VALUATION OF CREDIT DEFAULT SWAPTIONS AND CREDIT DEFAULT INDEX SWAPTIONS

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1 Credit Default Swaps and Swaptions

We provide simple and rigorous, albeit fairly general, derivations of valuation formulae for credit default swaptions (Section 1) and credit default index swaptions (Section 2). Results of this work cover as special cases the pricing formulae derived previously by Jamshidian [15], Pedersen [19], Brigo and Morini [11], and, more recently, by Morini and Brigo [18] (see also Doctor and Goulden [12], Hull and White [13], Jackson [14] and Liu and Jaeckel [17]). For a more detailed discussion of existing pricing methods for credit default (index) swaptions, we refer to Armstrong [1] and Armstrong and Rutkowski [2]. Most results presented in this work are independent of a particular convention regarding the specification of the fee and protection legs and thus they can be applied in valuation of other credit derivatives that exhibit similar features (e.g., options on CDO tranches).

In addition to general representations for swaptions' prices, we derive explicit valuation formulae based on a specification of dynamics of a suitably defined spread processes (this can be seen as an example of the *top-down approach*). As an alternative, in a single name case, one can also produce pricing formulae based on the dynamics of default intensity (see, e.g., Brigo and Alfonsi [9], Brigo and Cousot [10] or Brigo and El-Bachir [8]). For a credit default index swaption, this alternative *bottom-up approach* would be less practically appealing, since it would require a specification of the joint dynamics of a family of individual loss processes for a large portfolio of reference credit names, and the resulting dynamics of the spread process would become rather complicated.

Let us mention that the issue of hedging is dealt with only marginally in this work. For more details on hedging of single- and multi-name credit derivatives with traded credit default swaps, the interested reader is referred to Bielecki et al. [4]–[6].

A *credit default swap* (CDS) is an over-the-counter contract between two counterparties – the protection buyer and the protection seller – in which protection against the risk of default by the underlying reference entity is provided to the buyer. The protection buyer pays a premium at regular intervals to the protection seller in order to obtain the right to receive a contingent payment from the seller following a credit event by the reference entity. Examples of a credit event include default, restructuring, a ratings downgrade, or a failure to pay; for simplicity, we will refer to a credit event as being a default. If no default event takes place then the only cash flows are those from the protection buyer, who pays at periodic intervals (usually quarterly) a predetermined premium κ (termed CDS spread) until the contract expires. If, however, there is a default event, the protection buyer will cease those premium payments at the time of the event (including one final accrual payment at the default time). If a physical settlement has been agreed upon, the buyer will deliver any debt instrument (known as the reference obligation) of the reference entity to the seller, in exchange for a cash payment covering the reference obligations par value. If, on the other hand, a cash settlement has been agreed upon, then the seller will provide a cash payment equal to the market value of the reference obligation to the buyer. Once the loss to the buyer has been covered, the contract is terminated. A plain-vanilla credit default swaption is a European option on the value of the underlying forward credit default swap. An essential feature of this contract is that it is cancelled if default occurs prior to the swaption's maturity. It thus can formally be seen as an example of a *survival claim* (cf. Jamshidian [15]). Let us mention that, due to the presence of the *front-end protection*, this feature is no longer valid for a credit default index swaption examined in Section 2 of this work and thus this case requires a different approach.

1.1 Default Time and Reference Filtration

A strictly positive random variable τ defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is called a *random time*; it will also be later referred to as the *default time*. We introduce the jump process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ associated with τ and we denote by $\mathbb{H}$ the filtration generated by this process. We assume that we are given, in addition, some *reference filtration* $\mathbb{F}$ and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration*, since it is meant to convey the full information available to investors.

Remarks 1.1 It is worth stressing that an additional filtration is required to address the issue of spread risk. Indeed, credit risk models based the filtration $\mathbb{H}$ (or on its multi-name extension $\mathbb{H} = \mathbb{H}^1 \vee \ldots \vee \mathbb{H}^n$, as defined in Section 2.1) are only able to handle the *jump risk* associated with default events (cf. [4]), as opposed to the *credit spread risk*, which manifests itself by fluctuations of credit spreads for traded credit default swaps prior to and between default times of underlying credit names (cf. [5]).

Note that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time for any choice of a filtration $\mathbb{F}$, but it is not an $\mathbb{F}$-stopping time, in general. Let thus the process F be defined by the equality, for every $t \in \mathbb{R}_+$,

$$F_t = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t). \tag{1}$$

Let $G_t = 1 - F_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$ be the *survival process* with respect to the filtration $\mathbb{F}$ and let us temporarily assume that and $G_t > 0$ for every $t \in \mathbb{R}_+$ so that, in particular, τ is not an $\mathbb{F}$-stopping time. The assumption that the process G is strictly positive will be slightly weakened later on, since it is enough to postulate positivity of G for any date that precedes the maturity of a contract at hand.

