ON THE BRODY-HUGHSTON-MACRINA APPROACH TO MODELING OF DEFAULTABLE TERM STRUCTURE

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Introduction

This research is based on a recent working paper by D.C. Brody, L.P. Hughston and A. Macrina entitled "Beyond Hazard Rates: A New Framework for Credit-Risk Modelling" (Version: July 7, 2005). In what follows, we shall informally refer to the novel approach developed in Brody et al. [1] as the *BHM approach*. Critical parts of this new information-based framework in modeling defaultable bonds are re-examined and extended to a more general set-up.

There are two main approaches to the problems of pricing credit derivatives and/or credit risk management: the structural models and the reduced-form models.

In the *structural* approach, one makes explicit assumptions about the dynamics of a firm's assets, its capital structure, and its debt and share holders. A firm defaults if its assets are insufficient according to some measure. In this situation, a corporate liability can be characterized as an option on the firm assets.

The *reduced-form* approach is silent about why a firm defaults. Instead, the dynamics of default are exogenously given through a *default rate*, or *intensity*. In this approach, the values of credit-sensitive securities can be calculated as if they were default-free, using a credit-risk adjusted interest rate, that is, the risk-free rate plus the risk-neutral default intensity.

The *incomplete information* approach developed by Brody et al. [1] presents an attempt to combine the structural and reduced-form models. While avoiding their shortcomings, it picks the best features of both approaches: the economic and intuitive appeal of the structural approach, and the tractability and empirical fit of the reduced-form approach.

The general assumption of the paper by Brody et al. [1] is that interest rates are deterministic. And, for the sake of simplicity, the cash flow H_T from a defaultable bond at the maturity T is assumed to follow a discrete distribution. Both these assumptions are relaxed in the present work. However, the basic property of H_T that it should take values in the interval $[0, 1]$ is maintained, since it fits well the real-life feature of a corporate bond with a unit face value.

The new conceptual tool underpinning the information-based approach proposed by Brody et al. [1] is the so-called *market information process* ξ_t. It is formally defined in [1] by the expression

$$\xi_t = \sigma t H_T + \beta_{tT} \tag{1}$$

where the first term in the right-hand side represents the true information revealed, and a noise process β is introduced to model the uncertainty about the true value of H_T. Since at time T investors have a perfect information about the values of H_T (i.e., the uncertainty noise vanishes at time T), they suggested that the natural choice for β is a standard Brownian bridge.

In contrast to the original paper, the basic assumption in this work is that the terminal cash flow H_T follows a continuous distribution (although some results are valid for an arbitrary distribution of H_T). In addition, a large portion of this work is devoted to the case of random interest rates, which was not considered in the original paper.

This work is organized as follows. In Section 1, we introduce the market model and prove the Markov property for it. Explicit expression of defaultable bond price is also deduced. In Section 2, dynamics of the information process and dynamics of defaultable bond are given. For the later one, we consider both cases of deterministic and random interest rates. In Section 3, we deal with the valuation and hedging of derivatives. First, the self-financing trading strategies are introduced, and the model completeness is examined. Subsequently, valuation formulae for options on a defaultable bond are derived.

Let us describe briefly the contents of each section.

In Section 1, we present the market model, and we specify the pricing formula for defaultable bond. To this end, we introduce a random variable H_T representing the random cash flow occurring at time T. We mainly deal with the case where H_T follows a continuous probability distribution on the interval [0,1].

In order to build the model, a standard Brownian bridge process $\beta = \{\beta_{tT}\}_{0\leq t\leq T}$ is also introduced. As is well known, this process has the following mean and variance, for any $t \in [0,T]$,

$$\mathbb{E}_{\mathbb{P}}(\beta_{tT}) = 0, \quad \mathrm{Var}_{\mathbb{P}}(\beta_{tT}) = \frac{t(T-t)}{T}.$$

The basic features of the BHM model are summarized in Definition 1.1. The Markov property of market information process ξ is proved as in [1], by showing directly that

$$\mathbb{P}(\xi_t \leq x \,|\, \mathcal{F}^{\xi}_s) = \mathbb{P}(\xi_t \leq x \,|\, \xi_s), \quad \forall\, 0 \leq s \leq t \leq T.$$

By definition, the price of a defaultable bond price at time $t \in [0,T]$ equals

$$D(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{P}}(H_T \,|\, \xi_t)$$

so that, in particular, we have that $D(T,T) = H_T$. Here, $B(t,T)$ represents the price of a default-free discount bond, and $\mathbb{E}_{\mathbb{P}}(H_T \,|\, \xi_t)$ is the conditional expectation of the final cash flow, given the current market information. In Section 1.2, we compute the price of a defaultable bond, by deriving an explicit expression for this conditional expectation (see Corollary 1.1).

In Section 2, the properties of the price process of defaultable bond are examined. In Section 2.1, we show, along the similar lines as in Brody et al. [1], that an auxiliary process W, given by the formula

$$W_t = \xi_t + \int_0^t \frac{\xi_s - \sigma T \widehat{H}_s}{T-s}\, ds,$$

follows a standard Brownian motion under $\mathbb{P}$. In another words, it is a Brownian motion driven by the information process specified in our market model. This is a critical property which plays an important role in the determination of dynamics of a defaultable bond.

In Section 2.2, the dynamics of defaultable bond are derived under both deterministic and stochastic interest rates. We first derive the dynamics of the process $\widehat{H}_t := \mathbb{E}_{\mathbb{P}}(H_T|\,\xi_t)$, which is a prerequisite for finding the bond dynamics. Computations in the case of deterministic interest rates are rather straightforward, as soon as the Itô differential of the process $\widehat{H}_t$ is known. For an extended model with random interest rates, we introduce an additional Brownian noise, and we examine separately the dynamics under the forward and spot martingale measures.

Section 3 is devoted to the valuation and hedging of basic derivative securities. First, we focus on replicating strategies for derivative assets under both deterministic and random interest rates, and we establish the model completeness. Next, a generic valuation formula for a European option on a defaultable bond is derived (see Proposition 3.4).

A technical trick employed in the derivation of this formula, due to Brody et al. [1], is the change of probability measure. It ensures that the price can be conveniently represented in terms of independent random variables. A family of probability measures, $\{\mathbb{Q}_t\}_{t\in[0,T)}$, locally equivalent to the original probability measure $\mathbb{P}$, is defined in (43), using a suitable Radon-Nikodým density process (see Remark 3.1 for details). This family of probability measures has a striking property that, for any $t < T$, the information process ξ_u, $u \in [0,t]$, follows a Brownian bridge under $\mathbb{Q}_t$.

Using the above-mentioned change of a probability measure technique, the bond option pricing formulae under deterministic and stochastic interest rates are established. After deriving these quasi-explicit pricing formulae, we conclude this work by suggesting an approximation, which yields an explicit representation of the option price in terms of the Gaussian distribution (see Proposition 3.7). The accuracy of this approximation is not yet examined.

It is worth mentioning that our notation differs slightly from that of Brody et al. [1] in a few places. There should be no danger of confusion, but for convenience we point out the main differences. Brody et al. [1] use the notation $\mathbb{Q}$ for the risk neutral measure, $\mathbb{B}_T$ for the measure that makes the process ξ a Brownian bridge, P_{tT} for the price of a default-free discount bond, and B_{tT} for the price of a defaultable bond. In the present paper, we write $\mathbb{P}$ for the pricing measure, $\mathbb{Q}_t$ for the "bridge" measures, $B(t,T)$ for the default-free discount bond system, and $D(t,T)$ for a defaultable discount bond.

1 The Brody-Hughston-Macrina Model

We start by describing succinctly the basic features of a model put forward in Brody et al. [1]. Let H_T be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the cumulative distribution function F. In particular, we may have

$$F(x) = \int_{-\infty}^{x} p(y)\, dy, \quad \forall\, x \in \mathbb{R},$$

if H_T has the probability density function $p : \mathbb{R} \to \mathbb{R}_+$, and

$$F(x) = \Sigma_{x_i \le x} p_i, \quad \forall\, x \in \mathbb{R},$$

where $p_i = \mathbb{P}(H_T = x_i)$, if H_T is assumed to have a discrete probability distribution. To avoid dealing with trivial cases, we assume throughout that the distribution of H_T is non-degenerate, that is, the probability mass is not concentrated at a single point.

In the financial interpretation, a random variable H_T represents a *contingent claim* settling at time T, that is, a random cash flow occurring at time T. We postulate throughout that H_T takes values in the interval $[0, 1]$. Hence, the claim H_T can be interpreted as the payoff at maturity of a *defaultable bond* with maturity T and the face value 1. We write $D(T, T) = H_T$, and we denote by $D(t, T)$ the price of this bond at time $t \in [0, T]$. To determine the price process $\{D(t, T)\}_{0 \le t \le T}$, we need first to introduce some market model.

Let $\beta = \{\beta_{tT}\}_{0 \le t \le T}$ be the *Brownian bridge* process, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\beta_{0T} = 0$ and $\beta_{TT} = 0$ (see Karatzas and Shreve [2], Page 358). It is known that the mean and the covariance of the Brownian bridge are:

$$\mathbb{E}_{\mathbb{P}}(\beta_{tT}) = 0, \quad \forall\, 0 \le t \le T, \tag{2}$$

$$\mathbb{E}_{\mathbb{P}}(\beta_{sT}\beta_{tT}) = \frac{s(T-t)}{T}, \quad \forall\, 0 \le s \le t \le T. \tag{3}$$

We may, and do assume in what follows, that a Brownian bridge β_{tT} satisfies, for every $0 \le t \le u \le T$

$$\beta_{uT} - \beta_{tT} = -\int_t^u \frac{\beta_{sT}}{T-s}\, ds + \widetilde{W}_u - \widetilde{W}_t \tag{4}$$

where $\widetilde{W} = \{\widetilde{W}_t\}_{0 \le t \le T}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to its natural filtration $\widetilde{\mathbb{F}} = \mathbb{F}^{\widetilde{W}}$. It is useful to observe that the filtrations generated by $\widetilde{W}$ and β coincide.

In addition, we assume that the Brownian bridge β_{tT} is independent of the random variable H_T, and thus the Brownian motion $\widetilde{W}$ is independent of H_T as well. Let us introduce the *enlarged filtration* $\mathbb{G} = \widetilde{\mathbb{F}} \vee \mathcal{H}_T$ where $\mathcal{H}_T$ is the σ-field generated by H_T, i.e., for every $t \in [0, T]$,

$$\mathcal{G}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_T = \sigma\{\widetilde{W}_u;\, u \in [0, t],\, H_T\} = \sigma\{\beta_{uT};\, u \in [0, t],\, H_T\}.$$

It is worth noting that:
(i) the filtration $\widetilde{\mathbb{F}}$ is a sub-filtration of the filtration $\mathbb{G}$,
(ii) the process β ($\widetilde{W}$, respectively) is a Brownian bridge (Brownian motion, respectively) with respect to $\mathbb{G}$,
(iii) for any $t \in [0, T]$, the random variable H_T is $\mathcal{G}_t$-measurable.

The last property shows that the filtration $\mathbb{G}$ is too large, and thus we should introduce a suitable sub-filtration of $\mathbb{G}$ in order to get a non-trivial price process $D(t, T)$. To this end, let us introduce the following process

$$\xi_t = \sigma t H_T + \beta_{tT}. \tag{5}$$

Following Brody et al. [1], we assume that $\{\xi_t\}_{0 \le t \le T}$ is the *market information process*, in the sense, that the information flow available to investors is modelled as the natural filtration of ξ. Formally, the *information flow* thus equals to the filtration $\mathbb{F}^{\xi}$ where $\mathcal{F}^{\xi}_t = \sigma\{\xi_u; u \in [0, t]\}$ for $t \in [0, T]$.

The information flow $\mathbb{F}^\xi$ has the following obvious properties:
(i) $\mathbb{F}^\xi$ is a strict sub-filtration of $\mathbb{G}$, that is, $\mathbb{F}^\xi \subset \mathbb{G}$ and $\mathbb{F}^\xi \neq \mathbb{G}$,
(i) the processes β and $\widetilde{W}$ are not $\mathbb{F}^\xi$-adapted,
(iii) for any $t \in [0, T)$, the random variable H_T is not $\mathcal{F}^\xi_t$-measurable,
(iv) we have $\mathcal{F}^\xi_T = \mathcal{G}_T$, and thus the random variable H_T is $\mathcal{F}^\xi_T$-measurable.

In the next definition, we summarize the assumptions imposed on a model of defaultable term structure introduced by Brody et al. [1].

