COMPLETENESS OF A REDUCED-FORM CREDIT RISK MODEL WITH DISCONTINUOUS ASSET PRICES

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First draft: December 15, 2004

Revised version: August 20, 2005

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[*] The research of the first author was supported by NSF Grant 0202851 and by Moody's Corporation grant 5-55411.

[†] The research of the second author was supported by Moody's Corporation grant 5-55411.

Contents

1 Introduction

The goal of this work is to examine the issue of attainability of a generic defaultable claim within the reduced-form approach to credit risk modeling, as well as a closely related issue of market completeness. In contrast to our previous work Bielecki et al. (2004a), we consider here the case where the prices of default-free assets and the pre-default values of defaultable assets follow general (that is, not necessarily continuous) semimartingales.

In Section 2, we concentrate on trading in primary assets with discontinuous prices. Our main goals is to analyze the issue of replication of a generic contingent claim by means a self-financing trading strategy that is additionally subject to an algebraic constraint, interpreted as the balance condition. We examine the relationships between the completeness of a market model with unconstrained strategies, and a corresponding market with strategies satisfying an exogenous constraint, referred to as the *balance condition.* Though in this section we do not deal with specific issues related to the modeling of credit risk, it is nevertheless essential for a full appreciation of results of Section 3. For the proofs of all results presented in Section 2, the interested reader is referred to Bielecki et al. (2005a), where completeness of a generic market model is studied. Let us only mention that the case of credit risk model has peculiarities, which do not show up explicitly when dealing with a general set-up.

In Section 3, we also include defaultable assets in our portfolio. In this case, our primary goal is to examine the issue of replication of a generic defaultable contingent claim. The crucial property of a replicating strategy is that has to satisfy a suitable version of the balance condition (see equation (17) in Section 3). Its role is to ensure that a portfolio of default-free and defaultable assets is continuously rebalanced in such a way that, in case of default occurring at some random time, the post-default wealth will match the recovery payoff of a hedged defaultable claim.

In addition, we establish the basic results on completeness of a credit risk model within the reduced-form set-up. We conclude by analyzing particular examples of survival claims under deterministic and stochastic intensities. Let us emphasize that we focus on a generic samimartingale model, and we use throughout a purely probabilistic approach. An alternative PDE approach to the valuation and hedging of credit derivatives, applicable within a Markovian set-up, is studied in a companion paper Bielecki et al. (2005b). Also, we focus here on the case of a single default time; and extension to the case of several default times and basket credit derivatives can be dealt with in a similar way, by imposing a family of balance conditions, as briefly explained in Section 4.4 of Bielecki et al. (2005a).

Until recently, the important issue of hedging of credit derivatives in a reduced-form credit risk model was rather rarely addressed in the mathematical literature devoted to credit risk. Vaillant (2001) was among the first to examine this issue, but his study of hedging strategies is rather informal. Replication of defaultable claims was later studied at a very general level of abstraction by Bélanger el al. (2004) and Blanchet-Scalliet and Jeanblanc (2004), who focused on a suitable version of a predictable representation theorem. Also, in our previous paper Bielecki et al. (2004a), we examined various alternative approaches to hedging of defaultable claims, including examples of hedging under incompleteness.

The contribution of the present work is twofold. First, we deal here with hedging of defaultable claim in a credit risk model with discontinuous asset prices. It is well known that in the case of option pricing models, allowance for discontinuous asset prices can be seen as an efficient way of dealing with a smile (or a skew) of the implied volatility of traded options.

Second, we provide some insight to the issue of completeness of a reduced-form credit risk model. When dealing with the market completeness, we emphasize the distinct features of reduced-form models based on deterministic and stochastic default intensities. To the best of our knowledge, completeness of a reduced-form credit risk model was not previously analyzed in the existing literature, and thus this work can be seen as a first step in the study of this important issue.

2 Generic Market Model

Our goal in this section is to present some auxiliary results related to the concept of constrained trading strategies for a generic market model. As already mentioned, these results were established in a paper by Bielecki et al. (2005a). They are summarized here for the sake of a reader's convenience.

Let $Y^1_t, Y^2_t, \dots, Y^k_t$ represent cash values at time t of k assets. We postulate that the prices $Y^1, Y^2, \dots, Y^k$ follow semimartingales on some probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions. We set $X_{0-} = X_0$ for any stochastic process X, and we only consider semimartingales with càdlàg sample paths. We assume, in addition that at least one of the processes Y^1, $Y^2, \dots, Y^k$, say process Y^1, is strictly positive, so that it can be chosen as a *numéraire asset*. Unless explicitly stated otherwise, we do not assume that a market model is complete.

2.1 Unconstrained Trading Strategies

We consider trading within the interval $[0,T]$ for some finite horizon date $T > 0$. Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a generic trading strategy; in particular, the processes $\phi^1, \phi^2, \dots, \phi^k$ are $\mathbb{F}$-predictable. The wealth $V_t(\phi)$ at time t of the trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals

$$V_t(\phi) = \sum_{i=1}^k \phi^i_t Y^i_t, \quad \forall\, t \in [0,T], \tag{1}$$

and ϕ is said to be a *self-financing strategy* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^k \int_0^t \phi^i_u \, dY^i_u, \quad \forall\, t \in [0,T]. \tag{2}$$

Let Φ be the class of all self-financing trading strategies. By combining the last two formulae, we obtain the following expression for the dynamics of the wealth process of a strategy $\phi \in \Phi$

$$dV_t(\phi) = \Big(V_t(\phi) - \sum_{i=2}^k \phi^i_t Y^i_t \Big) (Y^1_t)^{-1} \, dY^1_t + \sum_{i=2}^k \phi^i_t \, dY^i_t.$$

The representation above shows that the wealth process $V(\phi)$ depends only on $k-1$ components of ϕ. Choosing Y^1 as a numéraire asset, and denoting $V^1_t(\phi) = V_t(\phi)(Y^1_t)^{-1}$, $Y^{i,1}_t = Y^i_t (Y^1_t)^{-1}$, we get the following well-known result showing that the self-financing feature of a trading strategy is invariant with respect to the choice of a numéraire asset. The proof of Lemma 2.1 is well known in the case of continuous semi-martingale. In the case of discontinuous processes, the proof of part (i) is given in Protter (2001) and the proof of part (ii) in Bielecki et al. (2005a).

Lemma 2.1 (i) *For any $\phi \in \Phi$, we have*

$$V^1_t(\phi) = V^1_0(\phi) + \sum_{i=2}^k \int_0^t \phi^i_u \, dY^{i,1}_u, \quad \forall\, t \in [0,T]. \tag{3}$$

(ii) *Conversely, let X be an $\mathcal{F}_T$-measurable random variable, and let us assume that there exists $x \in \mathbb{R}$ and $\mathbb{F}$-predictable processes ϕ^i, $i = 2, 3, \dots, k$ such that*

$$X = Y^1_T \left(x + \sum_{i=2}^k \int_0^T \phi^i_t \, dY^{i,1}_t \right).$$

Then there exists an $\mathbb{F}$-predictable process ϕ^1 such that the strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ is self-financing and replicates X. Moreover, the wealth process of ϕ satisfies $V_t(\phi) = V^1_t Y^1_t$, where the process V^1 is given by

$$V^1_t = x + \sum_{i=2}^k \int_0^t \phi^i_u \, dY^{i,1}_u, \quad \forall\, t \in [0,T]. \tag{4}$$

2.2 Constrained Trading Strategies

In this section, we make an additional assumption that the price process Y^k is strictly positive. Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing trading strategy satisfying the following constraint:

$$\sum_{i=l+1}^{k} \phi_t^i Y_{t-}^i = Z_t, \quad \forall\, t \in [0,T], \tag{5}$$

for some $1 \le l \le k-1$ and a predetermined, $\mathbb{F}$-predictable process Z. In the financial interpretation, equality (5) means that the portfolio ϕ is rebalanced in such a way that the total wealth invested in securities $Y^{l+1}, Y^{l+2}, \dots, Y^k$ matches a predetermined stochastic process. For this reason, the constraint (5) is referred to as the *balance condition.* In the next section, a suitable version of this condition will be used to ensure that a properly chosen portfolio of default-free assets insures against the liability associated with a premature default.

Our first goal is to extend part (i) in Lemma 2.1 to the case of constrained strategies. Let $\Phi_l(Z)$ stand for the class of all self-financing trading strategies satisfying the balance condition (5). They will be sometimes referred to as *constrained strategies.* Since any strategy $\phi \in \Phi_l(Z)$ is self-financing, we have

$$\Delta V_t(\phi) = \sum_{i=1}^{k} \phi_t^i \Delta Y_t^i = V_t(\phi) - \sum_{i=1}^{k} \phi_t^i Y_{t-}^i,$$

and thus we deduce from (5) that

$$V_{t-}(\phi) = \sum_{i=1}^{k} \phi_t^i Y_{t-}^i = \sum_{i=1}^{l} \phi_t^i Y_{t-}^i + Z_t.$$

Let us write $Y_t^{i,1} = Y_t^i (Y_t^1)^{-1}$, $Y_t^{i,k} = Y^i (Y_t^k)^{-1}$, $Z_t^1 = Z_t (Y_t^1)^{-1}$. It is apparent from Proposition 2.1 that the wealth process $V(\phi)$ of a strategy $\phi \in \Phi_l(Z)$ depends only on $k-2$ components of ϕ.

