t^1\big) \\ &= \Big(V_{t-}^1 - \sum_{i=2}^k \phi_t^i Y_{t-}^{i,1}\Big) dY_t^1 + \sum_{i=2}^k \phi_t^i \, dY_t^i, \end{aligned}$$

and our aim is to prove that

$$dV_t = \sum_{i=1}^{k} \phi_t^i \, dY_t^i.$$

The last equality holds if

$$\phi_t^1 = V_t^1 - \sum_{i=2}^{k} \phi_t^i Y_t^{i,1} = V_{t-}^1 - \sum_{i=2}^{k} \phi_t^i Y_{t-}^{i,1}, \tag{5}$$

i.e., if $\Delta V_t^1 = \sum_{i=2}^k \phi_t^i \Delta Y_t^{i,1}$, which is the case from the definition (4) of V^1. Note also that from the second equality in (5) it follows that the process ϕ^1 is indeed $\mathbb{F}$-predictable. Finally, the wealth process of ϕ satisfies $V_t(\phi) = V_t^1 Y_t^1$ for every $t \in [0,T]$, and thus $V_T(\phi) = X$. □

2.2 Constrained Trading Strategies

In this section, we make an additional assumption that the price process Y^k is strictly positive. Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing trading strategy satisfying the following constraint:

$$\sum_{i=l+1}^{k} \phi_t^i Y_{t-}^i = Z_t, \quad \forall\, t \in [0,T], \tag{6}$$

for some $1 \le l \le k-1$ and a predetermined, $\mathbb{F}$-predictable process Z. In the financial interpretation, equality (6) means that the portfolio ϕ should be rebalanced in such a way that the total wealth invested in securities $Y^{l+1}, Y^{l+2}, \dots, Y^k$ should match a predetermined stochastic process (for instance, we may assume that it is constant over time or follows a deterministic function of time). For this reason, the constraint (6) will be referred to as the *balance condition*.

Our first goal is to extend part (i) in Lemma 2.1 to the case of constrained strategies. Let $\Phi_l(Z)$ stand for the class of all self-financing trading strategies satisfying the balance condition (6). They will be sometimes referred to as *constrained strategies*. Since any strategy $\phi \in \Phi_l(Z)$ is self-financing, we have

$$\Delta V_t(\phi) = \sum_{i=1}^{k} \phi_t^i \Delta Y_t^i = V_t(\phi) - \sum_{i=1}^{k} \phi_t^i Y_{t-}^i,$$

and thus we deduce from (6) that

$$V_{t-}(\phi) = \sum_{i=1}^{k} \phi_t^i Y_{t-}^i = \sum_{i=1}^{l} \phi_t^i Y_{t-}^i + Z_t.$$

Let us write $Y_t^{i,1} = Y_t^i (Y_t^1)^{-1}$, $Y_t^{i,k} = Y^i (Y_t^k)^{-1}$, $Z_t^1 = Z_t (Y_t^1)^{-1}$. The following result extends Lemma 1.7 in Bielecki et al. (2004a) from the case of continuous semimartingales to the general case. It is apparent from Proposition 2.1 that the wealth process $V(\phi)$ of a strategy $\phi \in \Phi_l(Z)$ depends only on $k-2$ components of ϕ.

Proposition 2.1 *The relative wealth $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ of a strategy $\phi \in \Phi_l(Z)$ satisfies*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right) + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1}. \tag{7}$$

Proof. Let us consider discounted values of price processes $Y^1, Y^2, \dots, Y^k$, with Y^1 taken as a numeraire asset. By virtue of part (i) in Lemma 2.1, we thus have

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{k} \int_0^t \phi_u^i \, dY_u^{i,1}. \tag{8}$$

The balance condition (6) implies that

$$\sum_{i=l+1}^{k} \phi_t^i Y_{t-}^{i,1} = Z_t^1,$$

and thus

$$\phi_t^k = (Y_{t-}^{k,1})^{-1}\Big(Z_t^1 - \sum_{i=l+1}^{k} \phi_t^i Y_{t-}^{i,1}\Big). \tag{9}$$

By inserting (9) into (8), we arrive at the desired formula (7). □

Let us take $Z = 0$, so that $\phi \in \Phi_l(0)$. Then the balance condition becomes $\sum_{i=l+1}^{k} \phi_t^i Y_{t-}^i = 0$, and (7) reduces to

$$dV_t^1(\phi) = \sum_{i=2}^{l} \int_0^t \phi_t^i \, dY_t^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_t^i \left(dY_t^{i,1} - \frac{Y_{t-}^{i,1}}{Y_{t-}^{k,1}} \, dY_t^{k,1} \right). \tag{10}$$

2.3 Case of Continuous Semimartingales

For the sake of notational simplicity, we denote by $Y^{i,k,1}$ the process given by the formula

$$Y_t^{i,k,1} = \int_0^t \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right) \tag{11}$$

so that (7) becomes

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \, dY_u^{i,k,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1}. \tag{12}$$

In Bielecki et al. (2004a), we postulated that the primary assets $Y^1, Y^2, \ldots, Y^k$ follow strictly positive continuous semimartingales, and we introduced the auxiliary processes $\widehat{Y}_t^{i,k,1} = Y_t^{i,k} e^{-\alpha_t^{i,k,1}}$, where

$$\alpha_t^{i,k,1} = \langle \ln Y^{i,k}, \ln Y^{1,k} \rangle_t = \int_0^t (Y_u^{i,k})^{-1} (Y_u^{1,k})^{-1} \, d\langle Y^{i,k}, Y^{1,k} \rangle_u.$$

In Lemma 1.7 in Bielecki et al. (2004a) (see also Vaillant (2001)), we have shown that, under continuity of $Y^1, Y^2, \ldots, Y^k$, the discounted wealth of a self-financing trading strategy ϕ that satisfies the constraint $\sum_{i=l+1}^{k} \phi_t^i Y_t^i = Z_t$ can be represented as follows:

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} + \int_0^t \frac{Z_u^1}{Y_u^{k,1}} \, dY_u^{k,1}, \tag{13}$$

where we write $\widehat{\phi}_t^{i,k,1} = \phi_t^i (Y_t^{1,k})^{-1} e^{\alpha_t^{i,k,1}}$. The following simple result reconciles expression (12) established in Proposition 2.1 with representation (13) derived in Bielecki et al. (2004a).

Lemma 2.2 *Assume that the prices Y^1, Y^i and Y^k follow strictly positive continuous semimartingales. Then we have*

$$Y_t^{i,k,1} = \int_0^t (Y_u^{1,k})^{-1} e^{\alpha_u^{i,k,1}} \, d\widehat{Y}_u^{i,k,1}$$

and

$$dY_t^{i,k,1} = (Y_t^{1,k})^{-1} \big(dY_t^{i,k} - Y_t^{i,k} \, d\alpha_t^{i,k,1} \big).$$

Proof. In the case of continuous semimartingales, formula (11) becomes

$$Y_t^{i,k,1} = \int_0^t \left(dY_u^{i,1} - \frac{Y_u^{i,1}}{Y_u^{k,1}}\, dY_u^{k,1} \right) = \int_0^t \big(dY_u^{i,1} - Y_u^{i,k}\, d(Y_u^{1,k})^{-1} \big).$$

On the other hand, an application of Itô's formula yields

$$d\widehat{Y}_t^{i,k,1} = e^{-\alpha_t^{i,k,1}} \big(dY_t^{i,k} - (Y_t^{1,k})^{-1} d\langle Y^{i,k}, Y^{1,k}\rangle_t \big)$$

and thus

$$(Y_t^{1,k})^{-1} e^{\alpha_t^{i,k,1}}\, d\widehat{Y}_u^{i,k,1} = (Y_t^{1,k})^{-1} \big(dY_t^{i,k} - (Y_t^{1,k})^{-1} d\langle Y^{i,k}, Y^{1,k}\rangle_t \big).$$

One checks easily that for any two continuous semimartingales, say X and Y, we have

$$Y_t^{-1} \big(dX_t - Y_t^{-1} d\langle X, Y\rangle_t \big) = d(X_t Y_t^{-1}) - X_t\, dY_t^{-1},$$

provided that Y is strictly positive. To conclude the derivation of the first formula, it suffices to apply the last identity to processes $X = Y^{i,k}$ and $Y = Y^{1,k}$. For the second formula, note that

$$dY_t^{i,k,1} = (Y_t^{1,k})^{-1} e^{\alpha_t^{i,k,1}}\, d\widehat{Y}_t^{i,k,1} = (Y_t^{1,k})^{-1} e^{\alpha_t^{i,k,1}}\, d(Y_t^{i,k} e^{-\alpha_t^{i,k,1}}) = (Y_t^{1,k})^{-1} \big(dY_t^{i,k} - Y_t^{i,k}\, d\alpha_t^{i,k,1} \big),$$

as required. □

It is obvious that the processes $Y^{i,k,1}$ and $\widehat{Y}^{i,k,1}$ are uniquely specified by the joint dynamics of Y^1, Y^i and Y^k. The following result shows that the converse is also true.