Definition 1.1 The *Brody-Hughston-Macrina model* (the *BHM model*, for short) is determined through the following postulates:
(i) a random variable H_T and a Brownian motion $\widetilde{W}$ are given on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and they are assumed to be independent,
(ii) the filtration $\mathbb{F} = \mathbb{F}^\xi = \{\mathcal{F}^\xi_t\}_{0 \leq t \leq T}$ is generated by the market information process ξ given by the formula

$$\xi_t = \sigma t H_T + \beta_{tT}, \quad \forall\, t \in [0, T], \tag{6}$$

where the Brownian bridge β_{tT} is given by (4),
(iii) the short term interest rate r is deterministic (unless explicitly stated otherwise), so that the price $B(t, T)$ of a default-free discount bond equals

$$B(t, T) = e^{-\int_t^T r(u)\, du}, \quad \forall\, t \in [0, T],$$

(iv) the probability measure $\mathbb{P}$ is the *risk-neutral probability*, in the sense, that the price $D(t, T)$ of a defaultable bond with maturity T equals

$$D(t, T) = B(t, T)\ \mathbb{E}_{\mathbb{P}}(H_T | \mathcal{F}_t), \quad \forall\, t \in [0, T]. \tag{7}$$

In particular, the price at time 0 of a defaultable bond with maturity T equals

$$D(0, T) = B(0, T) \int_0^1 x\, dF(x). \tag{8}$$

Remark 1.1 In an extension of the BHM model to the case of random interest rates (see Section 2.2.2 below), the information flow will also encompass the observations of the price process $\{B(t, T)\}_{0 \leq t \leq T}$ of a default-free discount bond. Thus, in that case, we shall set $\mathbb{F} = \mathbb{F}^\xi \vee \mathbb{F}^B$, or more explicitly,

$$\mathcal{F}_t = \mathcal{F}^\xi_t \vee \mathcal{F}^B_t = \sigma\{\xi_u,\, B(u, T); u \in [0, t]\}, \quad \forall\, t \in [0, T].$$

In addition, we shall assume that the filtration $\mathbb{F}^B$ satisfies $\mathbb{F}^B \subset \mathbb{F}^W \vee \mathbb{F}^{\widehat{W}}$ where W is an $\mathbb{F}^\xi$-Brownian motion defined in Lemma 2.1 below, and $\widehat{W}$ is a Brownian motion independent of W. Moreover, under mild assumptions, we will have

$$\mathbb{F}^W \vee \mathbb{F}^{\widehat{W}} = \mathbb{F}^\xi \vee \mathbb{F}^{\widehat{W}} = \mathbb{F}^{D^*} \vee \mathbb{F}^{B^*}$$

where the first equality follows from Proposition 2.1. The importance of the last equality will become clear in Section 2.2.4. Let us only mention here that $D^*(t, T)$ and $B^*(t, T)$ are discounted bond prices.

1.1 Markov Property of Market Information Process

First, we need to examine the basic properties of the market information process ξ. We are going to verify that ξ has the Markov property, that is, for every $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}$,

$$\mathbb{P}(\xi_t \leq x \,|\, \mathcal{F}^\xi_s) = \mathbb{P}(\xi_t \leq x \,|\, \xi_s).$$

Lemma 1.1 *The process ξ has the Markov property with respect to its natural filtration $\mathbb{F}^{\xi}$.*

Proof. We shall follow rather closely the original proof of Brody et al. [1]. It suffices to check that

$$\mathbb{P}(\xi_t \le x \,|\, \xi_{s_0}, \xi_{s_1}, \xi_{s_2}, \dots, \xi_{s_k}) = \mathbb{P}(\xi_t \le x \,|\, \xi_{s_0}), \quad \forall\, x \in \mathbb{R},$$

for any dates $s_k < s_{k-1} < \dots < s_1 < s_0 < t$. To this end, we observe that the random variables ξ_{s_0} and ξ_t are independent of $\eta_1, \eta_2, \dots, \eta_k$ (see Karatzas and Shreve [2], Page 359), where we denote

$$\eta_i = \beta_{s_i T}/s_i - \beta_{s_{i+1} T}/s_{i+1}.$$

Also, it is easily seen that

$$\mathbb{P}(\xi_t \le x \,|\, \xi_{s_0}, \xi_{s_1}, \xi_{s_2}, \dots, \xi_{s_k}) = \mathbb{P}(\xi_t \le x \,|\, \xi_{s_0}, \eta_1, \dots, \eta_k).$$

We claim that

$$\mathbb{P}(\xi_t \le x \,|\, \xi_{s_0}, \eta_1, \dots, \eta_k) = \mathbb{P}(\xi_t \le x \,|\, \xi_{s_0}).$$

Indeed,

$$\begin{aligned}
\mathbb{P}(\xi_t \le x \,|\, \xi_{s_0}, \eta_1, \dots, \eta_k) &= \frac{\mathbb{P}(\xi_t \le x, \xi_{s_0} \le y_{s_0}, \eta_1 \le y_{s_1}, \dots, \eta_k \le y_{s_k})}{\mathbb{P}(\xi_{s_0} \le y_{s_0}, \eta_1 \le y_{s_1}, \dots, \eta_k \le y_{s_k})} \\
&= \frac{\mathbb{P}(\xi_t \le x, \xi_{s_0} \le y_{s_0})\mathbb{P}(\eta_1 \le y_{s_1}) \dots \mathbb{P}(\eta_k \le y_{s_k})}{\mathbb{P}(\xi_{s_0} \le y_{s_0})\mathbb{P}(\eta_1 \le y_{s_1}) \dots \mathbb{P}(\eta_k \le y_{s_k})} \\
&= \frac{\mathbb{P}(\xi_t \le x, \xi_{s_0} \le y_{s_0})}{\mathbb{P}(\xi_{s_0} \le y_{s_0})} = \mathbb{P}(\xi_t \le x \,|\, \xi_{s_0}).
\end{aligned}$$

This completes the proof. □

Remark 1.2 The Markov property of the information process ξ holds for an arbitrary distribution of H_T. In the proof of Lemma 1.1 it was essential, however, to assume that β_{tT} is Brownian bridge (as opposed to any continuous process satisfying $\beta_{0T} = \beta_{TT} = 0$), as well as the fact that in definition (6) of ξ the random variable H_T is multiplied by a linear function of t. A simple extension of these assumptions can be done by the time change technique.

Remark 1.3 In the case of random interest rates, we need to introduce another source of randomness. Let $\widehat{W}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, independent of ξ and H_T. Then ξ has the Markov property with respect to the joint filtration $\mathbb{F} = \mathbb{F}^{\xi} \vee \mathbb{F}^{\widehat{W}}$, that is, for every $0 \le s \le t \le T$ and $x \in \mathbb{R}$,

$$\mathbb{P}(\xi_t \le x \,|\, \mathcal{F}_s) = \mathbb{P}(\xi_t \le x \,|\, \xi_s).$$

1.2 Pricing Formula for a Defaultable Bond

Recall that the price of a defaultable bond is defined by the expression (see (7))

$$D(t,T) = B(t,T)\, \mathbb{E}_{\mathbb{P}}(H_T |\, \mathcal{F}_t^{\xi}) = B(t,T)\, \mathbb{E}_{\mathbb{P}}(H_T |\, \xi_t)$$

where the second equality follows from the equality $\xi_T = \sigma H_T T$ and the Markov property of the market information process ξ (see Lemma 1.1). To find the pricing formula for a defaultable bond, it thus suffices to evaluate the conditional expectation $H(\xi_t, t, T) := \mathbb{E}_{\mathbb{P}}(H_T |\, \xi_t)$. We shall first work under the assumption that H_T is continuously distributed. The posterior probability density function of H_T, given that $\xi_t = z$, is represented by the Bayes formula as follows

$$p(x|z) = \frac{p(x)p(z|x)}{\int_0^1 p(y)p(z|y)\, dy}$$

where, for brevity, the dependence of the densities $p(x|z), p(z|x)$ and $p(x)$ on a time parameter t is suppressed. A representation analogous to (10) was derived by Brody et al. [1] for the case of a discretely distributed random variable H_T.

Lemma 1.2 *The conditional expectation of H_T given ξ_t can be represented as*

$$\widehat{H}_t := \mathbb{E}_{\mathbb{P}}(H_T|\,\xi_t) = H(\xi_t, t, T) \tag{9}$$

where $H : \mathbb{R} \times [0, T] \to (0, 1)$ is a jointly continuous function given by the formula

$$H(z, t, T) = \frac{\int_0^1 xp(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2 x^2 t)}\,dx}{\int_0^1 p(y)e^{\frac{T}{T-t}(\sigma yz - \frac{1}{2}\sigma^2 y^2 t)}\,dy}. \tag{10}$$

The $\mathbb{F}^\xi$-adapted process $\widehat{H}$ is strictly positive, continuous and bounded, with $\widehat{H}_T = H_T$.

Proof. Since $\xi_t = \sigma t H_T + \beta_{tT}$, it is clear that the conditional distribution of ξ_t given that $H_T = x$ is Gaussian, with mean value σxt and variance $\frac{t(T-t)}{T}$. Note that this property does not depend on a probability distribution of H_T, so that it is always satisfied for a generic value x belonging to the support of the distribution of H_T (this allows us to state Lemma 1.3). We conclude that that the conditional density of ξ_t given that $H_T = x$ equals

$$p_{\xi_t|\,H_T}(z|x) := p(z|x) = \frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}}e^{-\frac{T}{2t(T-t)}(z-\sigma xt)^2}.$$

Consequently, under the assumption that H_T is continuously distributed, we obtain

$$\begin{aligned}
p_{H_T|\,\xi_t}(x|z) \quad &:= \quad p(x|z) = \frac{p(x)p(z|x)}{\int_0^1 p(y)p(z|y)\,dy} \\
&= \frac{p(x)\frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}}e^{-\frac{T}{2t(T-t)}(z-\sigma xt)^2}}{\int_0^1 p(y)\frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}}e^{-\frac{T}{2t(T-t)}(z-\sigma yt)^2}\,dy} \\
&= \frac{p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2 x^2 t)}}{\int_0^1 p(y)e^{\frac{T}{T-t}(\sigma yz - \frac{1}{2}\sigma^2 y^2 t)}\,dy}.
\end{aligned}$$

We thus have, for $t \in [0, T)$,

$$\begin{aligned}
H(z, t, T) \quad &= \quad \mathbb{E}_{\mathbb{P}}(H_T|\,\xi_t = z) \\
&= \int_0^1 xp(x|z)\,dx \\
&= \frac{\int_0^1 xp(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2 x^2 t)}\,dx}{\int_0^1 p(y)e^{\frac{T}{T-t}(\sigma yz - \frac{1}{2}\sigma^2 y^2 t)}\,dy}.
\end{aligned}$$

Clearly, for a fixed T, the function $H(z, t, T) : \mathbb{R} \times [0, T) \to (0, 1)$ is jointly continuous in (z, t). Hence, the process $\widehat{H}$ is strictly positive, continuous on $[0, T)$, and bounded. The last equality in the statement of the lemma is also clear, since $\xi_T = \sigma T H_T$. To complete the proof, we need to check that

$$\lim_{t\to T} H(\xi_t, t, T) = \lim_{t\to T} \mathbb{E}_{\mathbb{P}}(H_T|\,\xi_t) = H_T,$$

or equivalently,

$$\lim_{t\to T} \mathbb{E}_{\mathbb{P}}(\xi_T|\,\xi_t) = \xi_T.$$

The last convergence can be deduced from the fact that the Markov process ξ is integrable and continuous. We have thus shown that the process $\widehat{H}$ is indeed continuous on $[0, T]$. □

Corollary 1.1 *The price of a defaultable bond equals*

$$D(t, T) = B(t, T)H(\xi_t, t, T) = B(t, T)\,\frac{\int_0^1 xp(x)e^{\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t)}\,dx}{\int_0^1 p(y)e^{\frac{T}{T-t}(\sigma y\xi_t - \frac{1}{2}\sigma^2 y^2 t)}\,dy}. \tag{11}$$

If the random variable H_T has an arbitrary cumulative distribution function F then Lemma 1.2 can be reformulated as follows.

Lemma 1.3 *The conditional expectation of H_T given ξ_t can be represented as*

$$\widehat{H}_t := \mathbb{E}_{\mathbb{P}}(H_T \,|\, \xi_t) = H(\xi_t, t, T) \tag{12}$$

where $H : \mathbb{R} \times [0,T] \to (0,1)$ is a jointly continuous function given by the formula

$$H(z,t,T) = \frac{\int_0^1 x e^{\frac{T}{T-t}(\sigma x z - \frac{1}{2}\sigma^2 x^2 t)}\, dF(x)}{\int_0^1 e^{\frac{T}{T-t}(\sigma y z - \frac{1}{2}\sigma^2 y^2 t)}\, dF(y)}. \tag{13}$$

The $\mathbb{F}^\xi$-adapted process $\widehat{H}$ is strictly positive, continuous and bounded, with $\widehat{H}_T = H_T$.

Remark 1.4 As already mentioned, in the case of random interest rates, we will introduce an auxiliary Brownian motion $\widehat{W}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, independent both of the process ξ and of the random variable H_T. We will then have (see Section 3.1.2)

$$\widehat{H}_t := \mathbb{E}_{\mathbb{P}}(H_T \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(H_T \,|\, \mathcal{F}_t^\xi \vee \mathcal{F}_t^{\widehat{W}}) = \mathbb{E}_{\mathbb{P}}(H_T \,|\, \mathcal{F}_t^\xi) = H(\xi_t, t, T). \tag{14}$$

1.3 Special Case

We shall now consider an example of a random variable H_T taking values in $[0,1]$. Let us assume that H_T is uniformly distributed, specifically, $H_T \sim \mathrm{U}(a,b)$ for some constants $0 \le a \le b \le 1$. The proof of the next result presents no difficulties, and thus it is omitted.