Proposition 2.1 *The relative wealth $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ of a strategy $\phi \in \Phi_l(Z)$ satisfies*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right) + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1}. \tag{6}$$

2.3 Replication with Constrained Strategies

The next result is essentially a converse to Proposition 2.1. Also, it extends part (ii) of Lemma 2.1 to the case of constrained trading strategies. As in Section 2.2, we assume that $1 \le l \le k-1$, and Z is a predetermined, $\mathbb{F}$-predictable process. For the sake of notational simplicity, we shall write

$$Y_t^{i,k,1} = \int_0^t \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right). \tag{7}$$

Proposition 2.2 *Let an $\mathcal{F}_T$-measurable random variable X represent a contingent claim that settles at time T. Assume that there exist $\mathbb{F}$-predictable processes ϕ^i, $i = 2, 3, \dots, k-1$ such that*

$$X = Y_T^1 \left(x + \sum_{i=2}^{l} \int_0^T \phi_t^i \, dY_t^{i,1} + \sum_{i=l+1}^{k-1} \int_0^T \phi_t^i \, dY_t^{i,k,1} + \int_0^T \frac{Z_t^1}{Y_{t-}^{k,1}} \, dY_t^{k,1} \right). \tag{8}$$

Then there exist the $\mathbb{F}$-predictable processes ϕ^1 and ϕ^k such that the strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ belongs to $\Phi_l(Z)$ and replicates X. The wealth process of ϕ equals, for every $t \in [0,T]$,

$$V_t(\phi) = Y_t^1 \left(x + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \, dY_u^{i,k,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right). \tag{9}$$

Note that equality (8) is a necessary (by Proposition 2.1) and sufficient (by Proposition 2.2) condition for the existence of a constrained strategy replicating a given contingent claim X.

2.4 Synthetic Assets

Let us fix i, and let us analyze the auxiliary process $Y^{i,k,1}$ given by formula (7). We claim that this process can be interpreted as the relative wealth of a specific self-financing trading strategy associated with $Y^1, Y^2, \dots, Y^k$. Specifically, we will show that for any $i = 2, 3, \dots, k-1$ the process $\bar{Y}^{i,k,1}$, given by the formula

$$\bar{Y}^{i,k,1}_t = Y^1_t Y^{i,k,1}_t = Y^1_t \int_0^t \left(dY^{i,1}_u - \frac{Y^{i,1}_{u-}}{Y^{k,1}_{u-}} \, dY^{k,1}_u \right),$$

represents the price of a *synthetic asset.* For brevity, we shall frequently write $\bar{Y}^i$ instead of $\bar{Y}^{i,k,1}$.

2.4.1 Equivalence of Primary and Synthetic Assets

Our goal is to show that trading in primary assets is formally equivalent to trading in synthetic assets. The first result shows that the process $\bar{Y}^i$ can be obtained from primary assets Y^1, Y^i and Y^k through a simple self-financing strategy. This justifies the name synthetic asset given to $\bar{Y}^i$.

Lemma 2.2 *For any fixed $i = 2, 3, \dots, k-1$, let an $\mathcal{F}_T$-measurable random variable $\bar{Y}^i_T$ be given as*

$$\bar{Y}^i_T = Y^1_T Y^{i,k,1}_T = Y^1_T \int_0^T \left(dY^{i,1}_t - \frac{Y^{i,1}_{t-}}{Y^{k,1}_{t-}} \, dY^{k,1}_t \right). \tag{10}$$

Then there exists a strategy $\phi \in \Phi_1(0)$ that replicates the claim $\bar{Y}^i_T$. Moreover, we have, for every $t \in [0, T]$,

$$V_t(\phi) = Y^1_t Y^{i,k,1}_t = Y^1_t \int_0^t \left(dY^{i,1}_u - \frac{Y^{i,1}_{u-}}{Y^{k,1}_{u-}} \, dY^{k,1}_u \right) = \bar{Y}^i_t. \tag{11}$$

Note that to replicate the claim $\bar{Y}^i_T = \bar{Y}^{i,k,1}_T$, it suffices to invest in primary assets Y^1, Y^i and Y^k. Essentially, we start with zero initial endowment, we keep at any time one unit of the ith asset, we rebalance the portfolio in such a way that the total wealth invested in the ith and kth assets is always zero, and we put the residual wealth in the first asset. Hence, we deal here with a specific strategy such that the risk of the ith asset is perfectly offset by rebalancing the investment in the kth asset, and our trades are financed by taking positions in the first asset.

Note that the process $Y^{i,1}$ satisfies the following SDE

$$Y^{i,1}_t = Y^{i,1}_0 + \bar{Y}^{i,1}_t + \int_0^t \frac{Y^{i,1}_{u-}}{Y^{k,1}_{u-}} \, dY^{k,1}_u, \tag{12}$$

which is known to possess a unique strong solution. Hence, the relative price $Y^{i,1}_t$ at time t is uniquely determined by the initial value $Y^{i,1}_0$ and processes $\bar{Y}^{i,1}$ and $Y^{k,1}$. Consequently, the price Y^i_t at time t of the ith primary asset is uniquely determined by the initial value Y^i_0, the prices Y^1, Y^k of primary assets, and the price $\bar{Y}^i$ of the ith synthetic asset.

Lemma 2.3 *Filtrations generated by the primary assets $Y^1, Y^2, \dots, Y^k$ and by the price processes $Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$ coincide.*

Lemma 2.3 suggests that for any choice of the underlying filtration $\mathbb{F}$ (such that $\mathbb{F}^Y \subseteq \mathbb{F}$), trading in assets $Y^1, Y^2, \dots, Y^k$ is essentially equivalent to trading in $Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$. Let us first formally define the equivalence of market models.

Definition 2.1 We say that the two models, $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ say, are *equivalent with respect to a filtration* $\mathbb{F}$ if both models are defined on a common probability space and primary assets in $\mathcal{M}$ can be obtained by trading in primary assets in $\widetilde{\mathcal{M}}$ and vice versa, under the assumption that trading strategies are $\mathbb{F}$-predictable.

Note that we do not assume that models $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ have the same number of primary assets. The next result justifies our claim of equivalence of primary and synthetic assets.

Corollary 2.1 *Models* $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ *and* $\bar{\mathcal{M}} = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi)$ *are equivalent.*

2.4.2 Replicating Strategies with Synthetic Assets

In view of Lemma 2.2, the replicating trading strategy for a contingent claim X, for which (8) holds, can be conveniently expressed in terms of primary securities $Y^1, Y^2, \dots, Y^l$ and Y^k, and synthetic assets $\bar{Y}^{l+1}, \bar{Y}^{l+2}, \dots, \bar{Y}^{k-1}$. To this end, we represent (8)-(9) in the following way:

$$X = Y_T^1 \left(x + \sum_{i=2}^{l} \int_0^T \phi_t^i \, dY_t^{i,1} + \sum_{i=l+1}^{k-1} \int_0^T \phi_t^i \, d\bar{Y}_t^{i,1} + \int_0^T \frac{Z_t^1}{Y_{t-}^{k,1}} \, dY_t^{k,1} \right) \tag{13}$$

where $\bar{Y}_t^{i,1} = \bar{Y}_t^i / Y_t^1 = Y_t^{i,k,1}$, and

$$V_t(\phi) = Y_t^1 \left(x + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \, d\bar{Y}_u^{i,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} \, dY_u^{k,1} \right). \tag{14}$$

Corollary 2.2 *Let* X *be an* $\mathcal{F}_T$*-measurable random variable such that* (13) *holds for some* $\mathbb{F}$*-predictable process* Z *and some* $\mathbb{F}$*-predictable processes* $\phi^2, \phi^3, \dots, \phi^{k-1}$. *Let* $\psi^i = \phi^i$ *for* $i = 2, 3, \dots, k-1$,

$$\psi_t^k = \frac{Z_t^1}{Y_{t-}^{k,1}} = \frac{Z_t}{Y_{t-}^k},$$

and

$$\psi_t^1 = V_t^1 - \sum_{i=2}^{l} \psi_t^i Y_t^{i,1} - \sum_{i=l+1}^{k-1} \psi_t^i \bar{Y}_t^{i,1} - \psi_t^k Y_t^{k,1} = V_{t-}^1 - \sum_{i=2}^{l} \psi_t^i Y_{t-}^{i,1} - \sum_{i=l+1}^{k-1} \psi_t^i \bar{Y}_{t-}^{i,1} - \psi_t^k Y_{t-}^{k,1}.$$

Then $\psi = (\psi^1, \psi^2, \dots, \psi^k)$ *is a self-financing trading strategy in assets* $Y^1, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$. *Moreover,* ψ *satisfies* $\psi_t^k Y_{t-}^k = Z_t$, $t \in [0, T]$, *and it replicates* X.