Corollary 2.1 *The price Y_t^i at time t is uniquely specified by the initial value Y_0^i and either*
(i) the joint dynamics of processes Y^1, Y^k and $\widehat{Y}^{i,k,1}$, or
(ii) the joint dynamics of processes Y^1, Y^k and $Y^{i,k,1}$.

Proof. Since $\widehat{Y}_t^{i,k,1} = Y_t^{i,k} e^{-\alpha_t^{i,k,1}}$, we have

$$\alpha_t^{i,k,1} = \langle \ln Y^{i,k}, \ln Y^{1,k}\rangle_t = \langle \ln \widehat{Y}^{i,k,1}, \ln Y^{1,k}\rangle_t,$$

and thus

$$Y_t^i = Y_t^k \widehat{Y}_t^{i,k,1} e^{\alpha_t^{i,k,1}} = Y_t^k \widehat{Y}_t^{i,k,1} e^{\langle \ln \widehat{Y}^{i,k,1}, \ln Y^{1,k}\rangle_t}.$$

This completes the proof of part (i). For the second part, note that the process $Y^{i,1}$ satisfies

$$Y_t^{i,1} = Y_0^{i,1} + Y_t^{i,k,1} + \int_0^t \frac{Y_u^{i,1}}{Y_u^{k,1}}\, dY_u^{k,1}. \tag{14}$$

It is well known that the SDE

$$X_t = X_0 + H_t + \int_0^t X_u\, dY_u,$$

where H and Y are continuous semimartingales (with $H_0 = 0$) has the unique, strong solution given by the formula

$$X_t = \mathcal{E}_t(Y) \Big(X_0 + \int_0^t \mathcal{E}_u^{-1}(Y)\, dH_u - \int_0^t \mathcal{E}_u^{-1}(Y)\, d\langle Y, H\rangle_u \Big).$$

Upon substitution, this proves (ii). □

3 Replication with Constrained Strategies

The next result is essentially a converse to Proposition 2.1. Also, it extends part (ii) of Lemma 2.1 to the case of constrained trading strategies. As in Section 2.2, we assume that $1 \le l \le k-1$, and Z is a predetermined, $\mathbb{F}$-predictable process.

Proposition 3.1 *Let an $\mathcal{F}_T$-measurable random variable X represent a contingent claim that settles at time T. Assume that there exist $\mathbb{F}$-predictable processes ϕ^i, $i = 2, 3, \dots, k-1$ such that*

$$X = Y_T^1 \left(x + \sum_{i=2}^{l} \int_0^T \phi_t^i \, dY_t^{i,1} + \sum_{i=l+1}^{k-1} \int_0^T \phi_t^i \, dY_t^{i,k,1} + \int_0^T \frac{Z_t^1}{Y_{t-}^{k,1}} \, dY_t^{k,1} \right). \tag{15}$$

Then there exist the $\mathbb{F}$-predictable processes ϕ^1 and ϕ^k such that the strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ belongs to $\Phi_l(Z)$ and replicates X. The wealth process of ϕ equals, for every $t \in [0,T]$,

$$V_t(\phi) = Y_t^1 \left(x + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \, dY_u^{i,k,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right). \tag{16}$$

Proof. As expected, we first set (note that ϕ^k is $\mathbb{F}$-predictable)

$$\phi_t^k = \frac{1}{Y_{t-}^k} \Big(Z_t - \sum_{i=l+1}^{k-1} \phi_t^i Y_{t-}^i \Big) \tag{17}$$

and

$$V_t^1 = x + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \, dY_u^{i,k,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1}.$$

Arguing along the same lines as in the proof of Proposition 2.1, we obtain

$$V_t^1 = V_0^1 + \sum_{i=2}^{k} \int_0^t \phi_u^i \, dY_u^{i,1}.$$

Now, we define

$$\phi_t^1 = V_t^1 - \sum_{i=2}^{k} \phi_t^i Y_t^{i,1} = (Y_t^1)^{-1} \Big(V_t - \sum_{i=2}^{k} \phi_t^i Y_t^i \Big),$$

where $V_t = V_t^1 Y_t^1$. As in the proof of Lemma 2.1, we check that

$$\phi_t^1 = V_{t-}^1 - \sum_{i=2}^{k} \phi_t^i Y_{t-}^{i,1},$$

and thus the process ϕ^1 is $\mathbb{F}$-predictable. It is clear that the strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ is self-financing and its wealth process satisfies $V_t(\phi) = V_t$ for every $t \in [0,T]$. In particular, $V_T(\phi) = X$, so that ϕ replicates X. Finally, equality (17) implies (6), and thus $\phi \in \Phi_l(Z)$. □

Note that equality (15) is a necessary (by Proposition 2.1) and sufficient (by Proposition 3.1) condition for the existence of a constrained strategy replicating a given contingent claim X.

3.1 Modified Balance Condition

It is tempting to replace the constraint (6) by a more convenient condition:

$$\sum_{i=l+1}^{k} \phi_t^i Y_t^i = Z_t, \quad \forall\, t \in [0,T], \tag{18}$$

where Z is a predetermined, $\mathbb{F}$-predictable process. If a self-financing trading strategy ϕ satisfies the modified balance condition (18) then for the relative wealth process we obtain (cf. (7))

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \left(dY_u^{i,1} - \frac{Y_u^{i,1}}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right) + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1}. \quad (19)$$

Note that in many cases the integrals above are meaningful, so that a counterpart of Proposition 2.1 with the modified balance condition can be formulated. To get a counterpart of Proposition 3.1, we need to replace (15) by the equality

$$X = Y_T^1 \left(x + \sum_{i=2}^{l} \int_0^T \phi_t^i \, dY_t^{i,1} + \sum_{i=l+1}^{k-1} \int_0^T \phi_t^i \left(dY_t^{i,1} - \frac{Y_t^{i,1}}{Y_{t-}^{k,1}} \, dY_t^{k,1} \right) + \int_0^T \frac{Z_t^1}{Y_{t-}^{k,1}} \, dY_t^{k,1} \right), \quad (20)$$

where $\phi^3, \phi^4, \ldots, \phi^k$ are $\mathbb{F}$-predictable processes. We define

$$V_t^1 = x + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \left(dY_u^{i,1} - \frac{Y_u^{i,1}}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right) + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1},$$

and we set

$$\phi_t^k = \frac{1}{Y_t^k} \Big(Z_t - \sum_{i=l+1}^{k-1} \phi_t^i Y_t^i \Big), \quad \phi_t^1 = V_t^1 - \sum_{i=2}^{k} \phi_t^i Y_t^{i,1}.$$

Suppose, for the sake of argument, that the processes ϕ^1 and ϕ^k defined above are $\mathbb{F}$-predictable. Then the trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ is self-financing on $[0, T]$, replicates X, and satisfies the constraint (18). Note, however, that the predictability of ϕ^1 and ϕ^k is far from being obvious, and it is rather difficult to provide non-trivial and practically appealing sufficient conditions for this property.