Lemma 1.4 *Let β be a standard Brownian bridge so that*

$$\beta_{tT} \sim N\left(0, \frac{t(T-t)}{T}\right),$$

and $H_T \sim \mathrm{U}(a,b)$ for some $0 \le a \le b \le 1$. Using the Bayes formula, we find that the density of $H_T \,|\, \xi_t = z$ equals, for every $x \in [a,b]$,

$$p(x|z) = \frac{\sigma\sqrt{tT}}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{\sigma^2 tT}{2(T-t)}\left(x - \frac{z}{\sigma t}\right)^2\right), \tag{15}$$

and it equals 0 otherwise. Consequently,

$$\begin{aligned} H(z,t,T) &= \mathbb{E}_{\mathbb{P}}(H_T \,|\, \xi_t = z) = \int_a^b x p(x|z)\, dx \\ &= \int_a^b x \frac{\sigma\sqrt{tT}}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{\sigma^2 tT}{2(T-t)}\left(x - \frac{z}{\sigma t}\right)^2\right) dx. \end{aligned}$$

In practical implementations, it is more natural to assume that H_T has a mixed distribution, say, $0 < \alpha := \mathbb{P}(H_T = 1) < 1$ and, for every $0 \le x_1 \le x_2 < 1$

$$\mathbb{P}(H_T \in [x_1, x_2]) = (1-\alpha) \int_{x_1}^{x_2} \frac{1}{b-a} \mathbb{1}_{[a,b]}(x)\, dx$$

for some constants $0 \le a < b \le 1$. In that case, the bond pricing formula is a combination of formula (1.4) and the pricing formula established in [1] for the case of a discrete distribution.

2 Dynamics of a Defaultable Bond

The main goal in this section is the derivation of the dynamics of the defaultable bond price $D(t,T)$. For this purpose, we shall first examine the dynamics of the information process.

2.1 Dynamics of the Information Process

In the first step, we shall find the dynamics of ξ with respect to a Brownian motion $\widetilde{W}$ such that (see formula (4))

$$\begin{cases} d\beta_{tT} = -\frac{\beta_{tT}}{T-t}\,dt + d\widetilde{W}_t, \\ \beta_{0T} = 0. \end{cases}$$

We have

$$\begin{aligned} d\xi_t &= \sigma H_T\,dt + d\beta_{tT} \\ &= \sigma H_T\,dt - \frac{\beta_{tT}}{T-t}\,dt + d\widetilde{W}_t \\ &= \sigma H_T\,dt - \frac{(\xi_t - \sigma H_T t)}{T-t}\,dt + d\widetilde{W}_t \\ &= \frac{\sigma H_T(T-t) - \xi_t + \sigma H_T t}{T-t}\,dt + d\widetilde{W}_t \\ &= \frac{1}{T-t}(\sigma H_T T - \xi_t)\,dt + d\widetilde{W}_t. \end{aligned}$$

Hence, the information process ξ is a continuous semimartingale, and its quadratic variation satisfies $\langle \xi \rangle_t = t$ for every $t \in [0,T]$.

2.1.1 Information-driven Brownian Motion

It appears that the information process ξ can be represented as a solution to a stochastic differential equation driven by some Brownian motion W. Recall that $\widehat{H}_t = \mathbb{E}_{\mathbb{P}}(H_T \,|\, \xi_t)$ for $t \in [0,T]$.

Lemma 2.1 *Let the process W be defined by the formula*

$$W_t = \xi_t + \int_0^t \frac{\xi_s - \sigma T \widehat{H}_s}{T-s}\,ds. \tag{16}$$

Then W is a standard Brownian motion under $\mathbb{P}$ with respect to the filtration $\mathbb{F}^\xi$.

Proof. We have, for every $t \le u < T$,

$$W_u - W_t = \xi_u - \xi_t + \int_t^u \frac{\xi_s - \sigma T \widehat{H}_s}{T-s}\,ds$$

where

$$\xi_u - \xi_t = \sigma(u-t)H_T + \beta_{uT} - \beta_{tT}$$

and (see (4))

$$\beta_{uT} - \beta_{tT} = -\int_t^u \frac{\beta_{sT}}{T-s}\,ds + \widetilde{W}_u - \widetilde{W}_t.$$

Consequently,

$$W_u - W_t = \sigma(u-t)H_T - \int_t^u \frac{\beta_{sT}}{T-s}\,ds + \int_t^u \frac{\sigma s H_T + \beta_{sT} - \sigma T \widehat{H}_s}{T-s}\,ds + \widetilde{W}_u - \widetilde{W}_t,$$

so that

$$W_u - W_t = \int_t^u \frac{\sigma(T-s)H_T + \sigma s H_T - \sigma T \widehat{H}_s}{T-s}\, ds + \widetilde{W}_u - \widetilde{W}_t.$$

Finally,

$$W_u - W_t = \int_t^u \frac{\sigma T H_T - \sigma T \widehat{H}_s}{T-s}\, ds + \widetilde{W}_u - \widetilde{W}_t.$$

The increment $\widetilde{W}_u - \widetilde{W}_t$ is independent of the σ-field $\mathcal{F}_t^\xi$. Therefore,

$$\mathbb{E}_{\mathbb{P}}(W_u - W_t \,|\, \mathcal{F}_t^\xi) = \mathbb{E}_{\mathbb{P}}\left(\int_t^u \frac{\sigma T}{T-s} \,(H_T - \widehat{H}_s)\, ds \,\Big|\, \mathcal{F}_t^\xi \right) = 0.$$

The last equality can be established using the following chain of equalities, in which we write $t_i^n = t + (u-t)i/n$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(\int_t^u \frac{\sigma T}{T-s} \big(H_T - \widehat{H}_s\big)\, ds \,\Big|\, \mathcal{F}_t^\xi \right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\sigma T}{T-t_i^n} \big(H_T - \widehat{H}_{t_i^n}\big)\,(t_{i+1}^n - t_i^n) \,\Big|\, \mathcal{F}_t^\xi \right) \\
&= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left(\sum_{i=0}^{n-1} \frac{\sigma T}{T-t_i^n} \big(H_T - \mathbb{E}_{\mathbb{P}}(H_T|\xi_{t_i^n})\big)\,(t_{i+1}^n - t_i^n) \,\Big|\, \mathcal{F}_t^\xi \right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbb{E}_{\mathbb{P}}\left(\frac{\sigma T}{T-t_i^n} \big(\mathbb{E}_{\mathbb{P}}(H_T|\mathcal{F}_{t_i^n}^\xi) - \mathbb{E}_{\mathbb{P}}(H_T|\mathcal{F}_{t_i^n}^\xi)\big)\,(t_{i+1}^n - t_i^n) \,\Big|\, \mathcal{F}_t^\xi \right) = 0.
\end{aligned}$$

We have used here, in particular, the bounded convergence theorem and the tower property of a conditional expectation.

We have thus shown that W is a continuous martingale with respect to the filtration $\mathbb{F}^\xi$. To conclude the proof, it suffices to note that $\langle W \rangle_t = \langle \xi \rangle_t = t$ and thus, by Lévy's theorem, W is a standard Brownian motion. □

Remark 2.1 Let $\widehat{W}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, independent of ξ. Then W is also a Brownian motion with respect to the filtration $\mathbb{F} = \mathbb{F}^\xi \vee \mathbb{F}^{\widehat{W}}$. Moreover, the two Brownian motions W and $\widehat{W}$ are independent.

2.1.2 SDE for Market Information Process

Let us now examine the stochastic differential equation (SDE) satisfied by the information process. Formula (16) implies that

$$\xi_t = W_t - \int_0^t \frac{\xi_s - \sigma T H(s, \xi_s, T)}{T-s}\, ds. \tag{17}$$

This can be seen as an SDE driven by W, since formally

$$d\xi_t = dW_t - \frac{\xi_t - \sigma T H(t, \xi_t, T)}{T-t}\, dt \tag{18}$$

with the initial condition $\xi_0 = 0$. The process ξ is manifestly a *weak* solution of (18), that is, $\mathbb{F}^W \subseteq \mathbb{F}^\xi$. In fact, we also have the following stronger result, which shows, in particular, that $\mathbb{F}^\xi \subseteq \mathbb{F}^W$, and thus $\mathbb{F}^\xi = \mathbb{F}^W$.

Proposition 2.1 *The process ξ is the unique strong solution of the SDE (18). Moreover, the equality $\mathbb{F}^\xi = \mathbb{F}^W$ holds.*

Proof. From Lemma 1.2, we know that $H(z,t,T)$ is a jointly continuous function of (z,t). Moreover, it can be shown that it is (locally in t) a Lipschitz continuous function with respect to z (see Remark 2.2 below). This implies the SDE (18) admits a unique strong (local) solution. Since ξ is a solution, it is clear that a global strong solution exists. This implies, in particular, that $\mathcal{F}_t^\xi \subseteq \mathcal{F}_t^W$ for every $t<T$. Since the inclusion $\mathcal{F}_t^\xi \supseteq \mathcal{F}_t^W$ is obvious from (16), we conclude that $\mathcal{F}_t^\xi = \mathcal{F}_t^W$ for every $t<T$, that is, the filtrations $\mathbb{F}^\xi$ and $\mathbb{F}^W$ coincide. □

2.2 Dynamics of a Defaultable Bond

The conditional variance of the cash-flow H_T given the market information at time t is defined as

$$\mathrm{Var}\,(H_T|\,\xi_t) = \int_0^1 \Big(x - \mathbb{E}_{\mathbb{P}}(H_T|\,\xi_t)\Big)^2 p(x|\xi_t)\,dx, \tag{19}$$

or equivalently,

$$\mathrm{Var}\,(H_T|\,\xi_t) = \int_0^1 (x-\widehat{H}_t)^2 p(x|\xi_t)\,dx. \tag{20}$$

We can also observe that the process $\mathrm{Var}\,(H_T|\,\xi_t)$ is strictly positive and $\mathbb{F}^\xi$-adapted, since $p(x|\xi_t)$ is the conditional density and it is positive.

Lemma 2.2 *Let the process $\widehat{H}_t = H(\xi_t,t,T)$ be given by (9). Then the dynamics of $\widehat{H}$ under $\mathbb{P}$ are*

$$d\widehat{H}_t = \frac{\sigma T}{T-t}\mathrm{Var}\,(H_T|\,\xi_t)\Big(d\xi_t + \frac{\xi_t - \sigma T \widehat{H}_t}{T-t}\,dt\Big). \tag{21}$$

Hence, the dynamics of $\widehat{H}$ can also be represented as follows

$$d\widehat{H}_t = \frac{\sigma T}{T-t}\mathrm{Var}\,(H_T|\,\xi_t)\,dW_t \tag{22}$$

where the Brownian motion W with respect to $\mathbb{F}^\xi = \mathbb{F}^W$ is defined by (16).