2.5 Model Completeness

We shall now examine the relationship between the arbitrage-free property and completeness of a market model in which trading is restricted a priori to self-financing strategies satisfying the balance condition. A good understanding of these features is crucial from the viewpoint of further applications to replication of defaultable claims.

2.5.1 Minimal Completeness of an Unconstrained Model

Let $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ be an arbitrage-free market model. Unless explicitly stated otherwise, Φ stands for the class of all $\mathbb{F}$-predictable, self-financing strategies. Note, however, that the number of traded assets and their selection may be different for each particular model. Consequently, the dimension of a strategy $\phi \in \Phi$ will depend on the number of traded assets in a given model. For the sake of brevity, this feature is not reflected in our notation.

Definition 2.2 We say that a model $\mathcal{M}$ is *complete with respect to* $\mathbb{F}$ (briefly, $\mathbb{F}$-*complete*) if any bounded $\mathcal{F}_T$-measurable contingent claim X is attainable in $\mathcal{M}$. Otherwise, a model $\mathcal{M}$ is said to be *incomplete* with respect to $\mathbb{F}$.

Definition 2.3 An $\mathbb{F}$-complete model $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ is *minimally complete* with respect to $\mathbb{F}$ if for any choice of trading strategies $\phi_i \in \Phi$, $i = 1, 2, \dots, k-1$, the *reduced model* $\widehat{\mathcal{M}}^{k-1} = (\widehat{Y}^1, \widehat{Y}^2, \dots, \widehat{Y}^{k-1}; \Phi)$ where $\widehat{Y}^i = V(\phi_i)$, is incomplete with respect to $\mathbb{F}$. In this case, we say that the *degree of completeness* of $\mathcal{M}$ equals k.

Let us stress that trading strategies in the reduced model $\widehat{\mathcal{M}}^{k-1}$ are predictable with respect to $\mathbb{F}$, rather than with respect to the filtration generated by the price processes $\widehat{Y}^1, \widehat{Y}^2, \dots, \widehat{Y}^{k-1}$. Hence, the number of traded asset in $\widehat{\mathcal{M}}^{k-1}$ is reduced, but the original information structure $\mathbb{F}$ is preserved. Minimal completeness of a model $\mathcal{M}$ means, in particular, that all primary assets $Y^1, Y^2, \dots, Y^k$ are needed if we wish to generate the class of all (bounded) $\mathcal{F}_T$-measurable claims through $\mathbb{F}$-predictable trading strategies. The next result shows that the degree of completeness is a well-defined notion, in the sense that it does not depend on the choice of traded assets, provided that the model completeness is preserved.

Lemma 2.4 *Let a model* $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ *be minimally complete with respect to* $\mathbb{F}$. *Let* $\widetilde{\mathcal{M}} = (\widetilde{Y}^1, \widetilde{Y}^2, \dots, \widetilde{Y}^k; \Phi)$, *where the processes* $\widetilde{Y}^i = V(\phi_i)$, $i = 1, 2, \dots, k$ *represent the wealth processes of some trading strategies* $\phi_1, \phi_2, \dots, \phi_k \in \Phi$. *If a model* $\widetilde{\mathcal{M}}$ *is complete with respect to* $\mathbb{F}$ *then it is also minimally complete with respect to* $\mathbb{F}$, *and thus its degree of completeness equals* k.

By combining Lemma 2.4 with Corollary 2.1, we obtain the following result.

Corollary 2.3 *A model* $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ *is minimally complete if and only if a model* $\bar{\mathcal{M}} = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi)$ *has this property.*

2.5.2 Completeness of a Constrained Model

Let $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ be an arbitrage-free market model, and let us denote by $\mathcal{M}_l(Z) = (Y^1, Y^2, \dots, Y^k; \Phi_l(Z))$ the associated model in which the class Φ is replaced by the class $\Phi_l(Z)$ of constrained strategies. We claim that if $\mathcal{M}$ is arbitrage-free and minimally complete with respect to the filtration $\mathbb{F} = \mathbb{F}^Y$, where $Y = (Y^1, Y^2, \dots, Y^k)$, then the constrained model $\mathcal{M}_l(Z)$ is arbitrage-free, but it is incomplete with respect to $\mathbb{F}$. Conversely, if the model $\mathcal{M}_l(Z)$ is arbitrage-free and complete with respect to $\mathbb{F}$, then the original model $\mathcal{M}$ is not minimally complete. To prove these claims, we need some preliminary results.

The following definition extends the notion of equivalence of security market models to the case of constrained trading.

Definition 2.4 We say that the two constrained models are *equivalent with respect to a filtration* $\mathbb{F}$ if they are defined on a common probability space and the class of all wealth processes of $\mathbb{F}$-predictable constrained trading strategies is the same in both models.

In view of the next result, the constrained models $\bar{\mathcal{M}}_{k-1}(Z)$ and $\mathcal{M}_l(Z)$ are equivalent. Typically, the latter model is easier to handle than the former.

Corollary 2.4 *A constrained model* $\mathcal{M}_l(Z) = (Y^1, Y^2, \dots, Y^k; \Phi_l(Z))$ *is equivalent to a constrained model* $\bar{\mathcal{M}}_{k-1}(Z) = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi_{k-1}(Z))$.

Proposition 2.3 (i) *Assume that a model* $\mathcal{M}$ *is arbitrage-free and minimally complete. Then for any* $\mathbb{F}$-*predictable process* Z *and any* $l = 1, 2, \dots, k-1$ *a constrained model* $\mathcal{M}_l(Z)$ *is arbitrage-free and incomplete.*
(ii) *Assume that a constrained model* $\mathcal{M}_l(Z)$ *associated with* $\mathcal{M}$ *is arbitrage-free and complete. Then* $\mathcal{M}$ *is either not arbitrage-free or not minimally complete.*

3 Credit Risk Model

Our next goal is to extend the results of the previous section to the case of a market model with default-free and defaultable primary assets. As already mentioned, replication of defaultable claims in credit risk models with continuous prices of default-free assets and pre-default values of defaultable assets was examined in Part I of Bielecki et al. (2004a).

3.1 Self-financing Trading Strategies

Let τ be a strictly positive random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, referred to as a *default time*. In order to exclude trivial cases, we assume that $\mathbb{Q}\{\tau > 0\} = 1$ and $\mathbb{Q}\{\tau \leq T\} > 0$. We introduce the jump process $H_t = \mathbb{1}_{\{\tau \leq t\}}$, and we denote by $\mathbb{H}$ the filtration generated by this process.

In addition, we are given a *reference filtration* $\mathbb{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ for every $t \in [0,T]$. We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as the *full filtration*; it includes the observations of default event. We assume that any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale. Such an assumption is sometimes called the *H hypothesis*. For more details on this assumption, we refer to Bielecki et al. (2004b).

We assume that we are given a family $Y^1, Y^2, \dots, Y^k$ of semimartingales defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$. We interpret $Y^1, Y^2, \dots, Y^m$ as price processes of m defaultable assets with a common default time τ. An extension to the case of several default times and basket credit derivatives is rather straightforward, but it is notationally cumbersome, and thus is it not discussed here. Processes $Y^{m+1}, Y^{m+2}, \dots, Y^k$ represent the prices of $k-m$ default-free assets; it is thus natural to postulate that they are $\mathbb{F}$-adapted. We also make the following standing assumption.

Assumption (A): For every $i = m+1, m+2, \dots, k$, we have $\Delta Y^i_\tau = Y^i_\tau - Y^i_{\tau-} = 0$.

Remark. It is important to stress that the term 'defaultable asset' should not be taken literally. For instance, in the case of the so-called *flight to quality*, the price of a risk-free bond jumps at the moment τ associated with some 'default event' (see, for instance, Collin-Dufresne et al. (2004)). Hence, it is convenient to formally classify a risk-free bond as a 'defaultable asset' if its price process is subject to a jump at default time τ.

The definition of a *self-financing* trading strategy ϕ is exactly the same as before (see formula (2)), except that now the components $\phi^1, \phi^2, \dots, \phi^k$ are $\mathbb{G}$-predictable, rather than $\mathbb{F}$-predictable. However, in this work we shall never deal with a trading strategy after the random time $\tau \wedge T$ representing the *effective maturity* of a defaultable claim (see, in particular, Definition 3.3 below). Hence, we may and do assume, without loss of generality, that the components of a trading strategy ϕ are $\mathbb{F}$-predictable processes (cf. Bielecki et al. (2004a)). We assume throughout that the market model $\mathcal{M} = (Y^1, Y^2, \dots, Y^k, \Phi)$ is arbitrage-free in the standard sense. For the sake of expositional simplicity, we make the following assumption regarding the recovery at default.