3.2 Synthetic Assets

Let us fix i, and let us analyze the auxiliary process $Y^{i,k,1}$ given by formula (11). We claim that this process can be interpreted as the relative wealth of a specific self-financing trading strategy associated with $Y^1, Y^2, \ldots, Y^k$. Specifically, we will show that for any $i = 2, 3, \ldots, k-1$ the process $\bar{Y}^{i,k,1}$, given by the formula

$$\bar{Y}_t^{i,k,1} = Y_t^1 Y_t^{i,k,1} = Y_t^1 \int_0^t \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right),$$

represents the price of a *synthetic asset.* For brevity, we shall frequently write $\bar{Y}^i$ instead of $\bar{Y}^{i,k,1}$. Note that the process $\bar{Y}^i$ is not strictly positive (in fact, $\bar{Y}_0^i = 0$).

3.2.1 Equivalence of Primary and Synthetic Assets

Our goal is to show that trading in primary assets is formally equivalent to trading in synthetic assets. The first result shows that the process $\bar{Y}^i$ can be obtained from primary assets Y^1, Y^i and Y^k through a simple self-financing strategy. This justifies the name synthetic asset given to $\bar{Y}^i$.

Lemma 3.1 *For any fixed $i = 2, 3, \ldots, k-1$, let an $\mathcal{F}_T$-measurable random variable $\bar{Y}_T^i$ be given as*

$$\bar{Y}_T^i = Y_T^1 Y_T^{i,k,1} = Y_T^1 \int_0^T \left(dY_t^{i,1} - \frac{Y_{t-}^{i,1}}{Y_{t-}^{k,1}} \, dY_t^{k,1} \right). \quad (21)$$

Then there exists a strategy $\phi \in \Phi_1(0)$ *that replicates the claim* $\bar{Y}^i_T$. *Moreover, we have, for every* $t \in [0,T]$,

$$V_t(\phi) = Y^1_t Y^{i,k,1}_t = Y^1_t \int_0^t \left(dY^{i,1}_u - \frac{Y^{i,1}_{u-}}{Y^{k,1}_{u-}}\, dY^{k,1}_u \right) = \bar{Y}^i_t. \tag{22}$$

Proof. To establish the existence of a strategy ϕ with the desired properties, it suffices to apply Proposition 3.1. We fix i and we start by postulating that $\phi^i = 1$ and $\phi^j = 0$ for any $2 \leq j \leq k-1$, $j \neq i$. Then equality (21) yields (15) with $X = \bar{Y}^i_T$, $x = 0$, $l = 1$ and $Z = 0$. Note that the balance condition becomes

$$\sum_{j=2}^{k} \phi^j_t Y^j_{t-} = Y^i_{t-} + \phi^k_t Y^k_{t-} = 0.$$

Let us define ϕ^1 and ϕ^k by setting

$$\phi^k_t = -\frac{Y^i_{t-}}{Y^k_{t-}}, \quad \phi^1_t = V^1_t - Y^{i,1}_t - \phi^k_t Y^{k,1}_t.$$

Note that we also have

$$\phi^1_t = V^1_{t-} - Y^{i,1}_{t-} - \phi^k_t Y^{k,1}_{t-} = V^1_{t-}.$$

Hence, ϕ^1 and ϕ^k are $\mathbb{F}$-predictable processes, the strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ is self-financing, and it satisfies (6) with $l = 1$ and $Z = 0$, so that $\phi \in \Phi_1(0)$. Finally, equality (22) holds, and thus $V_T(\phi) = \bar{Y}^i_T$. □

Note that to replicate the claim $\bar{Y}^i_T = \bar{Y}^{i,k,1}_T$, it suffices to invest in primary assets Y^1, Y^i and Y^k. Essentially, we start with zero initial endowment, we keep at any time one unit of the ith asset, we rebalance the portfolio in such a way that the total wealth invested in the ith and kth assets is always zero, and we put the residual wealth in the first asset. Hence, we deal here with a specific strategy such that the risk of the ith asset is perfectly offset by rebalancing the investment in the kth asset, and our trades are financed by taking positions in the first asset.

Note that the process $Y^{i,1}$ satisfies the following SDE (cf. (14))

$$Y^{i,1}_t = Y^{i,1}_0 + \bar{Y}^{i,1}_t + \int_0^t \frac{Y^{i,1}_{u-}}{Y^{k,1}_{u-}}\, dY^{k,1}_u, \tag{23}$$

which is known to possess a unique strong solution. Hence, the relative price $Y^{i,1}_t$ at time t is uniquely determined by the initial value $Y^{i,1}_0$ and processes $\bar{Y}^{i,1}$ and $Y^{k,1}$. Consequently, the price Y^i_t at time t of the ith primary asset is uniquely determined by the initial value Y^i_0, the prices Y^1, Y^k of primary assets, and the price $\bar{Y}^i$ of the ith synthetic asset. We thus obtain the following result.

Lemma 3.2 *Filtrations generated by the primary assets* $Y^1, Y^2, \ldots, Y^k$ *and by the price processes* $Y^1, Y^2, \ldots, Y^l, \bar{Y}^{l+1}, \ldots, \bar{Y}^{k-1}, Y^k$ *coincide.*

Lemma 3.2 suggests that for any choice of the underlying filtration $\mathbb{F}$ (such that $\mathbb{F}^Y \subseteq \mathbb{F}$), trading in assets $Y^1, Y^2, \ldots, Y^k$ is essentially equivalent to trading in $Y^1, Y^2, \ldots, Y^l, \bar{Y}^{l+1}, \ldots, \bar{Y}^{k-1}, Y^k$. Let us first formally define the equivalence of market models.

Definition 3.1 We say that the two unconstrained models, $\mathcal{M}$ and $\tilde{\mathcal{M}}$ say, are *equivalent with respect to a filtration* $\mathbb{F}$ if both models are defined on a common probability space and every primary asset in $\mathcal{M}$ can be obtained by trading in primary assets in $\tilde{\mathcal{M}}$ and vice versa, under the assumption that trading strategies are $\mathbb{F}$-predictable.

Note that we do not assume that models $\mathcal{M}$ and $\tilde{\mathcal{M}}$ have the same number of primary assets. The next result justifies our claim of equivalence of primary and synthetic assets.

Corollary 3.1 *Models* $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ *and* $\bar{\mathcal{M}} = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi)$ *are equivalent with respect to any filtration* $\mathbb{F}$ *such that* $\mathbb{F}^Y \subseteq \mathbb{F}$ *.*

Proof. In view of Lemma 3.1, it suffices to show that the price process of each primary asset Y^i for $i = l, l+1, \dots, k-1$ can be mimicked by trading in $Y^1, \bar{Y}^i$ and Y^k. To see this, note that for any fixed $i = l, l+1, \dots, k-1$, we have (see the proof of Lemma 3.1)

$$\bar{Y}^i_t = V_t(\phi) = \phi^1_t Y^1_t + Y^i_t + \phi^k_t Y^k_t$$

with

$$d\bar{Y}^i_t = dV_t(\phi) = \phi^1_t \, dY^1_t + dY^i_t + \phi^k_t \, dY^k_t .$$

Consequently,

$$Y^i_t = -\phi^1_t Y^1_t + \bar{Y}^i_t - \phi^k_t Y^k_t$$

and

$$dY^i_t = -\phi^1_t \, dY^1_t + d\bar{Y}^i_t - \phi^k_t \, dY^k_t .$$

This shows that the strategy $(-\phi^1, 1, -\phi^k)$ in $Y^1, \bar{Y}^i$ and Y^k is self-financing and its wealth equals Y^i. □