Proof. The Itô differential of $H(\xi_t,t,T)$ equals (recall that $p(x|z)$ depends on t as well)

$$\begin{aligned} dH(\xi_t,t,T) &= H_z(\xi_t,t,T)\,d\xi_t + H_t(\xi_t,t,T)\,dt + \frac{1}{2}H_{zz}(\xi_t,t,T)\,d\langle\xi\rangle_t \\ &= \left[\int_0^1 x\frac{\partial}{\partial z}p(x|\xi_t)\,dx\right]d\xi_t + \left[\int_0^1 x\frac{\partial}{\partial t}p(x|\xi_t)\,dx\right]dt + \left[\frac{1}{2}\int_0^1 x\frac{\partial^2}{\partial z^2}p(x|\xi_t)\,dx\right]d\langle\xi\rangle_t \\ &= \left[\int_0^1 xJ_1(t,\xi_t)\,dx\right]d\xi_t + \left[\int_0^1 xJ_2(t,\xi_t)\,dx\right]dt + \left[\frac{1}{2}\int_0^1 xJ_3(t,\xi_t)\,dx\right]d\langle\xi\rangle_t. \end{aligned}$$

Now we calculate the quantities $J_1(t,z)$, $J_2(t,z)$ and $J_3(t,z)$. First, let us denote

$$p(x|z) = \frac{p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2x^2t)}}{\int_0^1 p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2x^2t)}\,dx} = \frac{\Delta_1}{\Delta_2}$$

where

$$\Delta_1 = \Delta_1(x,z,t) := p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2x^2t)}$$

and

$$\Delta_2 = \Delta_2(z,t) := \int_0^1 \Delta_1(x,z,t)\,dx = \int_0^1 p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2x^2t)}\,dx.$$

If we also denote

$$\Delta_3 = \Delta_3(z,t) := \int_0^1 x\Delta_1(x,z,t)\,dx = \int_0^1 xp(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2 x^2 t)}\,dx,$$

then we find that

$$H(z,t,T) = \frac{\Delta_3}{\Delta_2}.$$

Calculation of J_1. Note that

$$\frac{\partial}{\partial z}\Delta_1 = \frac{\partial}{\partial z}\Big(p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2 x^2 t)}\Big) = \frac{\sigma xT}{T-t}\Delta_1,$$

$$\begin{aligned}\frac{\partial}{\partial z}\Delta_2 &= \frac{\partial}{\partial z}\Big(\int_0^1 p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2 x^2 t)}\,dx\Big) = \int_0^1 \frac{\partial}{\partial z}\Big(p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2 x^2 t)}\Big)\,dx \\ &= \int_0^1 \Delta_1\frac{\sigma xT}{T-t}\,dx = \frac{\sigma T}{T-t}\Delta_3.\end{aligned}$$

Hence, the partial derivative of $p(x|z)$ with respect to z is

$$\begin{aligned}J_1 &= \frac{\partial}{\partial z}p(x|z) = \frac{\partial}{\partial z}\frac{\Delta_1}{\Delta_2} = \frac{1}{\Delta_2}\frac{\partial}{\partial z}\Delta_1 + \Delta_1\Big(-\frac{1}{\Delta_2^2}\Big)\frac{\partial}{\partial z}\Delta_2 \\ &= \frac{1}{\Delta_2}\Big(\frac{\partial}{\partial z}\Delta_1 - \frac{\Delta_1}{\Delta_2}\frac{\partial}{\partial z}\Delta_2\Big) = \frac{1}{\Delta_2}\Big(\frac{\sigma xT}{T-t}\Delta_1 - \frac{\Delta_1}{\Delta_2}\frac{\sigma T}{T-t}\Delta_3\Big) \\ &= \frac{\Delta_1}{\Delta_2}\frac{\sigma T}{T-t}\Big(x - \frac{\Delta_3}{\Delta_2}\Big) = p(x|z)\frac{\sigma T}{T-t}\Big(x - H(z,t,T)\Big).\end{aligned}$$

Calculation of J_2. To compute J_2, we first note that

$$\frac{\partial}{\partial t}\Delta_1 = \frac{\partial}{\partial t}\Big(p(x)e^{\frac{T}{T-t}(\sigma xz - \frac{1}{2}\sigma^2 x^2 t)}\Big) = \Delta_1\frac{\sigma xT}{(T-t)^2}\Big(z - \frac{1}{2}\sigma xT\Big)$$

and

$$\frac{\partial}{\partial t}\Delta_2 = \frac{\partial}{\partial t}\Big(\int_0^1 \Delta_1\,dx\Big) = \int_0^1 \Delta_1\frac{\sigma xT}{(T-t)^2}\Big(z - \frac{1}{2}\sigma xT\Big)\,dx = \frac{\sigma T}{(T-t)^2}\Big(z\Delta_3 - \frac{1}{2}\sigma T\int_0^1 \Delta_1 x^2\,dx\Big).$$

Hence, the partial differential of $p(x|z)$ with respect to t is

$$\begin{aligned}J_2 &= \frac{\partial}{\partial t}p(x|z) \\ &= \frac{1}{\Delta_2}\Big(\frac{\partial}{\partial t}\Delta_1 - \frac{\Delta_1}{\Delta_2}\frac{\partial}{\partial t}\Delta_2\Big) \\ &= \frac{1}{\Delta_2}\Big[\Delta_1\frac{\sigma xT}{(T-t)^2}\Big(z - \frac{1}{2}\sigma xT\Big) - \frac{\Delta_1}{\Delta_2}\frac{\sigma T}{(T-t)^2}\Big(z\Delta_3 - \frac{1}{2}\sigma T\int_0^1 \Delta_1 x^2\,dx\Big)\Big] \\ &= \frac{\Delta_1}{\Delta_2}\frac{\sigma T}{(T-t)^2}\Big(xz - \frac{1}{2}\sigma x^2 T - \frac{\Delta_3}{\Delta_2}z + \frac{1}{2}\sigma T\int_0^1 \frac{\Delta_1}{\Delta_2}x^2\,dx\Big) \\ &= p(x|z)\frac{\sigma T}{(T-t)^2}\Big(xz - \frac{1}{2}\sigma x^2 T - zH(z,t,T) + \frac{1}{2}\sigma T\,\mathbb{E}_{\mathbb{P}}(H_T^2\,|\,\xi_t = z)\Big).\end{aligned}$$

Calculation of J_3. We have

$$\begin{aligned}J_3 &= \frac{\partial}{\partial z}J_1 \\ &= \frac{\partial}{\partial z}\Big[p(x|z)\frac{\sigma T}{T-t}\Big(x - H(z,t,T)\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma T}{T-t}\Big[\Big(x-H(z,t,T)\Big)\frac{\partial}{\partial z}p(x|z)+p(x|z)\frac{\partial}{\partial z}\Big(x-H(z,t,T)\Big)\Big] \\
&= \frac{\sigma T}{T-t}\Big[p(x|z)\frac{\sigma T}{T-t}\Big(x-H(z,t,T)\Big)^2-p(x|z)\int_0^1 xp(x|z)\frac{\sigma T}{T-t}\Big(x-H(z,t,T)\Big)\,dx\Big] \\
&= \frac{\sigma^2T^2}{(T-t)^2}p(x|z)\Big[\Big(x-H(z,t,T)\Big)^2-\int_0^1 xp(x|z)\Big(x-H(z,t,T)\Big)\,dx\Big] \\
&= \frac{\sigma^2T^2}{(T-t)^2}p(x|z)\Big[\Big(x-H(z,t,T)\Big)^2-\mathrm{Var}\,(H_T|\,\xi_t=z)\Big].
\end{aligned}
$$

We are in the position to compute $dH(\xi_t,t,T)$. Upon substitution, we obtain

$$
\begin{aligned}
dH(\xi_t,t,T) &= \Big[\int_0^1 xJ_1(t,\xi_t)\,dx\Big]\,d\xi_t+\Big[\int_0^1 xJ_2(t,\xi_t)\,dx\Big]\,dt+\Big[\frac{1}{2}\int_0^1 xJ_3(t,\xi_t)\,dx\Big]\,d\langle\xi_t\rangle_t \\
&= \Big[\int_0^1 xJ_1(t,\xi_t)\,dx\Big]\,d\xi_t+\Big[\int_0^1 x\Big(J_2(t,\xi_t)+\frac{1}{2}J_3(t,\xi_t)\Big)\,dx\Big]\,dt \\
&= \left[\int_0^1 xp(x|\xi_t)\frac{\sigma T}{T-t}\Big(x-H(\xi_t,t,T)\Big)\,dx\right]d\xi_t \\
&\quad+\int_0^1 x\left[p(x|\xi_t)\frac{\sigma T}{(T-t)^2}\Big[x\xi_t-\frac{\sigma T}{2}x^2-\xi_t H(\xi_t,t,T)+\frac{\sigma T}{2}\,\mathbb{E}_{\mathbb{P}}(H_T^2\,|\,\xi_t)\right. \\
&\quad\left.+\frac{\sigma T}{2}\Big(x-H(\xi_t,t,T)\Big)^2-\frac{\sigma T}{2}\mathrm{Var}\,(H_T|\,\xi_t)\Big]\,dx\right]dt \\
&= \frac{\sigma T}{T-t}\mathrm{Var}\,(H_T|\,\xi_t)\,d\xi_t+\frac{\sigma T}{(T-t)^2}\left[\xi_t\,\mathbb{E}_{\mathbb{P}}(H_T^2\,|\,\xi_t)-\frac{\sigma T}{2}\,\mathbb{E}_{\mathbb{P}}(H_T^3\,|\,\xi_t)-\xi_t H^2(\xi_t,t,T)\right. \\
&\quad+\frac{\sigma T}{2}H(\xi_t,t,T)\,\mathbb{E}_{\mathbb{P}}(H_T^2\,|\,\xi_t)+\frac{\sigma T}{2}\,\mathbb{E}_{\mathbb{P}}(H_T^3\,|\,\xi_t)-\sigma TH(\xi_t,t,T)\,\mathbb{E}_{\mathbb{P}}(H_T^2\,|\,\xi_t) \\
&\quad\left.+\frac{\sigma T}{2}H^3(\xi_t,t,T)-\frac{\sigma T}{2}H(\xi_t,t,T)\,\mathbb{E}_{\mathbb{P}}(H_T^2\,|\,\xi_t)+\frac{\sigma T}{2}H^3(\xi_t,t,T)\right]dt \\
&= \frac{\sigma T}{T-t}\mathrm{Var}\,(H_T|\,\xi_t)\,d\xi_t+\frac{\sigma T}{(T-t)^2}\Big[\xi_t\mathrm{Var}\,(H_T|\,\xi_t)-\sigma TH(\xi_t,t,T)\mathrm{Var}\,(H_T|\,\xi_t)\Big]\,dt \\
&= \frac{\sigma T}{T-t}\mathrm{Var}\,(H_T|\,\xi_t)\Big[\,d\xi_t+\frac{\xi_t}{T-t}\,dt-\frac{\sigma T}{T-t}H(\xi_t,t,T)\,dt\Big].
\end{aligned}
$$

Recall that

$$dW_t = d\xi_t+\frac{\xi_t}{T-t}\,dt-\frac{\sigma T}{T-t}H(\xi_t,t,T)\,dt.$$

Consequently, we also have that

$$dH(\xi_t,t,T)=\frac{\sigma T}{T-t}\mathrm{Var}\,(H_T|\,\xi_t)\,dW_t.$$

This ends the proof of the lemma. □

Remark 2.2 In the proof Lemma 2.2, we have found that

$$H_z(z,t,T)=\int_0^1 x\frac{\partial}{\partial z}p(x|z)\,dx=\frac{\sigma T}{T-t}\int_0^1 xp(x|z)\Big(x-H(z,t,T)\Big)\,dx=\frac{\sigma T}{T-t}\mathrm{Var}\,(H_T|\,\xi_t=z)$$

so that manifestly $H_z(z,t,T)>0$ for $t\in[0,T)$. This shows that, for any fixed $t\in[0,T)$, the price of a defaultable bond is a strictly increasing function of the information process. We also see that the function $H:\mathbb{R}\times[0,t]\to[0,1]$ is Lipschitz continuous with respect to z, for any $t<T$.

Corollary 2.1 *If the interest rates are deterministic then the price process $D(t,T)$ is a Markov process with respect to the filtration $\mathbb{F}^\xi$.*

2.2.1 Case of Deterministic Interest Rates

We shall first examine the dynamics of the process $D(t,T)$ under the assumption of deterministic interest rates. Let the function $r(t)$, $t \in [0,T]$, represent the short-term interest rate. Then we have

$$B(t,T) = e^{-\int_t^T r(u)\,du}, \quad \forall\, t \in [0,T]. \tag{23}$$

Proposition 2.2 *The dynamics of a bond price $D(t,T)$, under the risk-neutral probability $\mathbb{P}$, are*

$$dD(t,T) = r(t)D(t,T)\,dt + \frac{\sigma T}{T-t} B(t,T)\mathrm{Var}\,(H_T|\,\xi_t)\,dW_t \tag{24}$$

where the process

$$dW_t = d\xi_t + \frac{\xi_t}{T-t}\,dt - \frac{\sigma T}{T-t}\frac{D(t,T)}{B(t,T)}\,dt \tag{25}$$

is a Brownian motion under $\mathbb{P}$ with respect to $\mathbb{F}^\xi = \mathbb{F}^W$. Equivalently, we have

$$dD(t,T) = r(t)D(t,T)\,dt + \sigma B(t,T) Z_t\,dW_t \tag{26}$$

where the strictly positive process Z equals, for $t \in [0,T)$,

$$Z_t = \frac{T}{T-t}\mathrm{Var}\,(H_T|\,\xi_t) = \frac{T}{T-t}\int_0^1 \left(x - \frac{D(t,T)}{B(t,T)}\right)^2 p(x|\xi_t)\,dx. \tag{27}$$

Proof. By Itô's formula and Lemma 2.2, we get

$$\begin{aligned} dD(t,T) &= d\big(B(t,T)H(\xi_t,t,T)\big) \\ &= H(\xi_t,t,T)r(t)B(t,T)\,dt + B(t,T)\,dH(\xi_t,t,T) \\ &= r(t)D(t,T)\,dt + B(t,T)\,dH(\xi_t,t,T) \\ &= r(t)D(t,T)\,dt + \frac{\sigma T}{T-t}B(t,T)\mathrm{Var}\,(H_T|\,\xi_t)\,dW_t, \end{aligned}$$

as expected. To establish (25), it suffices to combine (16) with the equality $\widehat{H}_t = \frac{D(t,T)}{B(t,T)}$. Finally, formula (27) is an immediate consequence of (24) and the definition of the conditional variance. □

2.2.2 Case of Stochastic Interest Rates

If interest rates are random, it is natural to assume that we have two sources of uncertainty, where the first one drives the information process and the second drives interest rates. In this context, we find it convenient to denote by W^1 the Brownian motion W given by formula (16). Also, we postulate that $W^2 = \widehat{W}$ is a Brownian motion on (possibly extended) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent of both the Brownian motion W^1 and the random variable H_T. For brevity, we shall write $\overline{W} = (W^1, W^2)$.