Assumption (B): We assume throughout the defaultable assets $Y^1, Y^2, \dots, Y^m$ are subject to the zero recovery at default, so that $Y^i_t = 0$ for every $t \geq \tau$ and $i = 1, 2, \dots, m$. We write $\widetilde{Y}^i$ to denote the *pre-default value* of the ith defaultable asset, that is, the unique $\mathbb{F}$-predictable process such that $Y^i_t = (1 - H_t)\widetilde{Y}^i_t$ for every $t \in [0,T]$.

Under Assumption (B), prices $Y^1, Y^2, \dots, Y^m$ of defaultable assets vanish for $t \geq \tau$ and, typically, $\Delta Y^i_\tau = -Y^i_{\tau-} = -\widetilde{Y}^i_{\tau-} \neq 0$. Since the value of each defaultable asset jumps to zero after default, it makes no sense to assume that these assets are also traded after default. Hence, the pre-default value can be interpreted as the *traded value* of a defaultable asset. Let us stress, however, that the concept of a pre-default value is useful also in the case of non-zero recovery. In fact, it appears to be a convenient and natural object to work with if we are interested in hedging a defaultable claim up to (and including) default time. With this motivation in mind, we introduce the notion of the pre-default wealth of a trading strategy.

Definition 3.1 The *pre-default wealth* $\widetilde{V}(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals

$$\widetilde{V}_t(\phi) = \sum_{i=1}^{m} \phi_t^i \widetilde{Y}_t^i + \sum_{i=m+1}^{k} \phi_t^i Y_t^i, \quad \forall\, t \in [0,T].$$

A strategy ϕ is said to be *self-financing prior to default time* if

$$\widetilde{V}_t(\phi) = \widetilde{V}_0(\phi) + \sum_{i=1}^{m} \int_0^t \phi_u^i \, d\widetilde{Y}_u^i + \sum_{i=m+1}^{k} \int_0^t \phi_u^i \, dY_u^i, \quad \forall\, t \in [0,T].$$

Note that $\widetilde{V}_0(\phi) = V_0(\phi)$, since $\mathbb{Q}\{\tau > 0\} = 1$. Let us stress that if a trading strategy ϕ is self-financing prior to default time in the sense of Definition 3.1, then it may be easily extended to a self-financing strategy on the interval $[0,T]$. Indeed, we may and do postulate that all trading activities are stopped at the effective maturity of a claim, and thus the terminal wealth at time $\tau \wedge T$ equals

$$V_{\tau\wedge T}(\phi) = \sum_{i=1}^{k} \phi_{\tau\wedge T}^i Y_{\tau\wedge T}^i.$$

In particular, on the event $\{\tau > T\}$ we have

$$V_{\tau\wedge T}(\phi) = V_T(\phi) = \widetilde{V}_T(\phi) = \sum_{i=1}^{m} \phi_T^i \widetilde{Y}_T^i + \sum_{i=m+1}^{k} \phi_T^i Y_T^i.$$

Hence, we do not distinguish in what follows between the concepts of a self-financing trading strategy and a trading strategy that is self-financing prior to default time.

3.2 Replication of a Defaultable Claim

We are interested in determining a replicating strategy for a generic defaultable claim. Formally, a generic defaultable claim can thus be represented as a triple (X, Z, τ), with the following financial interpretation of its components. The *default time* τ specifying the random time of default and thus also the default events $\{\tau \le t\}$ for every $t \in [0,T]$. It is always assumed that τ is strictly positive with probability 1. The *promised payoff* X is an $\mathcal{F}_T$-measurable random variable representing the random payoff received by the owner of the claim at time T, if there was no default prior to or at time T. The actual payoff at time T associated with X thus equals $X\mathbb{1}_{\{\tau>T\}}$. An $\mathbb{F}$-predictable *recovery process* Z specifies the recovery payoff Z_τ received by the owner of a claim at time of default, provided that the default occurs prior to or at maturity date T.

According to the interpretation above, the *dividend process* D, which represents the cash flows associated with a defaultable claim (X, Z, τ), is given by the formula

$$D_t = X\mathbb{1}_{\{\tau>T\}}\mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} Z_u \, dH_u,$$

where the integral is the (stochastic) Stieltjes integral. All natural integrability conditions are implicitly assumed to hold with regard to X and Z, so that the price process given by the following definition exists.

Definition 3.2 The *ex-dividend price process* U of a defaultable claim (X, Z, τ) settling at time T is given as

$$U_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big), \quad \forall\, t \in [0,T[,$$

where $\mathbb{Q}^*$ is the *spot martingale measure* and B is the savings account, so that, $dB_t = r_t B_t \, dt$ for some $\mathbb{F}$-adapted short-term rate process r. We also set $U_T = U_T(X) + U_T(Z) = X\mathbb{1}_{\{\tau>T\}} + Z_T\mathbb{1}_{\{\tau=T\}}$.

We denote by $\widetilde{U}$ the *pre-default value* of a claim (X, Z, τ), that is, the unique $\mathbb{F}$-predictable process satisfying $U_t = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t$ for every $t \in [0, T]$. It is clear that $\widetilde{U}$ can be decomposed as follows: $\widetilde{U} = \widetilde{U}(X) + \widetilde{U}(Z)$. For detailed computations of $\widetilde{U}(X)$ and $\widetilde{U}(Z)$ in terms of the hazard process of τ the reader is referred, for instance, to Bielecki et al. (2004a). Let us only quote here the pricing formulae from Propositions 1 and 2 in Section 2.2 in Bielecki et al. (2004a), which hold under the assumption that τ admits the $\mathbb{F}$-intensity γ^* under the spot martingale measure $\mathbb{Q}^*$:

$$\widetilde{U}_t(X) = \mathbb{E}_{\mathbb{Q}^*} \Big(e^{-\int_t^u (r_v + \gamma^*_v)\, dv} X \,\Big|\, \mathcal{F}_t \Big) \tag{15}$$

and

$$\widetilde{U}_t(Z) = \mathbb{E}_{\mathbb{Q}^*} \Big(\int_t^T Z_u \gamma^*_u \, e^{-\int_t^u (r_v + \gamma^*_v)\, dv} \, du \,\Big|\, \mathcal{F}_t \Big). \tag{16}$$

It is worth stressing that we will not use the last two formulae, since our goal is to replicate a defaultable claim, and thus also to derive its relative value, using directly replication arguments. The risk-neutral valuation formulae (15)-(16) are not fully operational, unless we find a method of specifying the default intensity γ^* under the spot martingale measure $\mathbb{Q}^*$. Also, in most texts, such pricing formulae are not supported by any kind of a hedging strategy, and our goal is actually to focus on this issue. The following definition of replication is natural in the case of defaultable claims.

Definition 3.3 A self-financing trading strategy ϕ is a *replicating strategy* for a defaultable claim (X, Z, τ) if and only if the following conditions are satisfied:
(i) $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \le T\}$,
(ii) $V_T(\phi) = X$ on the set $\{\tau > T\}$,
(iii) $\widetilde{V}_t(\phi) = \widetilde{U}_t(X) + \widetilde{U}_t(Z)$ on the random interval $[\![0, \tau \wedge T[\![$.
We say that a defaultable claim is *attainable* if it admits at least one replicating strategy.

In fact, it is possible to show that if a self-financing trading strategy ϕ satisfies conditions (i)-(ii) of Definition 3.3, then it also satisfies condition (iii). In other words, if a strategy replicates a defaultable claim at its effective maturity $T \wedge \tau$ then the pre-default wealth of ϕ and the pre-default value of (X, Z, τ) coincide on this interval.

According to Assumption (B), the primary defaultable securities Y^i, $i = 1, 2, \dots, m$ are subject to zero recovery, that is, $Y^i_t = 0$ for $t \ge \tau$. It is thus natural to introduce the following counterpart of the balance condition (5):

$$\sum_{i=m+1}^{k} \phi^i_t Y^i_{t-} = Z_t, \quad \forall\, t \in [0, T], \tag{17}$$

where Z is a predetermined recovery process, associated with a defaultable claim we wish to replicate. The financial interpretation of the balance condition (17) is that at any time prior to maturity we should be ready to cover the recovery payoff Z, since default may occur at any date, and the random moment of its occurrence is unpredictable (formally, it is modeled as a totally inaccessible $\mathbb{G}$-stopping time). Of course, if defaultable assets are subject to non-zero recovery, condition (17) should be modified accordingly.

Let us examine the consequences of the balance condition (17). Note also that Assumption (A), combined with condition (17), imply that on the event $\{\tau \le T\}$ we have

$$V_\tau(\phi) = \sum_{i=m+1}^{k} \phi^i_\tau Y^i_\tau = Z_\tau. \tag{18}$$

Equality (18) coincides with the property (i) of Definition 3.3. This means that, under Assumption (A), a strategy ϕ satisfying the balance condition (17) satisfies also the first condition of Definition 3.3. For this reason, when dealing with such strategies it will be sufficient to focus on the second condition of Definition 3.3. As already mentioned, the last condition in this definition will be automatically satisfied.