3.2.2 Replicating Strategies with Synthetic Assets

In view of Lemma 3.1, the replicating trading strategy for a contingent claim X, for which (15) holds, can be conveniently expressed in terms of primary securities $Y^1, Y^2, \dots, Y^l$ and Y^k, and synthetic assets $\bar{Y}^{l+1}, \bar{Y}^{l+2}, \dots, \bar{Y}^{k-1}$. To this end, we represent (15)-(16) in the following way:

$$X = Y^1_T \left(x + \sum_{i=2}^{l} \int_0^T \phi^i_t \, dY^{i,1}_t + \sum_{i=l+1}^{k-1} \int_0^T \phi^i_t \, d\bar{Y}^{i,1}_t + \int_0^T \frac{Z^1_t}{Y^{k,1}_{t-}} \, dY^{k,1}_t \right) \tag{24}$$

where $\bar{Y}^{i,1}_t = \bar{Y}^i_t / Y^1_t = Y^{i,k,1}_t$, and

$$V_t(\phi) = Y^1_t \left(x + \sum_{i=2}^{l} \int_0^t \phi^i_u \, dY^{i,1}_u + \sum_{i=l+1}^{k-1} \int_0^t \phi^i_u \, d\bar{Y}^{i,1}_u + \int_0^t \frac{Z^1_u}{Y^{k,1}_{u-}} \, dY^{k,1}_u \right). \tag{25}$$

Corollary 3.2 *Let* X *be an* $\mathcal{F}_T$*-measurable random variable such that* (24) *holds for some* $\mathbb{F}$*-predictable process* Z *and some* $\mathbb{F}$*-predictable processes* $\phi^2, \phi^3, \dots, \phi^{k-1}$*. Let* $\psi^i = \phi^i$ *for* $i = 2, 3, \dots, k-1$*,*

$$\psi^k_t = \frac{Z^1_t}{Y^{k,1}_{t-}} = \frac{Z_t}{Y^k_{t-}},$$

and

$$\psi^1_t = V^1_t - \sum_{i=2}^{l} \psi^i_t Y^{i,1}_t - \sum_{i=l+1}^{k-1} \psi^i_t \bar{Y}^{i,1}_t - \psi^k_t Y^{k,1}_t = V^1_{t-} - \sum_{i=2}^{l} \psi^i_t Y^{i,1}_{t-} - \sum_{i=l+1}^{k-1} \psi^i_t \bar{Y}^{i,1}_{t-} - \psi^k_t Y^{k,1}_{t-}.$$

Then $\psi = (\psi^1, \psi^2, \dots, \psi^k)$ *is a self-financing trading strategy in assets* $Y^1, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$*. Moreover,* ψ *satisfies* $\psi^k_t Y^k_{t-} = Z_t$*,* $t \in [0, T]$*, and it replicates* X*.*

Proof. In view of (24), it suffices to apply Proposition 3.1 with $l = k - 1$. □

4 Model Completeness

We shall now examine the relationship between the arbitrage-free property and completeness of a market model in which trading is restricted a priori to self-financing strategies satisfying the balance condition.

4.1 Minimal Completeness of an Unconstrained Model

Let $\mathcal{M} = (Y^1, Y^2, \ldots, Y^k; \Phi)$ be an arbitrage-free market model. Unless explicitly stated otherwise, Φ stands for the class of all $\mathbb{F}$-predictable, self-financing strategies. Note, however, that the number of traded assets and their selection may be different for each particular model. Consequently, the number of components of a strategy $\phi \in \Phi$ will depend on the number of traded assets in a given model. For the sake of brevity, this feature is not reflected in our notation.

Definition 4.1 We say that a model $\mathcal{M}$ is *complete with respect to* $\mathbb{F}$ (briefly, $\mathbb{F}$-*complete*) if any bounded $\mathcal{F}_T$-measurable contingent claim X is attainable in $\mathcal{M}$. Otherwise, a model $\mathcal{M}$ is said to be *incomplete* with respect to $\mathbb{F}$.

Definition 4.2 An $\mathbb{F}$-complete model $\mathcal{M} = (Y^1, Y^2, \ldots, Y^k; \Phi)$ is *minimally complete* with respect to $\mathbb{F}$ if for any choice of trading strategies $\phi_i \in \Phi$, $i = 1, 2, \ldots, k-1$, the *reduced model* $\widehat{\mathcal{M}}^{k-1} = (\widehat{Y}^1, \widehat{Y}^2, \ldots, \widehat{Y}^{k-1}; \Phi)$ where $\widehat{Y}^i = V(\phi_i)$, is incomplete with respect to $\mathbb{F}$. In this case, we say that the *degree of completeness* of $\mathcal{M}$ equals k.

Let us stress that trading strategies in the reduced model $\widehat{\mathcal{M}}^{k-1}$ are predictable with respect to $\mathbb{F}$, rather than with respect to the filtration generated by price processes $\widehat{Y}^1, \widehat{Y}^2, \ldots, \widehat{Y}^{k-1}$. Hence, by moving from $\mathcal{M}$ to $\widetilde{\mathcal{M}}^{k-1}$ we reduce the number of traded asset, but we preserve the original information structure $\mathbb{F}$. Minimal completeness of a model $\mathcal{M}$ means, in particular, that all primary assets $Y^1, Y^2, \ldots, Y^k$ are needed if we wish to generate the class of all (bounded) $\mathcal{F}_T$-measurable claims through $\mathbb{F}$-predictable trading strategies. The following lemma is thus an immediate consequence of Definition 4.2.

Lemma 4.1 *Assume that a model $\mathcal{M} = (Y^1, Y^2, \ldots, Y^k; \Phi)$ is complete, but not minimally complete, with respect to $\mathbb{F}$. Then there exists at least one primary asset Y^i, which is redundant, in the sense that there exists a complete reduced model $\widehat{\mathcal{M}}^l$ for some $l \leq k-1$ such that $Y^i \neq \widehat{Y}^j$ for $j = 1, 2, \ldots, l$.*

Complete models that are not minimally complete do not seem to describe adequately the real-life features of financial markets (in fact, it is frequently argued that the real-life markets are not even complete). Also, from the theoretical perspective, there is no advantage in keeping a redundant asset among primary securities. For this reasons, in what follows, we shall restrict our attention to market models $\mathcal{M}$ that are either incomplete or minimally complete. Lemma 4.2 shows that the degree of completeness is a well-defined notion, in the sense that it does not depend on the choice of traded assets, provided that the model completeness is preserved.

Lemma 4.2 *Let a model $\mathcal{M} = (Y^1, Y^2, \ldots, Y^k; \Phi)$ be minimally complete with respect to $\mathbb{F}$. Let $\widetilde{\mathcal{M}} = (\widetilde{Y}^1, \widetilde{Y}^2, \ldots, \widetilde{Y}^k; \Phi)$, where the processes $\widetilde{Y}^i = V(\phi_i)$, $i = 1, 2, \ldots, k$ represent the wealth processes of some trading strategies $\phi_1, \phi_2, \ldots, \phi_k \in \Phi$. If a model $\widetilde{\mathcal{M}}$ is complete with respect to $\mathbb{F}$ then it is also minimally complete with respect to $\mathbb{F}$, and thus its degree of completeness equals k.*

Proof. The proof relies on simple algebraic considerations. By assumption, for every $i = 1, 2, \ldots, k$, we have

$$d\widetilde{Y}_t^i = \sum_{j=1}^{k} \phi_t^{ij}\, dY_t^j,$$

for some family ϕ^{ij}, $i, j = 1, 2, \ldots, k$ of $\mathbb{F}$-predictable stochastic processes. Assume that a model $\widetilde{\mathcal{M}}$ is complete, but not minimally complete. Then there exists $l \leq k-1$ and trading strategies ψ^m, $m = 1, 2, \ldots, l$, such that the reduced model $\widehat{\mathcal{M}}^l = (\widehat{Y}^1, \widehat{Y}^2, \ldots, \widehat{Y}^l; \Phi)$, with asset prices satisfying

$$d\widehat{Y}_t^m = \sum_{i=1}^{l} \psi_t^{mi}\, d\widetilde{Y}_t^i,$$

is complete. Clearly, we have

$$d\widehat{Y}_t^m = \sum_{i=1}^{l} \psi_t^{mi} \sum_{j=1}^{k} \phi_t^{ij}\, dY_t^j = \sum_{j=1}^{k} \zeta_t^{mj}\, dY_t^j,$$

so that there exist trading strategies ζ^m, $m = 1, 2, \dots, l$, in primary assets $Y^1, Y^2, \dots, Y^k$ such that $\widehat{Y}^m = V(\zeta^m)$ for $m = 1, 2, \dots, l$. This contradicts the assumption that the model $\mathcal{M}$ is minimally complete. □

By combining Lemma 4.2 with Corollary 3.1, we obtain the following result.