Under the present assumptions, using (14) and (22), we thus obtain

$$d\widehat{H}_t = \frac{\sigma T}{T-t}\,\mathrm{Var}\,(H_T|\,\xi_t)\,dW_t^1 = \frac{\widehat{\sigma} T}{T-t}\,\mathrm{Var}\,(H_T|\,\xi_t)\,d\overline{W}_t$$

where we denote $\widehat{\sigma} = (\sigma, 0)$.

In view of pricing formula (7), it is clear that the probability measure $\mathbb{P}$ should now be interpreted as the *forward martingale measure*, frequently denoted as $\mathbb{P}_T$ in the literature (see, for instance, Musiela and Rutkowski [3]). Consequently, it is natural to postulate that the price process $B(t,T)$ of a T-maturity default-free discount bond satisfies under $\mathbb{P}$

$$dB(t,T) = B(t,T)\Big(\big(r_t + b_1^2(t,T) + b_2^2(t,T)\big)\,dt + b(t,T)\,d\overline{W}_t\Big) \tag{28}$$

for some $\mathbb{F}^{\overline{W}}$-adapted short-term rate process $\{r_t\}_{0\le t\le T}$ and some $\mathbb{R}^2$-valued volatility function (or process) $b(t,T)=(b_1(t,T),b_2(t,T))$. Equivalently,

$$dB(t,T)=B(t,T)\Big(r_t+b_1^2(t,T)+b_2^2(t,T)\Big)\,dt+B(t,T)\Big(b_1(t,T)\,dW_t^1+b_2(t,T)\,dW_t^2\Big).$$

Under the spot martingale measure $\mathbb{P}^*$, we thus have

$$dB(t,T)=B(t,T)\Big(r_t\,dt+b_1(t,T)\,dW_t^{*1}+b_2(t,T)\,dW_t^{*2}\Big)$$

where processes W^{*1} and W^{*2}, given by

$$W_t^{*i}=W_t^i+\int_0^t b_i(u,T)\,du,\quad i=1,2,$$

are independent Brownian motions under $\mathbb{P}^*$.

2.2.3 Dynamics under the Forward Martingale Measure

As explained above, the probability measure $\mathbb{P}$ is now interpreted as the forward martingale measure $\mathbb{P}_T$, in the sense, that the price $D(t,T)$ of a defaultable bond with maturity T equals (cf. (7))

$$D(t,T)=B(t,T)\,\mathbb{E}_{\mathbb{P}}(H_T\,|\,\mathcal{F}_t)=B(t,T)\widehat{H}_t,\quad \forall\, t\in[0,T]. \tag{29}$$

We are in the position to derive the dynamics of $D(t,T)$ in the framework of the market model with random interest rates.

Proposition 2.3 *When interest rates are assumed to be random, the dynamics of a defaultable bond price under the forward martingale measure $\mathbb{P}=\mathbb{P}_T$ are*

$$\begin{aligned} dD(t,T) &= \big(r_t+b_1^2(t,T)+b_2^2(t,T)\big)D(t,T)\,dt+B(t,T)b_1(t,T)\sigma Z_t\,dt \\ &\quad + B(t,T)Z_t\widehat{\sigma}\,d\overline{W}_t+D(t,T)b(t,T)\,d\overline{W}_t \\ &= \big(r_t+b_1^2(t,T)+b_2^2(t,T)\big)D(t,T)\,dt+B(t,T)b_1(t,T)\sigma Z_t\,dt \\ &\quad + B(t,T)\sigma Z_t\,dW_t^1+D(t,T)\big(b_1(t,T)\,dW_t^1+b_2(t,T)\,dW_t^2\big)\end{aligned}$$

where the strictly positive process Z_t equals, for $t\in[0,T)$,

$$Z_t=\frac{T}{T-t}\,\mathrm{Var}\,(H_T\,|\,\xi_t)=\frac{T}{T-t}\int_0^1\left(x-\frac{D(t,T)}{B(t,T)}\right)^2 p(x|\xi_t)\,dx. \tag{30}$$

The initial value equals $D(0,T)=B(0,T)\,\mathbb{E}_{\mathbb{P}}(H_T)$.

Proof. By Itô's formula, Lemma 2.2 and the current assumptions, we obtain

$$\begin{aligned} dD(t,T) &= d(B(t,T)\widehat{H}_t) \\ &= \widehat{H}_t\,dB(t,T)+B(t,T)\,d\widehat{H}_t+\langle B(\cdot,T),\widehat{H}\rangle_t \\ &= \widehat{H}_t\Big(\big(r_t+b_1^2(t,T)+b_2^2(t,T)\big)B(t,T)\,dt+B(t,T)b(t,T)\,d\overline{W}_t\Big) \\ &\quad + B(t,T)\Big(\frac{\widehat{\sigma}T}{T-t}\,\mathrm{Var}\,(H_T\,|\,\xi_t)\,d\overline{W}_t\Big)+B(t,T)b_1(t,T)\frac{\sigma T}{T-t}\,\mathrm{Var}\,(H_T\,|\,\xi_t)\,dt \\ &= B(t,T)\Big[\big(r_t+b_1^2(t,T)+b_2^2(t,T)\big)\widehat{H}_t+\frac{b_1(t,T)\sigma T}{T-t}\,\mathrm{Var}\,(H_T\,|\,\xi_t)\Big]\,dt \\ &\quad + B(t,T)\Big[b(t,T)\widehat{H}_t\,d\overline{W}_t+\frac{\widehat{\sigma}T}{T-t}\,\mathrm{Var}\,(H_T\,|\,\xi_t)\,d\overline{W}_t\Big] \\ &= B(t,T)\Big[\big(r_t+b_1^2(t,T)+b_2^2(t,T)\big)\widehat{H}_t+b_1(t,T)\sigma Z_t\Big]\,dt \\ &\quad + B(t,T)\Big[b(t,T)\widehat{H}_t\,d\overline{W}_t+\widehat{\sigma}Z_t\,d\overline{W}_t\Big]\end{aligned}$$

where Z_t is given (30). □

It is worth noting that the cross-variation of processes $D(t,T)$ and $B(t,T)$ equals

$$d\langle D(\cdot,T),B(\cdot,T)\rangle_t = B^2(t,T)\sigma b_1(t,T)\,dt + B(t,T)D(t,T)|b(t,T)|^2\,dt.$$

2.2.4 Dynamics under the Spot Martingale Measure

Let $\mathbb{P}^*$ be the spot martingale measure, so that the process $W^* = (W^{*1}, W^{*2})$, where

$$W_t^{*i} = W_t^i + \int_0^t b_i(u,T)\,du,$$

is a Brownian motion under $\mathbb{P}^*$ with $d\langle W^{*1},W^{*2}\rangle_t = 0$. Under the spot martingale measure $\mathbb{P}^*$, we have

$$dB(t,T) = B(t,T)\Big(r_t\,dt + b(t,T)\,dW_t^*\Big)$$

and

$$dD(t,T) = r_t D(t,T)\,dt + B(t,T)Z_t\widehat{\sigma}\,dW_t^* + D(t,T)b(t,T)\,dW_t^*. \tag{31}$$

Let B be the savings account process, defined by the formula

$$B_t = \exp\Big(\int_0^t r_u\,du\Big), \quad \forall\, t\in[0,T].$$

Then the discounted price processes $B^*(t,T) = B(t,T)B_t^{-1}$ and $D^*(t,T) = D(t,T)B_t^{-1}$ satisfy

$$dB^*(t,T) = B^*(t,T)b(t,T)\,dW_t^* \tag{32}$$

and

$$dD^*(t,T) = B^*(t,T)Z_t\widehat{\sigma}\,dW_t^* + D^*(t,T)b(t,T)\,dW_t^*. \tag{33}$$

Remark 2.3 Note that for $\sigma = 0$ we have $D(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{P}}(H_T)$ for every $t\in[0,T]$. Of course, this case is trivial, and thus we shall assume that $\sigma > 0$ in what follows.

The next result was already announced in Remark 1.1.

Lemma 2.3 *Assume that $\sigma \neq 0$ and $b_2(t,T)\neq 0$ for every $t\in[0,T)$. Then we have*

$$\mathbb{F}^\xi \vee \mathbb{F}^{\widehat{W}} = \mathbb{F}^W \vee \mathbb{F}^{\widehat{W}} = \mathbb{F}^{W^*} = \mathbb{F}^{D^*} \vee \mathbb{F}^{B^*}. \tag{34}$$

Proof. The first equality is obvious, since $\mathbb{F}^\xi = \mathbb{F}^W$. The second follows from the fact that

$$W_t^{*1} = W_t + \int_0^t b_1(u,T)\,du, \quad W_t^{*2} = \widehat{W}_t + \int_0^t b_2(u,T)\,du.$$

Finally, the last equality in (34) can be easily established using, for instance, (32)-(31). It suffices to observe that the matrix

$$A_t = \begin{bmatrix} b_1(t,T) & b_2(t,T) \\ \sigma Z_t + b_1(t,T) & b_2(t,T) \end{bmatrix}$$

is non-singular for every $t\in[0,T]$. □

2.2.5 Special Case

Assume that $b_1(t,T) = 0$ for every $t\in[0,T]$, so that the bond price $B(t,T)$ is independent of the information process ξ. Then we have, under the forward martingale measure $\mathbb{P}$,

$$dD(t,T) = \big(r_t + b_2^2(t,T)\big)D(t,T)\,dt + B(t,T)\sigma Z_t\,dW_t^1 + D(t,T)b_2(t,T)\,dW_t^2$$

and, under the spot martingale measure $\mathbb{P}^*$,

$$dD(t,T) = r_t D(t,T)\,dt + B(t,T)\sigma Z_t\,dW_t^{*1} + D(t,T)b_2(t,T)\,dW_t^{*2}. \tag{35}$$

Consequently,

$$d\langle D(\cdot,T),B(\cdot,T)\rangle_t = B(t,T)D(t,T)b_2(t,T)\,dt.$$

3 Valuation of Bond Options

The goal of this section is to find the replicating strategies and values for an option on a defaultable bond, under both deterministic and stochastic interest rates.

3.1 Replicating Strategies and Completeness

We shall first examine the completeness of security market models introduced in the previous section.

3.1.1 Case of Deterministic Interest Rates

By a *contingent claim* we mean any non-negative random variable X, which is $\mathcal{F}_t^\xi$-measurable for some $t < T$. We interpret X as a random payoff occurring at time t. A trading strategy $\varphi = (\varphi^1, \varphi^2)$ in primary assets $D(u,T)$ and $B(u,t)$ is *self-financing on* $[0,t]$ if the wealth process $V(\varphi)$, defined as

$$V_u(\varphi) = \varphi_u^1 D(u,T) + \varphi_u^2 B(u,t),$$

satisfies, for every $u \in [0,t]$,

$$V_u(\varphi) = V_0(\varphi) + \int_0^u \varphi_v^1 \, dD(v,T) + \int_0^u \varphi_v^2 \, dB(v,t).$$

Moreover, φ is said to be *admissible* if $V_u(\varphi) \geq 0$ for every $u \leq t$. Let $\Phi_{[0,t]}$ be the class of all self-financing admissible trading strategies on $[0,t]$.

We shall show that any integrable claim $X \in \mathcal{F}_t^\xi$ is *attainable*, in the sense, that there exists an admissible trading strategy $\varphi = (\varphi^1, \varphi^2) \in \Phi_{[0,t]}$ replicating X, so that X can be represented as follows

$$X = V_0(\varphi) + \int_0^t \varphi_u^1 \, dD(u,T) + \int_0^t \varphi_u^2 \, dB(u,t)$$

or equivalently,

$$X = \frac{V_0(\varphi)}{B(0,t)} + \int_0^t \varphi_u^1 \, d\left(\frac{D(u,T)}{B(u,t)}\right).$$

Proposition 3.1 *Assume that $\sigma \neq 0$. Then, for any $t \in [0,T)$, the model $(B(u,t), D(u,T); \Phi_{[0,t]})$ is complete, that is, any non-negative and integrable contingent claim is attainable.*

Proof. We first observe that since $X \in \mathcal{F}_t^\xi = \mathcal{F}_t^W$, by virtue of the predictable representation property of the Brownian motion, there exists a process ψ and a constant c_0 such that

$$X = c_0 + \int_0^t \psi_u \, dW_u.$$

Consequently, using (22), we obtain

$$X = c_0 + \int_0^t \psi_u \left(\frac{\sigma T}{T-u} \mathrm{Var}\,(H_T |\, \xi_u)\right)^{-1} d\widehat{H}_u = c_0 + \int_0^t \varphi_u^1 \, d\left(\frac{D(u,T)}{B(u,t)}\right)$$

since

$$\widehat{H}_u = \frac{D(u,T)}{B(u,T)} = \alpha \, \frac{D(u,T)}{B(u,t)}$$

where we denote $\alpha = 1/B(t,T)$. □

3.1.2 Case of Stochastic Interest Rates

Recall that in this case, the uncertainty is modeled by two independent Brownian motions $\widehat{W}^1, \widehat{W}^2$ and, as usual, by a random variable H_T. We now consider bonds $B(u,T)$, $D(u,T)$ and the savings account B_u as traded primary securities. A strategy $\varphi = (\varphi^0, \varphi^1, \varphi^2)$ is *self-financing* on $[0,t]$ if its *wealth process* $V(\varphi)$, defined as

$$V_u(\varphi) = \varphi^0_u B_u + \varphi^1_u D(u,T) + \varphi^2_u B(u,T),$$

satisfies, for every $u \in [0,t]$,

$$V_u(\varphi) = V_0(\varphi) + \int_0^u \varphi^0_v \, dB_v + \int_0^u \varphi^1_v \, dD(v,T) + \int_0^u \varphi^2_v \, dB(v,T).$$

As before, φ is said to be *admissible* if $V_u(\varphi) \geq 0$ for every $u \leq t$. Let $\widehat{\Phi}_{[0,t]}$ stand for the class of all self-financing admissible trading strategies on $[0,t]$ in the present set-up.