3.2.1 Sufficient Conditions for Attainability of a Defaultable Claim

In what follow, we find it convenient to adopt the notational convention that $\widetilde{Y}^i = Y^i$ for $i = m+1, m+2, \dots, k$. This convention is justified by a simple observation that the we may formally postulate that the 'pre-default value' of a default-free asset coincides with its value.

We assume that at least two processes, $\widetilde{Y}^l$ and $\widetilde{Y}^p$ say, are strictly positive, where $l \in \{1, 2, \dots, m\}$ and $p \in \{m+1, m+2, \dots, k\}$. The choice of $\widetilde{Y}^l$ and $\widetilde{Y}^p$ depends on the hedging problem at hand. We shall argue in what follows that in order to replicate a defaultable claim one needs to have $l \in \{1, 2, \dots, m\}$. Hence, we adopt the convention that $l = 1$ and $p = k$. Let us denote

$$\widetilde{V}_t^1(\phi) = \frac{V_t(\phi)}{\widetilde{Y}_t^1}, \quad \widetilde{Y}_t^{k,1} = \frac{\widetilde{Y}_t^k}{\widetilde{Y}_t^1}, \quad \widetilde{Z}_t^1 = \frac{Z_t}{\widetilde{Y}_t^1}, \quad d\widetilde{Y}_t^{i,k,1} = d\widetilde{Y}_t^{i,1} - \frac{\widetilde{Y}_{t-}^{i,1}}{\widetilde{Y}_{t-}^{k,1}}\, d\widetilde{Y}_t^{k,1}.$$

Also let $\bar{Y}^i = \bar{Y}^{i,k,1} = \widetilde{Y}^1 \widetilde{Y}^{i,k,1} = \widetilde{Y}^1 Y^{i,k,1}$ represents the price of the ith synthetic asset. Recall that a synthetic assets is a virtual object, but we can always re-express any constrained trading strategy in terms of the original primary assets $Y^1, Y^2, \dots, Y^k$.

The following result, which extends Propositions 2.1 and 2.2 to the case of a market model with default-free and defaultable securities. Part (ii) in Proposition 3.1 furnishes sufficient conditions for the existence of a replicating strategy for a defaultable claim (see Corollary 3.1 below).

Proposition 3.1 (i) *Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing strategy satisfying* (17). *Then the pre-default wealth process $\widetilde{V}(\phi)$ is an $\mathbb{F}$-adapted process satisfying, for every $t \in [0, T]$,*

$$\widetilde{V}_t(\phi) = \widetilde{Y}_t^1 \left(\widetilde{V}_0^1(\phi) + \sum_{i=2}^{m} \int_0^t \phi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^t \phi_u^i \, d\widetilde{Y}_u^{i,k,1} + \int_0^t \frac{\widetilde{Z}_u^1}{\widetilde{Y}_{u-}^{k,1}} \, d\widetilde{Y}_u^{k,1} \right). \tag{19}$$

In addition, we also have $\widetilde{V}_{t-}(\phi) = \sum_{i=1}^m \phi_t^i \widetilde{Y}_{t-}^i + Z_t$ for every $t \in [0, T]$
(ii) *Suppose that there exist $\mathbb{F}$-predictable processes ϕ^i, $i = 2, 3, \dots, k-1$, such that*

$$X = \widetilde{Y}_T^1 \left(x + \sum_{i=2}^{m} \int_0^T \phi_t^i \, d\widetilde{Y}_t^{i,1} + \sum_{i=m+1}^{k-1} \int_0^T \phi_t^i \, d\widetilde{Y}_t^{i,k,1} + \int_0^T \frac{\widetilde{Z}_t^1}{\widetilde{Y}_{t-}^{k,1}} \, d\widetilde{Y}_t^{k,1} \right). \tag{20}$$

Then there exist the $\mathbb{F}$-predictable processes ϕ^1 and ϕ^k such that $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ is a self-financing strategy in assets $Y^1, \dots, Y^m, \bar{Y}^{m+1}, \dots, \bar{Y}^{k-1}, Y^k$. The pre-default wealth of ϕ satisfies the equality $\widetilde{V}_T(\phi) = X$, and ϕ satisfies the balance condition $\sum_{i=m+1}^k \phi_t^i \widetilde{Y}_{t-}^i = Z_t$. Hence, in view of the part (i) above, it holds that $\widetilde{V}_{\tau-}(\phi) = \sum_{i=1}^m \phi_\tau^i \widetilde{Y}_{\tau-}^i + Z_\tau$ on the event $\{\tau \le T\}$.

Proof. The proofs of parts (i) and (ii) rely on exactly the same arguments as the proofs of Propositions 2.1 and 2.2, respectively, and thus they are omitted (see also Bielecki et al. (2005a)). ☐

The following corollary is the main result regarding the existence of a replicating strategy for a generic defaultable claim (X, Z, τ) in a credit risk model $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$. Recall that we work under Assumptions (A)-(B).

Corollary 3.1 *Let ϕ be a trading strategy in the assets $Y^1, \dots, Y^m, \bar{Y}^{m+1}, \dots, \bar{Y}^{k-1}, Y^k$ introduced in part (ii) of Proposition 3.1. Then ϕ replicates a defaultable claim (X, Z, τ) and the equality $\widetilde{U}_0(X) + \widetilde{U}_0(Z) = \widetilde{Y}_0^1 x$ holds.*

Proof. Since the strategy ϕ is self-financing, we need only to verify is that $V_T(\phi) = X$ on the event $\{\tau > T\}$ and $V_\tau(\phi) = Z_\tau$ on the event $\{\tau \le T\}$. For the former equality, observe that from part (ii) of Proposition 3.1 it follows that the pre-default wealth process of the strategy satisfies $\widetilde{V}_T(\phi) = X$; consequently, $V_T(\phi) = X$ on the set $\{\tau > T\}$. The latter equality is an immediate consequence of the equality: $\widetilde{V}_{\tau-}(\phi) = V_{\tau-}(\phi) = \sum_{i=1}^m \phi^i_\tau \widetilde{Y}^i_{\tau-} + Z_\tau$ on the event $\{\tau \le T\}$, and the assumed zero recovery scheme for all defaultable assets $Y^1, Y^2, \ldots, Y^m$. We have checked that ϕ replicates (X, Z, τ). The second part of the statement follows by remark made after Definition 3.3, combined with the equality $\widetilde{V}_0(\phi) = \widetilde{Y}^1_0 x$. □

3.3 Attainability of a Survival Claim

The goal of this section is to provide concrete, but still fairly general, examples of models and replicating strategies for defaultable claims. Let us take $k = 3$ and $m = 1$, so that the asset Y^1 is defaultable, and the assets Y^2 and Y^3 are default-free. According to the convention set forth above, we denote by $\widetilde{Y}^1$ the pre-default value of the first asset.

For the sake of concreteness and computational convenience, we assume that $Y^1_t = D^0(t,T)$ is the price process of a defaultable unit zero-coupon bond with zero recovery, and $Y^3_t = B(t,T)$ is the price of a risk-free unit zero-coupon bond. The two bonds are assumed to have a common maturity date T.

According to the convention set forth above, we denote by $\widetilde{Y}^1$ the pre-default value of the defaultable asset Y^1. Hence, we have $\widetilde{Y}^1_T = Y^3_T = 1$, since obviously $\widetilde{D}^0(T,T) = B(T,T) = 1$. Moreover, we make the natural assumption that the processes $\widetilde{Y}^1$ and Y^3 are strictly positive.

Remark. Let us stress the risk-free bond $B(t,T)$ is classified here as a default-free asset. Hence, in view of Assumption (A), credit risk models in which the price of a risk-free bond changes abruptly at the moment of default are not covered by the results of this section.

We shall consider a *survival claim* $(X, 0, \tau)$, that is, a defaultable claim with zero recovery at default and the promised payoff X such that $X \ge 0$ and $\mathbb{Q}\{X > 0\} > 0$. It is rather clear that, under Assumption (A), it is not possible to replicate a survival claim using a self-financing trading strategy of the form $\phi = (0, \phi^2, \phi^3)$. In other words, a survival claim is not attainable in the default-free market model $\mathcal{M}^1 = (Y^2, Y^3; \Phi)$, in which both primary assets Y^2 and Y^3 are default-free.

Lemma 3.1 *A survival claim $(X, 0, \tau)$ is not attainable in the default-free market $\mathcal{M}^1$.*

Proof. Recall that we have assumed that $\mathbb{Q}\{\tau \le T\} > 0$. Assumption (A) implies that the wealth process $V_t(\phi) = \phi^2_t Y^2_t + \phi^3_t Y^3_t$ is continuous at the default time τ, since under this assumption we have $\Delta V_\tau(\phi) = \phi^2_\tau \Delta Y^2_\tau + \phi^3_\tau \Delta Y^3_\tau = 0$. The value of a survival claim jumps from a positive value to zero at default time, and thus it cannot be replicated by means of any trading strategy ϕ of the form $(0, \phi^2, \phi^3)$. □

Obviously, the price Y^1 of a defaultable bond drops from the positive pre-default value to zero at the time of default. Thus, in order to replicate a survival claim $(X, 0, \tau)$, it is natural to include a defaultable bond in hedging portfolio.