Corollary 4.1 *A model $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ is minimally complete if and only if a model $\bar{\mathcal{M}} = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi)$ has this property.*

As one might easily guess, the degree of a model completeness depends on the relationship between the number of primary assets and the number of independent sources of randomness. In the two models examined in Sections 5.1 and 5.2 below, we shall deal with $k = 4$ primary assets, but the number of independent sources of randomness will equal two and three for the first and the second model, respectively.

4.2 Completeness of a Constrained Model

Let $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ be an arbitrage-free market model, and let us denote by $\mathcal{M}_l(Z) = (Y^1, Y^2, \dots, Y^k; \Phi_l(Z))$ the associated model in which the class Φ is replaced by the class $\Phi_l(Z)$ of constrained strategies. We claim that if $\mathcal{M}$ is arbitrage-free and minimally complete with respect to the filtration $\mathbb{F} = \mathbb{F}^Y$, where $Y = (Y^1, Y^2, \dots, Y^k)$, then the constrained model $\mathcal{M}_l(Z)$ is arbitrage-free, but it is incomplete with respect to $\mathbb{F}$. Conversely, if the model $\mathcal{M}_l(Z)$ is arbitrage-free and complete with respect to $\mathbb{F}$, then the original model $\mathcal{M}$ is not minimally complete. To prove these claims, we need some preliminary results.

The following definition extends the notion of equivalence of security market models to the case of constrained trading.

Definition 4.3 We say that the two constrained models are *equivalent with respect to a filtration* $\mathbb{F}$ if they are defined on a common probability space and the class of all wealth processes of $\mathbb{F}$-predictable constrained trading strategies is the same in both models.

Corollary 4.2 *The constrained model*

$$\mathcal{M}_l(Z) = (Y^1, Y^2, \dots, Y^k; \Phi_l(Z))$$

is equivalent to the constrained model

$$\bar{\mathcal{M}}_{k-1}(Z) = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi_{k-1}(Z)).$$

Proof. It suffices to make use of Corollaries 3.1 and 3.2. □

Note that the model $\bar{\mathcal{M}}_{k-1}(Z)$ is easier to handle than $\mathcal{M}_l(Z)$. For this reason, we shall state the next result for the model $\mathcal{M}_l(Z)$ (which is of our main interest), but we shall focus on the equivalent model $\bar{\mathcal{M}}_{k-1}(Z)$ in the proof.

Proposition 4.1 (i) *Assume that the model $\mathcal{M}$ is arbitrage-free and minimally complete. Then for any $\mathbb{F}$-predictable process Z and any $l = 1, 2, \dots, k-1$ the constrained model $\mathcal{M}_l(Z)$ is arbitrage-free and incomplete.*
(ii) *Assume that the constrained model $\mathcal{M}_l(Z)$ associated with $\mathcal{M}$ is arbitrage-free and complete. Then $\mathcal{M}$ is either not arbitrage-free or not minimally complete.*

Proof. The arbitrage-free property of $\mathcal{M}_l(Z)$ is an immediate consequence of Corollary 4.2 and the fact that $\Phi_{k-1}(Z) \subset \Phi$. In view of Corollary 4.1, it suffices to check that the minimal completeness of $\bar{\mathcal{M}}$ implies that $\bar{\mathcal{M}}_{k-1}(Z)$ is incomplete. By assumption, there exists a bounded, $\mathcal{F}_T$-measurable claim X that cannot be replicated in $\bar{\mathcal{M}}^k = (Y^1, Y^2, \ldots, Y^l, \bar{Y}^{l+1}, \ldots, \bar{Y}^{k-1}; \Phi)$ (i.e., when trading in Y^k is not allowed). Let us consider the following random variable

$$Y = X + \int_0^T \frac{Z_t}{Y_t^k}\, dY_t^k.$$

We claim that Y cannot be replicated in $\bar{\mathcal{M}}_{k-1}(Z)$. Indeed, for any trading strategy $\phi \in \Phi_{k-1}(Z)$, we have

$$V_T(\phi) = V_0(\phi) + \sum_{i=1}^{l} \int_0^T \phi_t^i\, dY_t^i + \sum_{i=l+1}^{k-1} \int_0^T \phi_t^i\, d\bar{Y}_t^i + \int_0^T \frac{Z_t}{Y_t^k}\, dY_t^k,$$

and thus the existence of a replicating strategy for Y in $\bar{\mathcal{M}}_{k-1}(Z)$ will imply the existence of a replicating strategy for X in $\bar{\mathcal{M}}^k$, which contradicts our assumption. Part (ii) is a straightforward consequence of part (i). □

It is worth noting that the arbitrage-free property of $\mathcal{M}_l(Z)$ does not imply the same property for $\mathcal{M}$. As a trivial example, we may take $l = k-1$ and $Z = 0$, so that trading in the asset Y^k is in fact excluded in $\mathcal{M}_l(Z)$, but it is allowed in the larger model $\mathcal{M}$.

5 Jump-Diffusion Case

In order to make the results of Sections 2-4 more tangible, we shall now analyze the case of jump-diffusion processes. For the sake of concreteness and simplicity, we shall take $k = 4$. Needless to say that this assumption is not essential, and the similar considerations can be done for any sufficiently large number of primary assets.

We consider a model $\mathcal{M} = (Y^1, Y^2, \ldots, Y^4; \Phi)$ with discontinuous asset prices governed by the SDE

$$dY_t^i = Y_{t-}^i \big(\mu_i\, dt + \sigma_i\, dW_t + \kappa_i\, dM_t\big) \tag{26}$$

for $i = 1, \ldots, 4$, where $W_t = (W_t^1, W_t^2, \ldots, W_t^d)$, $t \in [0,T]$, is a d-dimensional standard Brownian motion and $M_t = N_t - \lambda t$, $t \in [0,T]$, is a compensated Poisson process under the actual probability $\mathbb{P}$. Let us stress that W and N are a Brownian motion and a Poisson process with respect to $\mathbb{F}$, respectively. This means, in particular, that they are independent processes. We shall assume that $\mathbb{F} = \mathbb{F}^{W,N}$ is the filtration generated by W and N.

The coefficients $\mu_i, \sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^d)$ and κ_i in (26) can be constant, deterministic or even stochastic (predictable with respect to the filtration $\mathbb{F}$). For simplicity, in what follows we shall assume that they are constant. In addition, we postulate that $\kappa_1 > -1$, so that $Y_t^1 > 0$ for every $t \in [0,T]$, provided that $Y_0^1 > 0$. Finally, let Z be a predetermined $\mathbb{F}$-predictable process. Recall that $\Phi_1(Z)$ is the class of all self-financing strategies that satisfy the balance condition

$$\sum_{i=2}^{4} \phi_t^i Y_{t-}^i = Z_t, \quad \forall\, t \in [0,T]. \tag{27}$$

Our goal is to present examples illustrating Proposition 4.1 and, more importantly, to show how to proceed if we wish to replicate a contingent claim using a trading strategy satisfying the balance condition. It should be acknowledged that in the previous sections we have not dealt at all with the issue of admissibility of trading strategies, and thus some relevant technical assumptions were not mentioned. Also, an important tool of an (equivalent) martingale measure was not yet employed.