Then a claim $X \in \mathcal{F}^\xi_t \vee \mathcal{F}^{\widehat{W}}_t = \mathcal{F}^{W^*}_t$ is said to be *attainable* if there exists $\varphi \in \widehat{\Phi}_{[0,t]}$ such that

$$X = V_0(\varphi) + \int_0^t \varphi^0_u \, dB_u + \int_0^t \varphi^1_u \, dD(u,T) + \int_0^t \varphi^2_u \, dB(u,T),$$

or equivalently, if there exists a constant $V_0(\varphi)$ and the two predictable processes φ^1 and φ^2 such that

$$X^* = V_0(\varphi) + \int_0^t \varphi^1_u \, dD^*(u,T) + \int_0^t \varphi^2_u \, dB^*(u,T) \tag{36}$$

where we write $X^* = XB_t^{-1}$.

Proposition 3.2 *Assume that $\sigma \neq 0$ and $b_2(t,T) \neq 0$ for every $t \in [0,T)$. Then, for any $t \in [0,T)$, the model $(B_u, B(u,T), D(u,T); \widehat{\Phi}_{[0,t]})$ is complete, that is, any contingent claim $X \geq 0$, such that $X^* = XB_t^{-1}$ is integrable, is attainable.*

Proof. For any self-financing strategy ϕ, the discounted wealth process $V_t^*(\varphi) = V_t(\varphi)B_t^{-1}$ satisfies

$$V_t^*(\varphi) = V_0(\varphi) + \int_0^t \varphi^1_u \, dD^*(u,T) + \int_0^t \varphi^2_u \, dB^*(u,T)$$

where (see equations (32)-(31))

$$dB^*(u,T) = B^*(u,T) b(u,T) \, dW^*_u \tag{37}$$

and

$$dD^*(u,T) = B^*(u,T) Z_u \widehat{\sigma} \, dW^*_u + D^*(t,T) b(t,T) \, dW^*_u \tag{38}$$

where

$$Z_u = \frac{T}{T-u} \int_0^1 \left(x - \frac{D(u,T)}{B(u,T)} \right)^2 p(x|\xi_u) \, dx > 0.$$

To conclude the proof, it suffices to make use of the predictable representation theorem for the Brownian motion $W^* = (W^{*1}, W^{*2})$. There exists a process $\psi = (\psi^1, \psi^2)$ and a constant c_0 such that

$$X^* = c_0 + \int_0^t \psi_u \, dW^*_u.$$

Using (37)-(38), we derive the existence of a process $\varphi = (\varphi^1, \varphi^2)$, such that

$$X^* = V_0(\varphi) + \int_0^t \varphi^1_u \, dD^*(u,T) + \int_0^t \varphi^2_u \, dB^*(u,T).$$

where $V_0(\varphi) = c_0$. □

3.2 Option on a Defaultable Bond

We consider a European call option on a T-maturity defaultable bond, with expiration date $t < T$ and strike K. Hence, the option's payoff at expiry is represented by the random variable $X = (D(t,T) - K)^+$. In the first step, we focus on the price of a call option at time 0. Subsequently, derive the option price C_u for any $u \in [0,t]$. We shall employ an equivalent change of a probability measure analogous to that introduced in Brody et al. [1].

3.2.1 Generic Pricing Formula at Time 0

Since the model is complete, a European call option can be replicated, and thus its arbitrage price at time 0 is given by the formula

$$C_0 = B(0,t)\,\mathbb{E}_{\mathbb{P}}\Big[(D(t,T) - K)^+\Big].$$

By inserting the pricing formula for $D(t,T)$ (see (11)) into the above formula, we get

$$\begin{aligned} C_0 &= B(0,t)\,\mathbb{E}_{\mathbb{P}}\bigg[\Big(\int_0^1 xB(t,T)\frac{p(x)e^{\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)}}{\int_0^1 p(y)e^{\frac{T}{T-t}(\sigma y\xi_t - \frac{1}{2}\sigma^2y^2t)}\,dy}\,dx - K\Big)^+\bigg] \\ &= B(0,t)\,\mathbb{E}_{\mathbb{P}}\bigg[\Big(\frac{1}{\eta_t}\int_0^1 xB(t,T)p(x)e^{\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)}\,dx - K\Big)^+\bigg] \\ &= B(0,t)\,\mathbb{E}_{\mathbb{P}}\bigg[\frac{1}{\eta_t}\Big(\int_0^1 (xB(t,T) - K)p(x)e^{\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)}\,dx\Big)^+\bigg] \end{aligned}$$

where we denote

$$\eta_t = \int_0^1 p(x)e^{\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)}\,dx. \tag{39}$$

Let us examine the properties of the process η given by (39). Our goal is to show that the process $\psi_t := (\eta_t)^{-1}$ can be used to introduce an equivalent probability measure on $(\Omega, \mathcal{F}_t^\xi)$ for every $t < T$.

Lemma 3.1 *The dynamics of the process η, given by (39), are*

$$d\eta_t = \eta_t\bigg[\frac{\sigma T}{(T-t)^2}\xi_t\widehat{H}_t\,dt + \frac{\sigma T}{T-t}\widehat{H}_t\,d\xi_t\bigg]. \tag{40}$$

Proof. The proof is similar to the proof of Lemma 2.2. Note that $\eta_t = \Delta_2 = \int_0^1 \Delta_1\,dx$ and $p(x|\xi_t) = \Delta_1/\Delta_2$ where the auxiliary quantities Δ_1 and Δ_2 are defined in the proof of Lemma 2.2. Using Itô's formula, we obtain (we omit the variables (x,ξ_t,t) in Δ_1 and (ξ_t,t) in Δ_2 and Δ_3)

$$\begin{aligned} d\eta_t &= d\Delta_2 = \frac{\partial}{\partial z}\Delta_2\,d\xi_t + \frac{\partial}{\partial t}\Delta_2\,dt + \frac{1}{2}\frac{\partial^2}{\partial z^2}\Delta_2\,d\langle\xi\rangle_t \\ &= \Big(\frac{\sigma T}{T-t}\Delta_3\Big)\,d\xi_t + \bigg[\frac{\sigma T}{(T-t)^2}\Big(\xi_t\Delta_3 - \frac{1}{2}\sigma T\int_0^1 \Delta_1 x^2\,dx\Big)\bigg]\,dt + \frac{1}{2}\frac{\partial}{\partial z}\Big(\frac{\sigma T}{T-t}\Delta_3\Big)\,dt \\ &= \Delta_2\bigg[\Big(\frac{\sigma T}{T-t}\frac{\Delta_3}{\Delta_2}\Big)\,d\xi_t + \Big[\frac{\sigma T}{(T-t)^2}\Big(\xi_t\frac{\Delta_3}{\Delta_2} - \frac{1}{2}\sigma T\int_0^1 \frac{\Delta_1}{\Delta_2}x^2\,dx\Big) + \frac{1}{2}\frac{\sigma^2T^2}{(T-t)^2}\int_0^1 \frac{\Delta_1}{\Delta_2}x^2\,dx\Big]\,dt\bigg] \\ &= \Delta_2\bigg[\frac{\sigma T}{T-t}\widehat{H}_t\,d\xi_t + \Big[\frac{\sigma T}{(T-t)^2}\int_0^1 p(x|\xi_t)\Big(\xi_t x - \frac{1}{2}\sigma Tx^2 + \frac{1}{2}\sigma Tx^2\Big)\,dx\Big]\,dt\bigg] \\ &= \eta_t\bigg[\frac{\sigma T}{(T-t)^2}\xi_t\widehat{H}_t\,dt + \frac{\sigma T}{T-t}\widehat{H}_t\,d\xi_t\bigg], \end{aligned}$$

which is the desired equality. □

Lemma 3.2 *The dynamics of the process* $\psi_t := (\eta_t)^{-1}$ *are*

$$d\psi_t = \psi_t \left[-\frac{\sigma T}{T-t}\widehat{H}_t \right] \left[d\xi_t + \frac{\xi_t}{T-t}\,dt - \frac{\sigma T}{T-t}\widehat{H}_t\,dt \right] \tag{41}$$

or equivalently,

$$d\psi_t = \psi_t \left[-\frac{\sigma T}{T-t}\widehat{H}_t \right] dW_t. \tag{42}$$

Proof. Using Itô's formula and dynamics (40), we obtain

$$\begin{aligned} d\psi_t &= d\left(\frac{1}{\eta_t}\right) \\ &= -\frac{1}{\eta_t^2}\,d\eta_t + \frac{1}{\eta_t^3}\left(\eta_t^2 \frac{\sigma^2 T^2}{(T-t)^2}\widehat{H}_t^2\right) d\langle \xi \rangle_t \\ &= -\frac{1}{\eta_t^2}\left[\eta_t \frac{\sigma T}{(T-t)^2}\xi_t \widehat{H}_t\,dt + \eta_t \frac{\sigma T}{T-t}\widehat{H}_t\,d\xi_t\right] + \frac{1}{\eta_t}\frac{\sigma^2 T^2}{(T-t)^2}\widehat{H}_t^2\,dt \\ &= \psi_t\left[-\frac{\sigma T}{T-t}\widehat{H}_t\right]\left[d\xi_t + \frac{\xi_t}{T-t}\,dt - \frac{\sigma T}{T-t}\widehat{H}_t\,dt\right], \end{aligned}$$

as expected. Recall that

$$dW_t = d\xi_t + \frac{\xi_t}{T-t}\,dt - \frac{\sigma T}{T-t}\widehat{H}_t\,dt$$

where W is a standard Brownian motion defined by (16). The dynamics of ψ with respect to W is thus given by (42). □

Corollary 3.1 *The process* ψ_t, $t \in [0,T)$ *is a Radon-Nikodým density process with respect to* $\mathbb{P}$, *that is, it follows a strictly positive* $\mathbb{P}$*-martingale with* $\psi_0 = 1$.

Proof. It is clear that ψ is a strictly positive process with $\psi_0 = 1$ where the last equality follows from the equality $\eta_0 = 1$ (see (39)). Moreover, ψ is a martingale under $\mathbb{P}$, since, in view of (42), it satisfies

$$d\psi_t = \psi_t \Lambda_t\,dW_t$$

where Λ is a bounded process for $t \in [0,u]$, for any fixed $0 \le u < T$ (recall that $\widehat{H}$ is a bounded process). □

Let us fix $t \in [0,T)$, and let define a probability measure $\mathbb{Q}_t$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_t^\xi)$, by setting

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}} \,|\, \mathcal{F}_t = \psi_t = \frac{1}{\eta_t}. \tag{43}$$

Then we obtain the following generic pricing formula under $\mathbb{Q}_t$

$$\begin{aligned} C_0 &= B(0,t)\,\mathbb{E}_{\mathbb{P}}\left[\psi_t\Big(\int_0^1 (xB(t,T)-K)p(x)e^{\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t)}\,dx\Big)^+\right] \\ &= B(0,t)\,\mathbb{E}_{\mathbb{Q}_t}\left[\Big(\int_0^1 (xB(t,T)-K)p(x)e^{\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t)}\,dx\Big)^+\right]. \end{aligned}$$

The last equality can be seen as a convenient probabilistic representation of the option price at time 0. Before applying this result to derive more explicit pricing formulae, let us first extend it to any time $u \in [0,t)$.