We postulate from now on that the random variable X, which represents the promised payoff of a survival claim, can be replicated by continuous trading in primary assets Y^2 and Y^3. This is formalized in the following assumption.

Assumption (C): The promised payoff X is a contingent claim settling at time T, which is attainable in the default-free market model $\mathcal{M}^1 = (Y^2, Y^3; \Phi)$. Hence, there exists a constant y and an $\mathbb{F}$-predictable process α such that

$$\frac{X}{Y^3_T} = y + \int_0^T \alpha_t \, dY^{2,3}_t. \tag{21}$$

We denote by $\pi_t(X)$ the arbitrage price of X at time $t \in [0,T]$ in $\mathcal{M}^1$. Note that $\pi_0(X) = Y^3_0 y$.

We are going to study the existence a replicating strategy for a survival claim in the credit risk model $\bar{\mathcal{M}} = (Y^1, \bar{Y}^2, Y^3; \Phi)$, and to examine a relationship between the arbitrage price $\pi_t(X)$ of the promised payoff X in $\mathcal{M}^1$ and the pre-default value $\widetilde{U}_t(X)$ of a survival claim $(X, 0, \tau)$ in $\bar{\mathcal{M}}$. Recall that by general results of Section 2, the model $\bar{\mathcal{M}} = (Y^1, \bar{Y}^2, Y^3; \Phi)$, is essentially equivalent to the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$.

By virtue of part (ii) in Proposition 3.1, it suffices to examine the existence of a constant x and an $\mathbb{F}$-predictable process ϕ^2 such that

$$\frac{X}{\widetilde{Y}^1_T} = x + \int_0^T \phi^2_t \, d\widetilde{Y}^{2,3,1}_t. \tag{22}$$

It is easily seen that $\widetilde{U}_0(X) = \widetilde{Y}^1_0 x$, so that formally

$$\widetilde{U}_0(X) = \pi_0(X) \, \frac{\widetilde{Y}^1_0 x}{\widetilde{Y}^3_0 y} = \pi_0(X) \widetilde{Y}^{1,3}_0 \, \frac{x}{y},$$

provided that $\pi_0(X) \neq 0$ (so that $y \neq 0$). We wish to make the last equality more explicit, and to extend it to any $t \in [0, T]$.

3.3.1 Case of a Deterministic Default Intensity

A quite explicit pricing result for a survival claim can be obtained if we assume that the relative value $\widetilde{Y}^{3,1} = \widetilde{Y}^3/\widetilde{Y}^1 = Y^3/\widetilde{Y}^1$ follows a continuous process of finite variation. Recall that we formally set $\widetilde{Y}^i_t = Y^i_t$ for $i = 2, 3$, but for the defaultable zero-coupon bond, we have $Y^1_t = (1 - H_t)\widetilde{Y}^1_t$. Also, recall that $\bar{Y}^2 = \bar{Y}^{2,3,1} = \widetilde{Y}^1 \widetilde{Y}^{2,3,1} = \widetilde{Y}^1 Y^{2,3,1}$. The next proposition is an essential generalization of a result obtained in Bielecki et al. (2004a). We work under Assumption (C), so that the process α and the price $\pi_t(X)$ are as specified in this assumption.

Proposition 3.2 *Assume that $\widetilde{Y}^{3,1} = \widetilde{Y}^3/\widetilde{Y}^1$ is a continuous process of finite variation. Then a survival claim $(X, 0, \tau)$ can be replicated in the market model $\bar{\mathcal{M}}$ by means of a strategy ϕ such that $\phi^2_t = \alpha_t \widetilde{Y}^{1,3}_t$ and the processes ϕ^1, ϕ^3 are given by (26) below. The pre-default value of $(X, 0, \tau)$ equals*

$$\widetilde{U}_t(X) = \widetilde{Y}^{1,3}_t \pi_t(X) = \widetilde{Y}^1_t F_X(t, T), \quad \forall \, t \in [0, T],$$

where $F_X(t, T) = \pi_t(X)/B(t, T)$ is the forward price of X.

Proof. It suffices to consider the case $t = 0$. Let α be the first component of a replicating strategy for the promised payoff X in $\mathcal{M}^1$ (see formula (21)), so that we have, for every $t \in [0, T]$,

$$\frac{\pi_t(X)}{\widetilde{Y}^3_t} = \frac{\pi_0(X)}{\widetilde{Y}^3_0} + \int_0^t \alpha_u \, dY^{2,3}_u = \frac{\pi_0(X)}{\widetilde{Y}^3_0} + \int_0^t \alpha_u \, d\widetilde{Y}^{2,3}_u. \tag{23}$$

Let us set $\phi^2_t = \alpha_t \widetilde{Y}^{1,3}_t$. We claim that

$$\frac{X}{\widetilde{Y}^1_T} = \frac{\pi_0(X)}{\widetilde{Y}^3_0} + \int_0^T \phi^2_t \left(d\widetilde{Y}^{2,1}_t - \frac{\widetilde{Y}^{2,1}_{t-}}{\widetilde{Y}^{3,1}_{t-}} \, d\widetilde{Y}^{3,1}_t \right) = \frac{\pi_0(X)}{\widetilde{Y}^3_0} + \int_0^T \phi^2_t \, d\widetilde{Y}^{2,3,1}_t. \tag{24}$$

To establish (24), it suffices to note that we have

$$d\widetilde{Y}^{2,3,1}_t = d\widetilde{Y}^{2,1}_t - \frac{\widetilde{Y}^{2,1}_{t-}}{\widetilde{Y}^{3,1}_{t-}} \, d\widetilde{Y}^{3,1}_t = \widetilde{Y}^{3,1}_t d\widetilde{Y}^{2,3}_t + d[\widetilde{Y}^{2,3}, \widetilde{Y}^{3,1}]_t = \widetilde{Y}^{3,1}_t d\widetilde{Y}^{2,3}_t, \tag{25}$$

since $\widetilde{Y}^{3,1}$ is assumed to be a continuous process of finite variation. Combining (23) with (25), we obtain

$$d(\pi_t(X)/\widetilde{Y}^3_t) = \alpha_t \, d\widetilde{Y}^{2,3}_t = \alpha_t \widetilde{Y}^{1,3}_t \, d\widetilde{Y}^{2,3,1}_t.$$

Finally, since $\widetilde{Y}^1_T = \widetilde{Y}^3_T = 1$, we obtain

$$\frac{X}{\widetilde{Y}^1_T} = \frac{X}{\widetilde{Y}^3_T} = \frac{\pi_0(X)}{\widetilde{Y}^3_0} + \int_0^T \alpha_t \, d\widetilde{Y}^{2,3}_t = \frac{\pi_0(X)}{\widetilde{Y}^3_0} + \int_0^T \alpha_t \widetilde{Y}^{1,3}_t \, d\widetilde{Y}^{2,3,1}_t,$$

so that (24) holds with $\phi^2_t = \alpha_t \widetilde{Y}^{1,3}_t$. Note that (24) is a special case of (20) with $k = 3$ and $Z^1 = 0$. From Corollary 3.1, it follows that the replicating strategy $\phi = (\phi^1, \phi^2, \phi^3)$ for a survival claim $(X, 0, \tau)$ can be constructed as in part (ii) of Proposition 3.1. Specifically, we define the process $\widetilde{V}$ by the formula

$$\widetilde{V}_t = \widetilde{Y}^1_t \left(\frac{\pi_0(X)}{\widetilde{Y}^3_0} + \int_0^t \alpha_u \widetilde{Y}^{1,3}_u \, d\widetilde{Y}^{2,3,1}_u \right),$$

and we set (cf. Proposition 2.2)

$$\phi^1_t = (\widetilde{Y}^1_t)^{-1} \Big(\widetilde{V}_t - \phi^2_t \widetilde{Y}^2_t - \phi^3_t \widetilde{Y}^3_t \Big) = \widetilde{V}^1_t, \quad \phi^3_t = -\phi^2_t \widetilde{Y}^{2,3}_t. \tag{26}$$

Then the pre-default wealth of the strategy ϕ coincides with $\widetilde{V}$. In particular, the price at time 0 of a survival claim equals $\widetilde{U}_0(X) = \widetilde{V}_0 = \widetilde{Y}^{1,3}_0 \pi_0(X)$. □

Remark. Recall that Y^1 and Y^3 are prices of defaultable and default-free zero-coupon bonds respectively. Within the reduced-form set-up, the assumption that $\widetilde{Y}^{3,1} = \widetilde{Y}^3/\widetilde{Y}^1 = Y^3/\widetilde{Y}^1$ follows a continuous process of finite variation is satisfied only when the hazard process of the default time τ with respect to $\mathbb{F}$ is deterministic (see Bielecki et al. (2004b)). Hence, if we wish to deal with the case of stochastic intensity process, we need to relax this assumption.