5.1 Complete Constrained Model

In this subsection, it it assumed that $d = 1$, so that we have two independent sources of randomness, a one-dimensional Brownian motion W and a Poisson process N. We shall verify directly that, under natural additional conditions, the model $\mathcal{M}_1(Z)$ is arbitrage-free and complete with respect to $\mathbb{F}$, but the original model $\mathcal{M}$ is not minimally complete, so that a redundant primary asset exists in $\mathcal{M}$.

Lemma 5.1 *Assume that* $\delta := \det A \neq 0$, *where*

$$A = \begin{bmatrix} \sigma_2 - \sigma_4 & \kappa_2 - \kappa_4 \\ \sigma_3 - \sigma_4 & \kappa_3 - \kappa_4 \end{bmatrix}.$$

Then there exists a unique probability measure $\widetilde{\mathbb{P}}$, *equivalent to* $\mathbb{P}$ *on* $(\Omega, \mathcal{F}_T)$, *and such that the relative prices* $\bar{Y}^{2,1} = \bar{Y}^2/Y^1$ *and* $\bar{Y}^{3,1} = \bar{Y}^3/Y^1$ *of synthetic assets* $\bar{Y}^2$ *and* $\bar{Y}^3$ *are* $\widetilde{\mathbb{P}}$*-martingales.*

Proof. Let us write $\widehat{W}_t = W_t - \sigma_1 t$ and $\widehat{M}_t = M_t - \lambda\kappa_1 t$. By straightforward calculations, the relative value of the synthetic asset $\bar{Y}^i$ satisfies, for $i = 2, 3$,

$$d\bar{Y}^{i,1}_t = dY^{i,4,1}_t = Y^{i,1}_{t-}\Big((\mu_i - \mu_4)\,dt + (\sigma_i - \sigma_4)(dW_t - \sigma_1 dt) + \frac{\kappa_i - \kappa_4}{1 + \kappa_1}\,(dM_t - \lambda\kappa_1 dt)\Big), \tag{28}$$

or equivalently,

$$d\bar{Y}^{i,1}_t = Y^{i,1}_{t-}\Big((\mu_i - \mu_4)\,dt + (\sigma_i - \sigma_4)\,d\widehat{W}_t + \frac{\kappa_i - \kappa_4}{1 + \kappa_1}\,d\widehat{M}_t\Big).$$

By virtue of Girsanov's theorem, there exists a unique probability measure $\widehat{\mathbb{P}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, and such that the processes $\widehat{W}$ and $\widehat{M}$ follow $\mathbb{F}$-martingales under $\widehat{\mathbb{P}}$. Under our assumption $\delta := \det A \neq 0$, the equations

$$\mu_4 - \mu_i = (\sigma_4 - \sigma_i)\theta + \frac{\kappa_4 - \kappa_i}{1 + \kappa_1}\,\nu\lambda, \quad i = 2, 3, \tag{29}$$

uniquely specify θ and ν. Using once again Girsanov's theorem, we show that there exists a unique probability measure $\widetilde{\mathbb{P}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, and such that the processes $\widetilde{W}_t = \widehat{W}_t - \theta t = W_t - (\sigma_1 + \theta)t$ and

$$\widetilde{M}_t = \widehat{M}_t - \lambda\nu t = N_t - \lambda(1 + \kappa_1 + \nu)t$$

are $\mathbb{F}$-martingales under $\widetilde{\mathbb{P}}$. We then have, for $i = 2, 3$ and every $t \in [0, T]$,

$$d\bar{Y}^{i,1}_t = Y^{i,1}_{t-}\Big((\sigma_i - \sigma_4)\,d\widetilde{W}_t + \frac{\kappa_i - \kappa_4}{1 + \kappa_1}\,d\widetilde{M}_t\Big).$$

Note that N follows under $\widetilde{\mathbb{P}}$ a Poisson process with the constant intensity $\lambda(1 + \kappa_1 + \nu)$, and thus $\widetilde{M}$ is the compensated Poisson process under $\widetilde{\mathbb{P}}$. Moreover, under the present assumptions, the processes $\widetilde{W}$ and $\widetilde{M}$ are independent under $\widetilde{\mathbb{P}}$. □

From now on, we postulate that $\delta = \det A \neq 0$ and $\kappa_i > -1$ for every $i = 1, 2, \dots, 4$. Under this assumption, the filtration $\mathbb{F}$ coincides with the filtration $\mathbb{F}^Y$ generated by primary assets.

In the next result, we provide sufficient conditions for the existence of a replicating strategy satisfying the balance condition (27). Essentially, Proposition 5.1 shows that the model $\mathcal{M}_1(Z) = (Y^1, \bar{Y}^2, \bar{Y}^3, Y^4; \Phi_1(Z))$ is complete with respect to $\mathbb{F}$.

Proposition 5.1 *Let* X *be an* $\mathcal{F}_T$*-measurable contingent claim that settles at time* T. *Assume that the random variable* $\widehat{X}$, *given by the formula*

$$\widehat{X} = \frac{X}{Y^1_T} - \int_0^T \frac{Z_t}{Y^4_{t-}}\,dY^{4,1}_t, \tag{30}$$

is square-integrable under $\widetilde{\mathbb{P}}$, where $\widetilde{\mathbb{P}}$ is the unique probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that the relative prices $\bar{Y}^{2,1}$ and $\bar{Y}^{3,1}$ are $\widetilde{\mathbb{P}}$-martingales. Then X can be replicated in the model $\mathcal{M}_1(Z)$.

Proof. To prove the existence of a replicating strategy for X in the class $\Phi_1(Z)$, we may use either Proposition 3.1 (if we wish to work with traded assets Y^1, Y^2, Y^3, Y^4) or Corollary 3.2 and Lemma 5.1 (if we prefer to work with $Y^1, \bar{Y}^2, \bar{Y}^3, Y^4$). The second choice seems to be more convenient, and thus we shall focus on the existence a trading strategy $\psi = (\psi^1, \psi^2, \ldots, \psi^4)$ with the properties described in Corollary 3.2. In view of (24) and Corollary 3.2, it suffices to check that there exist a constant x, and $\mathbb{F}$-predictable processes ϕ^2 and ϕ^3 such that

$$\widehat{X} = x + \sum_{i=2}^{3} \int_0^T \phi_t^i \, d\bar{Y}_t^{i,1}. \tag{31}$$

To show that such processes exist, we shall use Lemma 5.1. It is crucial to observe that the pair $(\widetilde{W}, \widetilde{M})$, which was obtained in the proof of Lemma 5.2 from the original pair (W, M) by means of Girsanov's transformation, enjoys the predictable representation property (see, for example, Jacod and Shiryaev (2002), Sections III.4 and III.5). Since $\widehat{X}$ is square-integrable under $\widetilde{\mathbb{P}}$, there exists a constant x and $\mathbb{F}$-predictable processes ξ and ς such that

$$\widehat{X} = x + \int_0^T \xi_t \, d\widetilde{W}_t + \int_0^T \varsigma_t \, d\widetilde{M}_t.$$

Observe that

$$d\widetilde{W}_t = \delta^{-1} \left((\kappa_3 - \kappa_4) \frac{d\bar{Y}_t^{2,1}}{Y_{t-}^{2,1}} - (\kappa_2 - \kappa_4) \frac{d\bar{Y}_t^{3,1}}{Y_{t-}^{3,1}} \right) =: \Theta_t^2 \, d\bar{Y}_t^{2,1} + \Theta_t^3 \, d\bar{Y}_t^{3,1}$$

and

$$d\widetilde{M}_t = (1 + \kappa_1) \delta^{-1} \left((\sigma_2 - \sigma_4) \frac{d\bar{Y}_t^{3,1}}{Y_{t-}^{3,1}} - (\sigma_3 - \sigma_4) \frac{d\bar{Y}_t^{2,1}}{Y_{t-}^{2,1}} \right) =: \Psi_t^2 \, d\bar{Y}_t^{2,1} + \Psi_t^3 \, d\bar{Y}_t^{3,1}.$$

Hence, upon setting

$$\phi_t^2 = \xi_t \Theta_t^2 + \varsigma_t \Psi_t^2, \quad \phi_t^3 = \xi_t \Theta_t^3 + \varsigma_t \Psi_t^3,$$

we obtain the desired representation (31) for $\widehat{X}$. To complete the proof of the proposition, it suffices to make use of Corollary 3.2. □

Remark. If we take the class $\Phi_2(Z)$ of constrained strategies, instead of the class $\Phi_1(Z)$, then we need to show the existence of $\mathbb{F}$-predictable processes ϕ^2 and ϕ^3 such that

$$\widehat{X} = x + \int_0^T \phi_t^2 \, dY_t^{2,1} + \int_0^T \phi_t^3 \, d\bar{Y}_t^{3,1}. \tag{32}$$

To this end, it suffices to focus on an equivalent probability measure under which the relative prices $Y^{2,1}$ and $\bar{Y}^{3,1}$ are $\mathbb{F}$-martingales, and to follow the same steps as in the proof of Proposition 5.1.