3.2.2 Generic Pricing Formula at Time $u \in [0,t)$

It is well known that the price of a European call option at time $u \in [0,t)$ can be represented as follows

$$C_u = B(u,t)\,\mathbb{E}_{\mathbb{P}}\Big[\big(D(t,T)-K\big)^+\,\Big|\,\mathcal{F}_u\Big]. \tag{44}$$

Lemma 3.3 *Let us fix $t \in [0,T)$, and let us define the process U^* by the formula*

$$U^*_u = W_u + \int_0^u \frac{\sigma T}{T-v}\widehat{H}_v\,dv, \quad \forall\, u \in [0,t]. \tag{45}$$

Also, let a probability measure $\mathbb{Q}_t$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}^\xi_t)$ be given by means of the Radon-Nikodým density

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}}\,|\,\mathcal{F}_t = \psi_t = \frac{1}{\eta_t} \tag{46}$$

where η_t is defined in (39). Then U^* is a standard Brownian motion on the space $(\Omega, \mathcal{F}, \mathbb{Q}_t)$.

Proof. From Lemma 3.1, we know that η_t can be written as

$$\frac{1}{\eta_t} = \exp\left[\int_0^t \Big(-\frac{\sigma T}{T-u}\widehat{H}_u\Big)\,dW_u - \frac{1}{2}\int_0^t \frac{\sigma^2T^2}{(T-u)^2}\widehat{H}^2_u\,du\right]. \tag{47}$$

Hence, the proof can be easily completed using Girsanov's theorem. □

The next result examines the properties of the information process ξ_u, $u \in [0,t]$ under $\mathbb{Q}_t$.

Proposition 3.3 *For any fixed $t \in [0,T)$, the process ξ_u, $u \in [0,t]$ follows a Brownian bridge on the space $(\Omega, \mathcal{F}_t, \mathbb{Q}_t)$.*

Proof. Recall that

$$dW_u = d\xi_u + \frac{\xi_u}{T-u}\,du - \frac{\sigma T}{T-u}\widehat{H}_u\,du, \tag{48}$$

and

$$dU^*_u = dW_u + \frac{\sigma T}{T-u}\widehat{H}_u\,du. \tag{49}$$

Therefore

$$d\xi_u = dU^*_u - \frac{1}{T-u}\xi_u\,du. \tag{50}$$

Since U^*_u, $u \in [0,t]$, is a standard Brownian motion on the space $(\Omega, \mathcal{F}_t, \mathbb{Q}_t)$, the last equality is just the definition of a Brownian bridge on the space $(\Omega, \mathcal{F}_t, \mathbb{Q}_t)$. □

Remark 3.1 Let us stress that a probability measure $\mathbb{Q}_t$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_t)$ and the family $\mathbb{Q}_t$, $t \in [0,T)$ is consistent, in the sense that $\mathbb{Q}_t\,|\,\mathcal{F}_s = \mathbb{Q}_s$ for every $0 \le s \le t < T$. However, we cannot claim that the information process ξ_t, $t \in [0,T]$, follows a Brownian bridge under some probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_t)$. On the contrary, if $\widehat{\mathbb{Q}}$ is some probability measure such that ξ_t, $t \in [0,T]$, follows a Brownian bridge under $\widehat{\mathbb{Q}}$ then $\widehat{\mathbb{Q}}$ and $\mathbb{P}$ are singular if $H_T > 0$. Indeed, we then have $\mathbb{P}(H_T > 0) = 1$ and $\widehat{\mathbb{Q}}(\xi_T = 0) = \widehat{\mathbb{Q}}(H_T = 0) = 1$.

In the next two results, we fix $t \in [0,T)$ and, for the sake of brevity, we write $\mathbb{Q} = \mathbb{Q}_t$.

Corollary 3.2 *The price of a call option at time $u \in [0,t)$ is given by the following expression*

$$C_u = \frac{B(u,t)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}\left[\Big(\int_0^1 (xB(t,T)-K)p(x)\exp\Big[\frac{\sigma x T}{T-u}\xi_u + \sigma x T Y_t - \frac{1}{2}\frac{\sigma^2x^2tT}{T-t}\Big]\,dx\Big)^+\,\Big|\,\mathcal{F}_u\right]$$

where

$$Y_t = \frac{\xi_t}{T-t} - \frac{\xi_u}{T-u}.$$

Proof. Using (44), the Bayes formula, and Lemma 3.3, we obtain

$$C_u = \frac{B(u,t)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}\bigg[\Big(\int_0^1 (xB(t,T)-K)p(x)e^{\frac{T}{T-t}(\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t)}\,dx\Big)^+\,\Big|\,\mathcal{F}_u\bigg]. \tag{51}$$

To obtain the desired representation for the price C_u, we write

$$Y_t = \frac{\xi_t}{T-t} - \frac{\xi_u}{T-u}, \tag{52}$$

and we substitute $\xi_t = (T-t)\Big(Y_t + \frac{\xi_u}{T-u}\Big)$ into (51). □

Representation of C_u in terms of ξ_u and Y_t is convenient for computational purposes, since the random variable Y_t has the Gaussian distribution $N\Big(0, \frac{t-u}{(T-t)(T-u)}\Big)$ under $\mathbb{Q}$ and it is independent of $\mathcal{F}_u$. In fact, it can be shown that Y_t is a random variable independent of ξ_t and ξ_u (see Karatzas and Shreve [2], Page 359).

Proposition 3.4 *The price C_u admits the following representation under $\mathbb{Q}$*

$$C_u = \frac{B(u,t)}{\eta_u}\Big[\mathbb{E}_{\mathbb{Q}}\big(B(t,T)\zeta_t\mathbb{1}_A\,\big|\,\mathcal{F}_u\big) - \mathbb{E}_{\mathbb{Q}}\big(K\eta_t\mathbb{1}_A\,\big|\,\mathcal{F}_u\big)\Big] \tag{53}$$

where $A = \{B(t,T)\zeta_t > K\eta_t\}$ and

$$\zeta_t = \int_0^1 xp(x)\exp\Big[\frac{\sigma xT}{T-u}\xi_u + \sigma xTY_t - \frac{1}{2}\frac{\sigma^2x^2tT}{T-t}\Big]\,dx.$$

Proof. We have

$$\begin{aligned}
C_u &= \frac{B(u,t)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}\bigg[\Big(\int_0^1 (xB(t,T)-K)p(x)\exp\Big[\frac{\sigma xT}{T-u}\xi_u + \sigma xTY_t - \frac{1}{2}\frac{\sigma^2x^2tT}{T-t}\Big]\,dx\Big)^+\,\Big|\,\mathcal{F}_u\bigg]\\
&= \frac{B(u,t)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}\bigg[\Big(B(t,T)\int_0^1 xp(x)\exp\Big[\frac{\sigma xT}{T-u}\xi_u + \sigma xTY_t - \frac{1}{2}\frac{\sigma^2x^2tT}{T-t}\Big]\,dx\\
&\quad -K\int_0^1 p(x)\exp\Big[\frac{\sigma xT}{T-u}\xi_u + \sigma xTY_t - \frac{1}{2}\frac{\sigma^2x^2tT}{T-t}\Big]\,dx\Big)^+\,\Big|\,\mathcal{F}_u\bigg]\\
&= \frac{B(u,t)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}\Big[\Big(B(t,T)\zeta_t - K\eta_t\Big)^+\,\Big|\,\mathcal{F}_u\Big].
\end{aligned}$$

Let $A = \{B(t,T)\zeta_t > K\eta_t\}$. Then C_u can be represented using the indicator function $\mathbb{1}_A$ as follows

$$\begin{aligned}
C_u &= \frac{B(u,t)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}\Big[\Big(B(t,T)\zeta_t - K\eta_t\Big)\mathbb{1}_A\,\Big|\,\mathcal{F}_u\Big]\\
&= \frac{B(u,t)}{\eta_u}\Big[\mathbb{E}_{\mathbb{Q}}\big(B(t,T)\zeta_t\mathbb{1}_A\,\big|\,\mathcal{F}_u\big) - \mathbb{E}_{\mathbb{Q}}\big(K\eta_t\mathbb{1}_A\,\big|\,\mathcal{F}_u\big)\Big]
\end{aligned}$$

and thus the proposition is established. □

In Sections 3.2.3 and 3.2.4 below, the generic pricing formula of Proposition 3.4 is employed to derive more explicit representations for the option price under deterministic and stochastic interest rates, respectively.

3.2.3 Case of Deterministic Interest Rates

Let us first recall the concept of the forward price.

Definition 3.1 The *forward price* of a T-maturity defaultable bond is given by the formula

$$F_D(u,t,T) = \frac{D(u,T)}{B(u,t)}, \quad \forall\, u \in [0,t]. \tag{54}$$

By the definition above, the forward price of a call option equals

$$F_C(u,t) = \frac{C_u}{B(u,t)}, \quad \forall\, u \in [0,t]. \tag{55}$$

As before, we fix a date $T \in [0,T)$ and we denote $\mathbb{Q} = \mathbb{Q}_t$. We have, for any $u \in [0,t]$,

$$\zeta_u = \int_0^1 xp(x) e^{\frac{T}{T-u}(\sigma x \xi_u - \frac{1}{2}\sigma^2 x^2 u)}\, dx,$$

$$\eta_u = \int_0^1 p(x) e^{\frac{T}{T-u}(\sigma x \xi_u - \frac{1}{2}\sigma^2 x^2 u)}\, dx.$$

For $u = t$, we shall use the following representations of ζ_t and η_t

$$\zeta_t = \int_0^1 xp(x) \exp\Big[\frac{\sigma x T}{T-u}\xi_u + \sigma x T Y_t - \frac{1}{2}\frac{\sigma^2 x^2 t T}{T-t}\Big]\, dx,$$

$$\eta_t = \int_0^1 p(x) \exp\Big[\frac{\sigma x T}{T-u}\xi_u + \sigma x T Y_t - \frac{1}{2}\frac{\sigma^2 x^2 t T}{T-t}\Big]\, dx,$$

where Y_t is given by (52). Then we have the following result.

Proposition 3.5 *The price of call option on a defaultable bond price under deterministic interest rates equals*

$$C_u = D(u,T)M_1(u,t,T) - KB(u,t)M_2(u,t,T) \tag{56}$$

where $A = \{B(t,T)\zeta_t > K\eta_t\}$ *and*

$$M_1(u,t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\frac{\zeta_t}{\zeta_u}\mathbb{1}_A \,\Big|\, \mathcal{F}_u\Big), \quad M_2(u,t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\frac{\eta_t}{\eta_u}\mathbb{1}_A \,\Big|\, \mathcal{F}_u\Big).$$

Equivalently,

$$F_C(u,t) = F_D(u,t,T)M_1(u,t,T) - KM_2(u,t,T). \tag{57}$$

Proof. Since the interest rates are assumed to be deterministic, using the definitions of $D(u,T)$ and $H(\xi_u,u,T)$, we obtain

$$D(u,T) = B(u,T)H(\xi_u,u,T) = B(u,T)\frac{\zeta_u}{\eta_u}, \quad \forall\, u \in [0,t].$$

Hence, using (53), we may represent the call option price at any time $u \in [0,t]$ as follows

$$\begin{aligned}
C_u &= \frac{B(u,t)}{\eta_u}\Big[B(t,T)\,\mathbb{E}_{\mathbb{Q}}(\zeta_t \mathbb{1}_A \,|\, \mathcal{F}_u) - \mathbb{E}_{\mathbb{Q}}(K\eta_t \mathbb{1}_A \,|\, \mathcal{F}_u)\Big] \\
&= \frac{B(u,T)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}(\zeta_t \mathbb{1}_A \,|\, \mathcal{F}_u) - K\frac{B(u,t)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}(\eta_t \mathbb{1}_A \,|\, \mathcal{F}_u) \\
&= \frac{D(u,T)}{\zeta_u}\,\mathbb{E}_{\mathbb{Q}}(\zeta_t \mathbb{1}_A \,|\, \mathcal{F}_u) - K\frac{B(u,t)}{\eta_u}\,\mathbb{E}_{\mathbb{Q}}(\eta_t \mathbb{1}_A \,|\, \mathcal{F}_u) \\
&= D(u,T)\,\mathbb{E}_{\mathbb{Q}}\Big(\frac{\zeta_t}{\zeta_u}\mathbb{1}_A \,\Big|\, \mathcal{F}_u\Big) - KB(u,t)\,\mathbb{E}_{\mathbb{Q}}\Big(\frac{\eta_t}{\eta_u}\mathbb{1}_A \,\Big|\, \mathcal{F}_u\Big) \\
&= D(u,T)M_1(u,t,T) - KB(u,t)M_2(u,t,T)
\end{aligned}$$

as desired. To derive (57), it suffices to divide both sides of (56) by $B(u,t)$. □

3.2.4 Case of Stochastic Interest Rates

As explained in Section 2.2.2, under stochastic interest rates the process $\widehat{H}$ has under $\mathbb{P}$ the following dynamics

$$d\widehat{H}_u = \frac{\sigma T}{T-u}\,\mathrm{Var}\,(H_T|\,\xi_u)\,dW^1_u = \frac{\widehat{\sigma} T}{T-u}\,\mathrm{Var}\,(H_T|\,\xi_u)\,d\overline{W}_u$$

with respect to a two-dimensional Brownian motion $\overline{W} = (W^1, W^2)$, where $\widehat{\sigma} = (\sigma, 0)$. Moreover

$$dB(u,T) = B(u,T)\Big(\big(r_u + b_1^2(u,T) + b_2^2(u,T)\big)\,du + b(u,T)\,d\overline{W}_u\Big)$$

for some $\mathbb{F}^{\overline{W}}$-adapted short-term rate process $\{r_u\}_{0\le u\le T}$ and some $\mathbb{R}^2$-valued volatility function $b(u,T) = (b_1(u,T), b_2(u,T))$.