3.3.2 Case of a Stochastic Default Intensity

In this section, we no longer assume that the process $\widetilde{Y}^{3,1}$ is a continuous process of finite variation. As mentioned above, our goal is to give results that cover the case of stochastic intensity of default. For the sake of concreteness, we postulate that $\widetilde{Y}^i$, $i = 1, 2, 3$ follow Itô processes:

$$d\widetilde{Y}^i_t = \widetilde{Y}^i_t \big(\mu^i_t \, dt + \sigma^i_t \, dW_t \big), \tag{27}$$

where μ^i and $\sigma^i = (\sigma^{i1}, \sigma^{i2}, \ldots, \sigma^{id})$ are $\mathbb{F}$-adapted processes and the underlying filtration $\mathbb{F}$ is generated by a d-dimensional Brownian motion $W = (W^1, W^2, \ldots, W^d)$ on $(\Omega, \mathcal{F}, \mathbb{Q})$.

Before proceeding to an analysis of particular models, let us recall a well-known auxiliary result (see, e.g., Lemma 1.6.7 in Karatzas and Shreve (1998)) that will prove useful in what follows.

Lemma 3.2 *Let $\widehat{W}_t$, $t \in [0, T]$, be a d-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \widehat{\mathbb{Q}})$ and let $\mathbb{F}$ be the natural filtration of $\widehat{W}$. Let*

$$\eta_t = \exp \Big(\int_0^t \theta_u \, d\widehat{W}_u - \frac{1}{2} \int_0^t \|\theta_u\|^2 \, du \Big),$$

where $\theta = (\theta^1, \theta^2, \ldots, \theta^d)$ is an $\mathbb{F}$-progressively measurable process, such that $\mathbb{E}_{\widehat{\mathbb{Q}}}(\eta_T) = 1$. Let $\widetilde{\mathbb{Q}}$ be a probability measure on $(\Omega, \mathcal{F}_T)$ given by $d\widetilde{\mathbb{Q}}/d\widehat{\mathbb{Q}} = \eta_T$, so that the process $\widetilde{W}_t = \widehat{W}_t - \int_0^t \theta_u \, du$, $t \in [0, T]$, follows a Brownian motion under $\widetilde{\mathbb{Q}}$. Let X be an $\mathcal{F}_T$-measurable random variable integrable with respect to $\widetilde{\mathbb{Q}}$. Let us set $\widetilde{X}_t = \mathbb{E}_{\widetilde{\mathbb{Q}}}(X \,|\, \mathcal{F}_t)$ and $\widehat{X}_t = \mathbb{E}_{\widehat{\mathbb{Q}}}(X \eta_T \,|\, \mathcal{F}_t)$, so that $\widetilde{X}_0 = \widehat{X}_0 = \mathbb{E}_{\widetilde{\mathbb{Q}}}(X)$. Then there exists an $\mathbb{F}$-progressively measurable process $\widehat{\phi}$ such that

$$\widehat{X}_t = \widehat{X}_0 + \int_0^t \widehat{\phi}_u \, d\widehat{W}_u, \quad \forall \, t \in [0, T].$$

Moreover

$$\widetilde{X}_t = \widetilde{X}_0 + \int_0^t \widetilde{\phi}_u \, d\widetilde{W}_u, \quad \forall\, t \in [0,T],$$

where $\widetilde{\phi}_t = \eta_t^{-1}(\widehat{\phi}_t - \theta_t \widehat{X}_t)$.

Complete case. We first consider the case of a stochastic intensity model driven by a one-dimensional Brownian motion W. Under the assumption that $\sigma_t^2 \neq \sigma_t^3$ for every $t \in [0,T]$, the default-free market model $\mathcal{M}^1 = (Y^2, Y^3; \Phi)$ is complete, and thus any contingent claim X is attainable in $\mathcal{M}^1$ (hence, the assumption (C) is satisfied). Before stating the next result, we find it convenient to introduce some notation. Using Itô's formula, it is easy to check that

$$d\widetilde{Y}_t^{2,3} = \widetilde{Y}_t^{2,3}(\sigma_t^2 - \sigma_t^3)\, d\widehat{W}_t,$$

where

$$d\widehat{W}_t = dW_t - \sigma_t^3\, dt + \frac{(\mu_t^2 - \mu_t^3)}{(\sigma_t^2 - \sigma_t^3)}\, dt. \tag{28}$$

Similarly, we obtain

$$d\widetilde{Y}_t^{2,3,1} = \widetilde{Y}_t^{2,1}(\sigma_t^2 - \sigma_t^3)\, d\widetilde{W}_t,$$

where

$$d\widetilde{W}_t = dW_t - \sigma_t^1\, dt + \frac{(\mu_t^2 - \mu_t^3)}{(\sigma_t^2 - \sigma_t^3)}\, dt. \tag{29}$$

Under mild regularity assumptions, there exist unique equivalent probability measures $\widehat{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_T)$ such that $\widehat{W}$ and $\widetilde{W}$ are Brownian motions under $\widehat{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$ respectively. It is easy to check that $d\widetilde{\mathbb{Q}}/d\widehat{\mathbb{Q}} = \eta_T$, where the process η is given by the formula

$$\eta_t = \exp\Big(\int_0^t (\sigma_u^1 - \sigma_u^3)\, d\widehat{W}_u - \frac{1}{2}\int_0^t (\sigma_u^1 - \sigma_u^3)^2\, du \Big). \tag{30}$$

Let us denote by $\widehat{X}$ an auxiliary contingent claim given as $\widehat{X} = X\eta_T$.

Proposition 3.3 *A survival claim* $(X, 0, \tau)$ *is attainable in* $\bar{\mathcal{M}}$ *and its pre-default value* $\widetilde{U}(X)$ *satisfies*

$$\frac{\widetilde{U}_t(X)}{\widetilde{Y}_t^1} = x + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,3,1},$$

where

$$\phi_t^2 = \eta_t^{-1}\left(\widehat{\alpha}_t \widetilde{Y}_t^{1,3} - \frac{\sigma_t^1 - \sigma_t^3}{\sigma_t^2 - \sigma_t^3}\, \widetilde{Y}_t^{1,2} F_{\widehat{X}}(t,T) \right), \tag{31}$$

the process η *is given by* (30), *the process* $\widehat{\alpha}$ *is such that*

$$\widehat{X} = \frac{\widehat{X}}{\widetilde{Y}_T^3} = \widehat{y} + \int_0^T \widehat{\alpha}_t \, d\widetilde{Y}_t^{2,3},$$

and $F_{\widehat{X}}(t,T)$ *is the forward price of a default-free claim* $\widehat{X}$, *that is,*

$$F_{\widehat{X}}(t,T) = \frac{\pi_t(\widehat{X})}{B(t,T)} = \frac{\pi_t(\widehat{X})}{\widetilde{Y}_t^3}.$$

The components ϕ^1 *and* ϕ^3 *of a replicating strategy for* $(X, 0, \tau)$ *are given by* (26), *and its pre-default value equals, for every* $t \in [0,T]$,

$$\widetilde{U}_t(X) = \widetilde{Y}_t^{1,3}\pi_t(\widehat{X}) = \widetilde{Y}_t^1 F_{\widehat{X}}(t,T). \tag{32}$$

Proof. The main tool used in the proof of the proposition is Lemma 3.2. First, we observe that there exists a process $\widehat{\phi}$ such that

$$\frac{X\eta_T}{\widetilde{Y}^3_T} = \widehat{X} = \widehat{y} + \int_0^T \widehat{\phi}_t \, d\widehat{W}_t = \widehat{y} + \int_0^T \widehat{\phi}_t \widetilde{Y}^{3,2}_t (\sigma^2_t - \sigma^3_t)^{-1} \, d\widetilde{Y}^{2,3}_t = \widehat{y} + \int_0^T \widehat{\alpha}_t \, d\widetilde{Y}^{2,3}_t,$$

where $\widehat{y} = \mathbb{E}_{\widehat{\mathbb{Q}}}(\widehat{X}) = \mathbb{E}_{\widetilde{\mathbb{Q}}}(X)$, and where we denote $\widehat{\alpha}_t = \widehat{\phi}_t \widetilde{Y}^{3,2}_t (\sigma^2_t - \sigma^3_t)^{-1}$. Note that the process $\widehat{X}_t = \mathbb{E}_{\widehat{\mathbb{Q}}}(\widehat{X} \,|\, \mathcal{F}_t)$ coincides with the forward price of $\widehat{X}$, i.e., $\widehat{X}_t = F_{\widehat{X}}(t,T)$. In view of Lemma 3.2, for the process $\widetilde{\phi}$ given by the formula

$$\widetilde{\phi}_t = \eta_t^{-1} \big(\widehat{\phi}_t - (\sigma^1_t - \sigma^3_t) F_{\widehat{X}}(t,T) \big),$$