In view of Lemma 5.1, the reduced model $\bar{\mathcal{M}}^4 = (Y^1, \bar{Y}^2, \bar{Y}^3; \Phi)$ admits a martingale measure $\widetilde{\mathbb{P}}$ corresponding to the choice of Y^1 as a numeraire asset, and thus it is arbitrage-free, under the usual choice of admissible trading strategies (e.g., the so-called *tame* strategies). By virtue of formula (7) in Proposition 2.1, for the arbitrage-free property of the model $\mathcal{M}_1(Z)$ to hold, it suffices, in addition, that the process

$$\int_0^t \frac{Z_u}{Y_{u-}^4} \, dY_u^{4,1}, \quad \forall \, t \in [0, T],$$

follows a martingale under $\widetilde{\mathbb{P}}$.

Note, however, that the above-mentioned property does not imply, in general, that the probability measure $\widetilde{\mathbb{P}}$ is a martingale measure for the relative price $Y^{4,1}$. Since

$$dY_t^{4,1} = Y_{t-}^{4,1}\Big((\mu_4-\mu_1)\,dt + (\sigma_4-\sigma_1)(dW_t-\sigma_1 dt) + \frac{\kappa_4-\kappa_1}{1+\kappa_1}\,(dM_t-\lambda\kappa_1 dt)\Big), \tag{33}$$

a martingale measure for the relative prices $\bar{Y}^{2,1}, \bar{Y}^{3,1}$ and $Y^{4,1}$ exists if and only if for the pair (θ,ν) that solves (29), we also have that

$$\mu_1-\mu_4 = (\sigma_4-\sigma_1)\theta + \frac{\kappa_4-\kappa_1}{1+\kappa_1}\,\nu\lambda.$$

This holds if and only if $\det \widehat{A}=0$, where $\widehat{A}$ is the following matrix

$$\widehat{A} = \begin{bmatrix} \mu_1-\mu_4 & \sigma_1-\sigma_4 & \kappa_1-\kappa_4 \\ \mu_2-\mu_4 & \sigma_2-\sigma_4 & \kappa_2-\kappa_4 \\ \mu_3-\mu_4 & \sigma_3-\sigma_4 & \kappa_3-\kappa_4 \end{bmatrix}.$$

Hence, the model $\bar{\mathcal{M}}$ (or, equivalently, the model $\mathcal{M}$) is not arbitrage-free, in general. In fact, $\mathcal{M}$ is arbitrage-free if and only if the primary asset Y^4 is redundant in $\mathcal{M}$. The following result summarizes our findings.

Proposition 5.2 *Let $\mathcal{M}$ be the model given by* (26). *Assume that $\kappa_i > -1$ for every $i=1,2,\dots,4$ and $\delta = \det A \neq 0$. Moreover, let the process*

$$\int_0^t \frac{Z_u}{Y_{u-}^4}\,dY_u^{4,1}$$

follow a martingale under $\widetilde{\mathbb{P}}$. Then the following statements hold.
(i) *The model $\mathcal{M}_1(Z)$ is arbitrage-free and complete, in the sense of Proposition 5.1.*
(ii) *If the model $\mathcal{M}$ is arbitrage-free then it is complete, in the sense that any $\mathcal{F}_T$-measurable random variable X such that $X(Y_T^1)^{-1}$ is square-integrable under $\widetilde{\mathbb{P}}$ is attainable in this model, but $\mathcal{M}$ is not minimally complete.*

Example 5.1 Consider, for instance, a call option written on the asset Y^4, so that $X=(Y_T^4-K)^+$, and let us assume that $Z_t = Y_{t-}^4$. Under assumptions of Proposition 5.2, models $\mathcal{M}$ and $\mathcal{M}_1(Z)$ are arbitrage-free and the asset Y^4 is redundant. It is thus rather clear that the option can be hedged by dynamic trading in primary assets Y^1, Y^2, Y^3 and by keeping at any time one unit of Y^4. Of course, the same conclusion applies to any European claim with Y^4 as the underlying asset.

5.2 Incomplete Constrained Model

We now assume that $d=2$, so that the number of independent sources of randomness is increased to three. In view of (26), we have, for $i=1,\dots,4$,

$$dY_t^i = Y_{t-}^i\big(\mu_i\,dt + \sigma_i^1\,dW_t^1 + \sigma_i^2\,dW_t^2 + \kappa_i\,dM_t\big).$$

We are going to check that under the set of assumptions making the unconstrained model $\mathcal{M}$ arbitrage-free and minimally complete, the constrained model $\mathcal{M}_l(Z)$ is also arbitrage-free, but it is incomplete. To this end, we first examine the existence and uniqueness of a martingale measure associated with the numeraire Y^1.

Lemma 5.2 *Assume that* $\det \widetilde{A} \neq 0$, *where the matrix $\widetilde{A}$ is given as*

$$\widetilde{A} = \begin{bmatrix} \sigma_1^1-\sigma_4^1 & \sigma_1^2-\sigma_4^2 & \kappa_1-\kappa_4 \\ \sigma_2^1-\sigma_4^1 & \sigma_2^2-\sigma_4^2 & \kappa_2-\kappa_4 \\ \sigma_3^1-\sigma_4^1 & \sigma_3^2-\sigma_4^2 & \kappa_3-\kappa_4 \end{bmatrix}.$$

Then there exists a unique probability measure $\widetilde{\mathbb{P}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, and such that the relative prices $\bar{Y}^{2,1} = \bar{Y}^2/Y^1$, $\bar{Y}^{3,1} = \bar{Y}^3/Y^1$ of synthetic assets $\bar{Y}^2, \bar{Y}^3$, and the relative price $Y^{4,1}$ of the primary asset Y^4 follow martingales under $\widetilde{\mathbb{P}}$.