It can be easily seen that the change of probability measure from $\mathbb{P}$ to $\mathbb{Q}_t$ only affects the processes W^1 and $\widehat{H}$. This means the Radon-Nikodým density of ψ_t can be employed in the case of stochastic interest rates as well. Using Proposition 3.4 and proceeding as in the proof of Proposition 3.5, we thus obtain the following result.

Proposition 3.6 *The price of call option on a defaultable bond price under stochastic interest rates equals*

$$C_u = D(u,T)\widetilde{M}_1(u,t,T) - KB(u,t)\widetilde{M}_2(u,t,T) \tag{58}$$

where $A = \{B(t,T)\zeta_t > K\eta_t\}$ *and*

$$\widetilde{M}_1(u,t,T) = \frac{B(u,t)}{D(u,T)}\,\mathbb{E}_{\mathbb{Q}}\Big(D(t,T)\frac{\eta_t}{\eta_u}\mathbb{1}_A\,\Big|\,\mathcal{F}_u\Big), \quad \widetilde{M}_2(u,t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\frac{\eta_t}{\eta_u}\mathbb{1}_A\,\Big|\,\mathcal{F}_u\Big).$$

Remark 3.2 Note here $\widetilde{M}_2(u,t,T) = M_2(u,t,T)$, but the equality $\widetilde{M}_1(u,t,T) = M_1(u,t,T)$ holds only when r is deterministic.

3.2.5 Approximation of the Option Price

In Section 2.2.2, we provided the dynamics of defaultable bond $D(u,T)$ in the case of stochastic interest rates under the forward and spot martingale measures. Now, we will directly use these results to derive an approximating formula for the option price.

From Proposition 2.3, we know that the dynamics of $D(u,T)$ under $\mathbb{P}$ are

$$\begin{aligned} dD(u,T) &= (r_u + |b(u,T)|^2)D(u,T)\,du + B(u,T)b(u,T)\widehat{\sigma}Z_u\,du \\ &\quad + B(u,T)Z_u\widehat{\sigma}\,d\overline{W}_u + D(u,T)b(u,T)\,d\overline{W}_u \end{aligned}$$

where $\overline{W} = (W^1, W^2)$ is a two-dimensional Brownian motion. Notice that $\widehat{H}_u > 0$ for $u \in [0,T)$, so that we can rewrite the last formula as follows (see (29))

$$\begin{aligned} dD(u,T) &= (r_u + |b(u,T)|^2)D(u,T)\,du + \frac{D(u,T)}{\widehat{H}_u}b(u,T)\widehat{\sigma}Z_u\,du \\ &\quad + \frac{D(u,T)}{\widehat{H}_u}Z_u\widehat{\sigma}\,d\overline{W}_u + D(u,T)b(u,T)\,d\overline{W}_u \\ &= D(u,T)\left[\Big(r_u + b(u,T)\Big(b(u,T) + \frac{Z_u\widehat{\sigma}}{\widehat{H}_u}\Big)\Big)\,du + \Big(\frac{Z_u\widehat{\sigma}}{\widehat{H}_u} + b(u,T)\Big)\,d\overline{W}_u\right]. \end{aligned}$$

Hence, the defaultable bond price is also given by the following expression

$$\begin{aligned} D(t,T) &= D(0,T)\exp\left[\int_0^t \Big(r_s + b(s,T)\Big(b(s,T) + \frac{Z_s\widehat{\sigma}}{\widehat{H}_s}\Big)\Big)\,ds + \int_0^t \Big(\frac{Z_s\widehat{\sigma}}{\widehat{H}_s} + b(s,T)\Big)\,d\overline{W}_s\right] \\ &= D(u,T)\exp\left[\int_u^t \Big(r_s + b(s,T)\Big(b(s,T) + \frac{Z_s\widehat{\sigma}}{\widehat{H}_s}\Big)\Big)\,ds + \int_u^t \Big(\frac{Z_s\widehat{\sigma}}{\widehat{H}_s} + b(s,T)\Big)\,d\overline{W}_s\right]. \end{aligned}$$

Since stochastic models of the term structure of interest rates can be constructed in many ways, the details of the approximating procedure may vary from case to case. For the sake of simplicity, we shall assume that the ratio $Z_s/\widehat{H}_s$ can be approximated by some deterministic function $h(s)$. In particular, we can simply take $h(s) = Z_0/\widehat{H}_0 = \mathrm{Var}\,(H_T)/\,\mathbb{E}_{\mathbb{P}}(H_T)$ for every $s \in [0,t]$. If we replace $Z_s/\widehat{H}_s$ by $h(s)$, the process appearing in the exponent of the pricing formula has a Gaussian distribution.

Specifically, when $h(s) \approx Z_s/\widehat{H}_s$ we denote

$$\begin{aligned} L(u,t,T) &:= \int_u^t \Big(r_s + b(s,T)\big(\widehat{\sigma}h(s) + b(s,T)\big)\Big)\,ds + \int_u^t \Big(\widehat{\sigma}h(s) + b(s,T)\Big)\,d\overline{W}_s \\ &\approx \int_u^t \Big(r_s + b(s,T)\Big(\frac{Z_s\widehat{\sigma}}{\widehat{H}_s} + b(s,T)\Big)\Big)\,ds + \int_u^t \Big(\frac{Z_s\widehat{\sigma}}{\widehat{H}_s} + b(s,T)\Big)\,d\overline{W}_s, \end{aligned}$$

so that $L(u,t,T)$ has a Gaussian conditional distribution $N(\mu_L(u,t,T), \sigma_L^2(u,t,T))$ with respect to the σ-field $\mathcal{F}_u$ under $\mathbb{P}$. This assumption leads to the following result, yielding a simple Gaussian approximation for the price of a European call option on a defaultable bond. Note that the approximation introduced below makes perfect sense both under deterministic and under stochastic interest rates. The accuracy of this approximation is yet to be examined, however.

Proposition 3.7 *Under the assumption that the conditional distribution of $L(u,t,T)$ with respect to the σ-field $\mathcal{F}_u$ under $\mathbb{P}$ is Gaussian $N(\mu_L(u,t,T), \sigma_L^2(u,t,T))$, the forward price $F^C(u,t)$ of a European call option at time $u \in [0,t)$ can be approximated as follows*

$$F_C(u,t) \approx D(u,T)\exp\Big(\mu_L(u,t,T) + \frac{1}{2}\sigma_L^2(u,t,T)\Big)N\Big(d_1(D(u,T),u,t,T)\Big) - KN\Big(d_2(D(u,T),u,t,T)\Big)$$

where N is the standard Gaussian cumulative distribution function and

$$\begin{aligned} d_1(D(u,T),u,t,T) &= \frac{\ln\Big(\frac{D(u,T)}{K}\Big) + \mu_L(u,t,T) + \sigma_L^2(u,t,T)}{\sigma_L(u,t,T)}, \\ d_2(D(u,T),u,t,T) &= \frac{\ln\Big(\frac{D(u,T)}{K}\Big) + \mu_L(u,t,T)}{\sigma_L(u,t,T)}, \end{aligned} \tag{59}$$

so that $d_2(D(u,T),u,t,T) = d_1(D(u,T),u,t,T) - \sigma_L(u,t,T)$.

Proof. Let $A = \{D(t,T) > K\}$. Then the price at time $u \in [0,t)$ of a call option can be written as

$$\begin{aligned} C_u &= B(u,t)\,\mathbb{E}_{\mathbb{P}}\big((D(t,T) - K)\mathbb{1}_A \,|\, \mathcal{F}_u\big) \\ &= B(u,t)\,\mathbb{E}_{\mathbb{P}}(D(t,T)\mathbb{1}_A \,|\, \mathcal{F}_u) - B(u,t)K\,\mathbb{E}_{\mathbb{P}}(\mathbb{1}_D \,|\, \mathcal{F}_u) \\ &\approx B(u,t)D(u,T)\,\mathbb{E}_{\mathbb{P}}(\exp(L(u,t,T))\mathbb{1}_A \,|\, \mathcal{F}_u) - B(u,t)K\,\mathbb{E}_{\mathbb{P}}(\mathbb{1}_A \,|\, \mathcal{F}_u). \end{aligned}$$

For the expectation in the second term, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A \,|\, \mathcal{F}_u) &= \mathbb{P}(D(t,T) > K \,|\, \mathcal{F}_u) \\ &= \mathbb{P}(D(u,T)\exp(L(u,t,T)) > K \,|\, \mathcal{F}_u) \\ &= \mathbb{P}\Big(L(u,t,T) > \ln\Big(\frac{K}{D(u,T)}\Big) \,\Big|\, \mathcal{F}_u\Big) \\ &= \mathbb{P}\left(\frac{L(u,t,T) - \mu_L(u,t,T)}{\sigma_L(u,t,T)} > \frac{\ln(\frac{K}{D(u,T)}) - \mu_L(u,t,T)}{\sigma_L(u,t,T)} \,\Big|\, \mathcal{F}_u\right) \\ &= N\left(\frac{\ln(\frac{D(u,T)}{K}) + \mu_L(u,t,T)}{\sigma_L(u,t,T)}\right). \end{aligned}$$

The expectation in the first term can be calculated as follows (we omit the parameters in $\mu_L(u,t,T)$ and $\sigma_L(u,t,T)$, and we denote $\widetilde{K} = K/D(u,T)$)

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}(\exp(L(u,t,T))\mathbb{1}_A \,|\, \mathcal{F}_u) = \int_{\ln \widetilde{K}}^{\infty} \exp(l) p(l)\, dl \\
&= \int_{\ln \widetilde{K}}^{\infty} \exp(l) \frac{1}{\sqrt{2\pi}\sigma_L} \exp\Big(-\frac{1}{2\sigma_L^2}(l-\mu_L)^2\Big)\, dl \\
&= \int_{\ln \widetilde{K}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_L} \exp\Big(-\frac{1}{2\sigma_L^2}\big(l^2 - 2l\mu_L + \mu_L^2 - 2l\sigma_L^2\big)\Big)\, dl \\
&= \exp\Big(-\frac{1}{2\sigma_L^2}\big(-(\mu_L+\sigma_L^2)^2 + \mu_L^2\big)\Big) \int_{\ln \widetilde{K}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_L} \exp\Big(-\frac{1}{2\sigma_L^2}\big(l - (\mu_L+\sigma_L^2)\big)^2\Big)\, dl \\
&= \exp\Big(\mu_L + \frac{1}{2}\sigma_L^2\Big) \int_{\frac{\ln \widetilde{K} - (\mu_L+\sigma_L^2)}{\sigma_L}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}y^2\Big)\, dy.
\end{aligned}$$

It is thus clear that

$$\mathbb{E}_{\mathbb{P}}\big(\exp(L(u,t,T))\mathbb{1}_A \,\big|\, \mathcal{F}_u\big) = \exp\Big(\mu_L + \frac{1}{2}\sigma_L^2\Big) N\left(\frac{\ln(\frac{D(u,T)}{K}) + \mu_L + \sigma_L^2}{\sigma_L}\right).$$

Combining the above calculations, and denoting

$$\begin{aligned}
d_1(D(u,T),u,t,T) &= \frac{\ln\Big(\frac{D(u,T)}{K}\Big) + \mu_L(u,t,T) + \sigma_L^2(u,t,T)}{\sigma_L(u,t,T)}, \\
d_2(D(u,T),u,t,T) &= \frac{\ln\Big(\frac{D(u,T)}{K}\Big) + \mu_L(u,t,T)}{\sigma_L(u,t,T)},
\end{aligned}$$

we finally obtain

$$\begin{aligned}
C_u &\approx B(u,t)D(u,T)\,\mathbb{E}_{\mathbb{P}}(\exp(L(u,t,T))\mathbb{1}_A \,|\, \mathcal{F}_u) - B(u,t)K\,\mathbb{E}_{\mathbb{P}}(\mathbb{1}_A \,|\, \mathcal{F}_u) \\
&= B(u,t)D(u,T)\exp\Big(\mu_L(u,t,T) + \frac{1}{2}\sigma_L^2(u,t,T)\Big) N\Big(d_1(D(u,T),u,t,T)\Big) \\
&\quad -B(u,t)KN\Big(d_2(D(u,T),u,t,T)\Big)
\end{aligned}$$

where $d_1(D(u,T),u,t,T)$ and $d_2(D(u,T),u,t,T)$ satisfy

$$d_1(D(u,T),u,t,T) - d_2(D(u,T),u,t,T) = \sigma_L(u,t,T).$$

This completes the proof of the proposition. □

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