we obtain

$$\frac{X}{\widetilde{Y}^1_T} = X = x + \int_0^T \widetilde{\phi}_t \, d\widetilde{W}_t = x + \int_0^T \widetilde{\phi}_t \widetilde{Y}^{1,2}_t (\sigma^2_t - \sigma^3_t)^{-1} \, d\widetilde{Y}^{2,3,1}_t = x + \int_0^T \phi^2_t \, d\widetilde{Y}^{2,3,1}_t,$$

where $x = \mathbb{E}_{\widetilde{\mathbb{Q}}}(X) = \widehat{y}$ and where we set $\phi^2_t = \widetilde{\phi}_t \widetilde{Y}^{1,2}_t (\sigma^2_t - \sigma^3_t)^{-1}$. To obtain (31), it suffices to combine the formulae above, and to use Corollary 3.1. For equality (32), note that

$$\widetilde{U}_0(X) = \widetilde{Y}^1_0 x = \widetilde{Y}^1_0 \widehat{y} = \widetilde{Y}^{1,3}_0 \pi_0(\widehat{X}),$$

as expected. □

Incomplete case. We shall now examine a more general situation in which a credit risk model is associated with an incomplete default-free market model. Results of this paragraph cover the case of a reduced-form model in which the default time admits a stochastic intensity adapted to the filtration generated by $W = (W^1, W^2, \ldots, W^d)$. We postulate that $\widetilde{Y}^i$, $i = 1,2,3$ are given by (27) for some $d \geq 2$. Using Itô's formula, we find that

$$d\widetilde{Y}^{2,3}_t = \widetilde{Y}^{2,3}_t (\sigma^2_t - \sigma^3_t) \, d\widehat{W}_t,$$

where the process $\widehat{W} = (\widehat{W}^1, \widehat{W}^2, \ldots, \widehat{W}^d)$ satisfies

$$(\sigma^2_t - \sigma^3_t) \, d\widehat{W}_t = (\sigma^2_t - \sigma^3_t) \, dW_t - \sigma^3_t (\sigma^2_t - \sigma^3_t) \, dt + (\mu^2_t - \mu^3_t) \, dt. \tag{33}$$

Similarly, we obtain

$$d\widetilde{Y}^{2,3,1}_t = \widetilde{Y}^{2,1}_t (\sigma^2_t - \sigma^3_t) \, d\widetilde{W}_t,$$

where $\widetilde{W} = (\widetilde{W}^1, \widetilde{W}^2, \ldots, \widetilde{W}^d)$ is such that $d\widetilde{W}_t = d\widehat{W}_t - (\sigma^1_t - \sigma^3_t) \, dt$, so that

$$(\sigma^2_t - \sigma^3_t) \, d\widetilde{W}_t = (\sigma^2_t - \sigma^3_t) \, dW_t - \sigma^1_t (\sigma^2_t - \sigma^3_t) \, dt + (\mu^2_t - \mu^3_t) \, dt. \tag{34}$$

Under mild technical assumptions, one can show the existence of the two probability measures, $\widehat{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$, equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$, and such that the processes $\widehat{W}$ and $\widetilde{W}$ that satisfy (33) and (34) are Brownian motions under $\widehat{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$ respectively. Note that $d\widetilde{\mathbb{Q}}/d\widehat{\mathbb{Q}} = \eta_T$, where the Radon-Nikodým density process η is defined by the formula

$$\eta_t = \exp\Big(\int_0^t (\sigma^1_u - \sigma^3_u) \, d\widehat{W}_u - \frac{1}{2} \int_0^t \|\sigma^1_u - \sigma^3_u\|^2 \, du \Big). \tag{35}$$

Since $d \geq 2$, it is easy to see that the reduced model $\mathcal{M}^1$ is not necessarily complete, in general. To proceed further, we shall postulate that the claim $\widehat{X} = X\eta_T$ is attainable in $\mathcal{M}^1$. This assumption is not restrictive if the model $\mathcal{M}^1$ is complete, since $\widehat{X}$ is an $\mathcal{F}_T$-measurable random variable, and thus it suffices to impose a suitable integrability condition.

Proposition 3.4 *Assume that there exists an* $\mathbb{F}$*-predictable process* β *such that*

$$\frac{\sigma_t^{1i} - \sigma_t^{3i}}{\sigma_t^{2i} - \sigma_t^{3i}} = \beta_t, \quad \forall\, i = 1, 2, \dots, d. \tag{36}$$

Let the claim $\widehat{X} = X\eta_T$ *be attainable in* $\mathcal{M}^1$*. Then a survival claim* $(X, 0, \tau)$ *is attainable in the market model* $\bar{\mathcal{M}}$ *and its pre-default value satisfies*

$$\frac{\widetilde{U}_t(X)}{\widetilde{Y}_t^1} = x + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,3,1},$$

where

$$\phi_t^2 = \eta_t^{-1} \left(\widehat{\alpha}_t \widetilde{Y}_t^{1,3} - \beta_t \widetilde{Y}_t^{1,2} F_{\widehat{X}}(t,T) \right), \tag{37}$$

the process η *is given by* (35)*, the process* $\widehat{\alpha}$ *is such that*

$$\widehat{X} = \frac{\widehat{X}}{\widetilde{Y}_T^3} = \widehat{y} + \int_0^T \widehat{\alpha}_t \, d\widetilde{Y}_t^{2,3},$$

and $F_{\widehat{X}}(t,T)$ *is the forward price of a default-free claim* $\widehat{X}$*. The components* ϕ^1 *and* ϕ^3 *of a replicating strategy for a survival claim are given by* (26)*, and its the pre-default value equals, for every* $t \in [0, T]$*,*

$$\widetilde{U}_t(X) = \widetilde{Y}_t^{1,3} \pi_t(\widehat{X}) = \widetilde{Y}_t^1 F_{\widehat{X}}(t,T). \tag{38}$$

Proof. Once again, the proof relies on Lemma 3.2. By assumption, there exists an $\mathbb{F}$-predictable process $\widehat{\alpha}$ such that

$$\frac{\widehat{X}}{\widetilde{Y}_T^3} = \widehat{X} = \widehat{y} + \int_0^T \widehat{\alpha}_t \, d\widetilde{Y}_t^{2,3} = \widehat{y} + \int_0^T \widehat{\alpha}_t \widetilde{Y}_t^{2,3} (\sigma_t^2 - \sigma_t^3) \, d\widehat{W}_t = \widehat{y} + \int_0^T \widehat{\phi}_t \, d\widehat{W}_t,$$

where we denote $\widehat{\phi}_t = \widehat{\alpha}_t \widetilde{Y}_t^{2,3} (\sigma_t^2 - \sigma_t^3)$. Note that $\widehat{y} = \mathbb{E}_{\widehat{\mathbb{Q}}}(\widehat{X})$ and the process $\widehat{X}_t = \mathbb{E}_{\widehat{\mathbb{Q}}}(\widehat{X} \,|\, \mathcal{F}_t)$ coincides with the forward price of $\widehat{X}$, that is, $\widehat{X}_t = F_{\widehat{X}}(t,T)$. Using Lemma 3.2, we conclude that for the process

$$\widetilde{\phi}_t = \eta_t^{-1} \big(\widehat{\phi}_t - (\sigma_t^1 - \sigma_t^3) F_{\widehat{X}}(t,T) \big), \tag{39}$$

we have

$$\frac{X}{\widetilde{Y}_T^1} = X = x + \int_0^T \widetilde{\phi}_t \, d\widetilde{W}_t,$$

where $x = \mathbb{E}_{\widetilde{\mathbb{Q}}}(X) = \mathbb{E}_{\widehat{\mathbb{Q}}}(\widehat{X}) = \widehat{y}$. In view of (36), for the process ϕ^2 given by (37), we obtain

$$\int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,3,1} = \int_0^T \widetilde{\phi}_t \, d\widetilde{W}_t.$$

To complete the proof, it suffices to combine the formulae above, and to use Corollary 3.1. □

Remark. (a) If $\sigma_t^1 = \sigma_t^3 = 0$, we are back in the set-up of Section 3.3.1. In this case, we may assume that W is a one-dimensional Brownian motion. In particular, the equality $\widehat{\mathbb{Q}} = \widetilde{\mathbb{Q}}$ (equivalently, $\eta_T = 1$) holds, and

$$d\widetilde{W}_t = d\widehat{W}_t = dW_t + \frac{\mu_t^2 - \mu_t^3}{\sigma_t^2} \, dt.$$

Moreover, in this case we have that

$$d\widetilde{Y}_t^{2,3,1} = -\sigma_t^2 \widetilde{Y}_t^{2,1} \, d\widetilde{W}_t = \widetilde{Y}_t^{2,1} ((\mu_t^2 - \mu_t^3) \, dt + \sigma_t^2 \, dW_t) = \widetilde{Y}_t^{3,1} \, d\widetilde{Y}_t^{2,3},$$

and the formulae of Proposition 3.4 (or Proposition 3.3) reduce to those of Proposition 3.2.

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