Proof. Let us write

$$\widehat{W}_t = W_t - \sigma_1 t = (W_t^1, W_t^2) - (\sigma_1^1, \sigma_1^2)t$$

and $\widehat{M}_t = M_t - \lambda\kappa_1 t$. By straightforward calculations, the relative values $\bar{Y}^{i,1}$, $i = 2,3$ and $Y^{4,1}$ satisfy

$$d\bar{Y}_t^{i,1} = Y_{t-}^{i,1}\Big((\mu_i - \mu_4)\,dt + (\sigma_i - \sigma_4)\,d\widehat{W}_t + \frac{\kappa_i - \kappa_4}{1+\kappa_1}\,d\widehat{M}_t\Big)$$

and

$$dY_t^{4,1} = Y_{t-}^{4,1}\Big((\mu_4 - \mu_1)\,dt + (\sigma_4 - \sigma_1)(dW_t - \sigma_1 dt) + \frac{\kappa_4 - \kappa_1}{1+\kappa_1}\,(dM_t - \lambda\kappa_1 dt)\Big).$$

By virtue of Girsanov's theorem, there exists a unique probability measure $\widehat{\mathbb{P}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, and such that the processes $\widehat{W}$ and $\widehat{M}$ follow $\mathbb{F}$-martingales under $\widehat{\mathbb{P}}$. Now, let $\theta = (\theta^1, \theta^2)$ and ν be uniquely specified by the conditions

$$\mu_4 - \mu_i = (\sigma_i - \sigma_4)\theta + \frac{\kappa_i - \kappa_4}{1+\kappa_1}\,\nu\lambda, \quad i = 2,3,4.$$

Another application of Girsanov's theorem yields the existence of a unique probability measure $\widetilde{\mathbb{P}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, such that the processes $\widetilde{W}_t = \widehat{W}_t - \theta t = W_t - (\sigma_1 + \theta)t$ and

$$\widetilde{M}_t = \widehat{M}_t - \lambda\nu t = N_t - \lambda(1 + \kappa_1 + \nu)t$$

are $\mathbb{F}$-martingales under $\widetilde{\mathbb{P}}$. We then have, for $i = 2,3$ and every $t \in [0,T]$,

$$d\bar{Y}_t^{i,1} = Y_{t-}^{i,1}\Big((\sigma_i - \sigma_4)\,d\widetilde{W}_t + \frac{\kappa_i - \kappa_4}{1+\kappa_1}\,d\widetilde{M}_t\Big) \tag{34}$$

while

$$dY_t^{4,1} = Y_{t-}^{4,1}\Big((\sigma_4 - \sigma_1)d\widetilde{W}_t + \frac{\kappa_4 - \kappa_1}{1+\kappa_1}\,d\widetilde{M}_t\Big). \tag{35}$$

Note that N follows under $\widetilde{\mathbb{P}}$ a Poisson process with the constant intensity $\lambda(1 + \kappa_1 + \nu)$, and thus $\widetilde{M}$ is the compensated Poisson process under $\widetilde{\mathbb{P}}$. Moreover, under the present assumptions, the processes $\widetilde{W}$ and $\widetilde{M}$ are independent under $\widetilde{\mathbb{P}}$. □

It is clear that the inequality $\det \widetilde{A} \neq 0$ is a necessary and sufficient condition for the arbitrage-free property of the model $\mathcal{M}$. Under this assumption, we also have $\mathbb{F} = \mathbb{F}^Y$ and, as can be checked easily, the model $\mathcal{M}$ is minimally complete.

In the next result, we provide sufficient conditions for the existence of a replicating strategy satisfying the balance condition (27) with some predetermined process Z. In particular, it is possible to deduce from Proposition 5.3 that the model $\mathcal{M}_1(Z)$ is incomplete with respect to $\mathbb{F}$.

Proposition 5.3 *Assume that* $\det \widetilde{A} \neq 0$. *Let X be an $\mathcal{F}_T$-measurable contingent claim that settles at time T. Assume that the random variable $\widehat{X}$, given by the formula*

$$\widehat{X} := \frac{X}{Y_T^1} - \int_0^T \frac{Z_t}{Y_{t-}^4}\,dY_t^{4,1}, \tag{36}$$

is square-integrable under $\widetilde{\mathbb{P}}$, where $\widetilde{\mathbb{P}}$ is the unique probability measure, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, such that the relative prices $\bar{Y}^{2,1}, \bar{Y}^{3,1}$ and $Y^{4,1}$ follow martingales under $\widetilde{\mathbb{P}}$. Then X can be replicated in $\mathcal{M}_1(Z)$ if and only if the process ϕ^4 given by formula (40) *below vanishes identically.*

Proof. We shall use similar arguments as in the proof of Proposition 5.1. In view of Corollary 3.2, we need to check that there exist a constant x, and $\mathbb{F}$-predictable processes ϕ^2 and ϕ^3 such that

$$\widehat{X} = x + \sum_{i=2}^{3} \int_0^T \phi_t^i \, d\bar{Y}_t^{i,1}. \tag{37}$$

Note that the pair $(\widetilde{W}, \widetilde{M})$ introduced in the proof of Lemma 5.2 has the predictable representation property. Since $\widehat{X}$ is square-integrable under $\widetilde{\mathbb{P}}$, there exists a constant x and $\mathbb{F}$-predictable processes ξ and ς such that

$$\widehat{X} = x + \int_0^T \xi_t \, d\widetilde{W}_t + \int_0^T \varsigma_t \, d\widetilde{M}_t. \tag{38}$$

In view of (34)-(35), we have

$$\begin{bmatrix} d\widetilde{W}_t^1 \\ d\widetilde{W}_t^2 \\ d\widetilde{M}_t \end{bmatrix} = \widetilde{A}^{-1} \begin{bmatrix} (Y_{t-}^{2,1})^{-1} \, d\bar{Y}_t^{2,1} \\ (Y_{t-}^{3,1})^{-1} \, d\bar{Y}_t^{3,1} \\ (Y_{t-}^{4,1})^{-1} \, dY_t^{4,1} \end{bmatrix},$$

so that there exist $\mathbb{F}$-predictable processes $\Psi^i, \Lambda^i, \Theta^i$, $i = 2, 3, 4$ such that

$$\begin{aligned} d\widetilde{W}_t^1 &= \Theta_t^2 \, d\bar{Y}_t^{2,1} + \Theta_t^3 \, d\bar{Y}_t^{3,1} + \Theta_t^4 \, dY_t^{4,1}, \\ d\widetilde{W}_t^2 &= \Lambda_t^2 \, d\bar{Y}_t^{2,1} + \Lambda_t^3 \, d\bar{Y}_t^{3,1} + \Lambda_t^4 \, dY_t^{4,1}, \\ d\widetilde{M}_t &= \Psi_t^2 \, d\bar{Y}_t^{2,1} + \Psi_t^3 \, d\bar{Y}_t^{3,1} + \Psi_t^4 \, dY_t^{4,1}. \end{aligned} \tag{39}$$

Let us set, for $i = 2, 3, 4$,

$$\phi_t^i = \xi_t^1 \Theta_t^i + \xi_t^2 \Lambda_t^i + \varsigma_t \Psi_t^i, \quad \forall \, t \in [0, T]. \tag{40}$$

Suppose first that $\phi_t^4 = 0$ for every $t \in [0, T]$. Then, by combining (38), (39) and (40), we end up with the desired representation (37) for $\widehat{X}$. To show the existence of a replicating strategy for X in $\mathcal{M}_1(Z)$, it suffices to apply Corollary 3.2. If, on the contrary, ϕ^4 does not vanish identically, equality (37) cannot hold for any choice of ϕ^2 and ϕ^3. The fact that ϕ^4 is non-vanishing for some claims follows from Proposition 4.1. □

In general, i.e., when the component ϕ^4 does not vanish, we get the following representation

$$\frac{X}{Y_T^1} = x + \sum_{i=2}^{3} \int_0^T \phi_t^i \, d\bar{Y}_t^{i,1} + \int_0^T \widetilde{\phi}_t^4 \, dY_t^{4,1}, \tag{41}$$

where we set $\widetilde{\phi}_t^4 = \phi_t^4 + Z_t (Y_{t-}^4)^{-1}$. Hence, as expected any contingent claim satisfying a suitable integrability condition is attainable in the unconstrained model $\mathcal{M}$.

Example 5.2 To get a concrete example of a non-attainable claim in $\mathcal{M}_1(Z)$, let us take $X = (Y_T^4 - K)^+$ and $Z_t = Y_{t-}^4$. Then, for $K = Y_0^4$, we obtain $\widehat{X} = (Y_0^4 - Y_T^4)^+ (Y_T^1)^{-1}$, and thus we formally deal with the put option written on Y^4 with strike Y_0^4. We claim that $\widehat{X}$ does not admit representation (37). Indeed, equality (37) implies that the hedge ratio of a put option with respect to the underlying asset equals zero. This may happen only if the underlying asset is redundant so that hedging can be done with other primary assets, and this is not the case in our model.

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