For any $\mathbb{Q}$-integrable and $\mathcal{F}_T$-measurable random variable Y we have the following well-known results (see, for instance, Chapter 5 in Bielecki and Rutkowski [3] or Jeanblanc and Rutkowski [16])

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{T<\tau\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \, G_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(G_T Y \,|\, \mathcal{F}_t). \tag{2}$$

The following lemma is also standard.

Lemma 1.1 *Assume that Y is some $\mathbb{G}$-adapted stochastic process. Then there exists a unique $\mathbb{F}$-adapted process $\widetilde{Y}$ such that, for every $t \in \mathbb{R}_+$,*

$$Y_t = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t. \tag{3}$$

The process $\widetilde{Y}$ is termed the pre-default value *of the process Y.*

Let us recall that we may attach to the strictly positive survival process G the *hazard process* Γ, defined as $\Gamma_t = -\ln G_t$. If the hazard process Γ is absolutely continuous, so that $\Gamma_t = \int_0^t \gamma_u \, du$ for some $\mathbb{F}$-predictable process γ, then we say that τ admits the $\mathbb{F}$-*intensity process* γ or simply, that γ is the intensity of default. In the special case when the default intensity γ is well defined, some formulae presented below can be represented in terms of this process. Neither the existence of the default intensity nor even the continuity of the survival process G are required for our further developments, however.

1.2 Forward-Start Credit Default Swap

We write $T_0 = T < T_1 < \cdots < T_J$ to denote the *tenor structure* of a forward-start credit default swap, where:

- $T_0 = T$ is the CDS inception date;
- T_J is the maturity date of the CDS;
- T_j is the jth fee payment date for $i = 1, 2, \ldots, J$.

Let $\mathbb{1}_A$ be the indicator function of an event A and let τ be a strictly positive random variable representing the moment of default of the reference credit name. We set $\beta(\tau) = T_{j-1}$ on the event $\{T_{j-1} \le \tau < T_j\}$ and we write $\alpha_j = T_j - T_{j-1}$ for every $j = 1, 2, \ldots, J$. Let B be an $\mathbb{F}$-adapted and strictly positive process modelling the savings account (or any other strictly positive numeraire). From now on, the underlying probability measure $\mathbb{Q}$ is interpreted as a martingale measure associated with the choice of B as the numeraire asset. Let Z be a bounded and $\mathbb{F}$-adapted stochastic process. In practical implementations of Definition 1.1, it is common to postulate that $Z = 1 - \delta$, where δ is the constant recovery rate.

Definition 1.1 The *forward credit default swap* issued at time $s \in [0,T]$, with unit notional, protection payment Z and $\mathcal{F}_s$-measurable spread κ is determined by its discounted payoff, which equals $D_t = P_t - \kappa A_t$ for every $t \in [s,T]$, where the discounted payoffs of the *protection leg* and the *fee leg* are given by

$$P_t = B_t Z_\tau B_\tau^{-1} \mathbb{1}_{\{T \le \tau \le T_J\}} \tag{4}$$

and

$$A_t = B_t \sum_{j=1}^{J} \alpha_j B_{T_j}^{-1} \mathbb{1}_{\{T_j < \tau\}} + B_t B_\tau^{-1} (\tau - T_{\beta(\tau)}) \mathbb{1}_{\{T < \tau \le T_J\}} \tag{5}$$

respectively. The *fair price* at time $t \in [s,T]$ of a forward credit default swap for the protection buyer equals

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}}(D_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(P_t \,|\, \mathcal{G}_t) - \kappa \mathbb{E}_{\mathbb{Q}}(A_t \,|\, \mathcal{G}_t).$$

Brigo and Morini [18] examine also a simplified version $\bar{D}_t$ of the actual discounted payoff D_t, which is given by the following expression

$$\bar{D}_t = (1-\delta) B_t \sum_{j=1}^{J} \alpha_j B_{T_j}^{-1} \mathbb{1}_{\{T_{j-1} < \tau \le T_j\}} - \kappa B_t \sum_{j=1}^{J} \alpha_j B_{T_j}^{-1} \mathbb{1}_{\{T_j < \tau\}} = \bar{P}_t - \kappa \bar{A}_t. \tag{6}$$

We do not require any specific convention of this kind for our further purposes, however. In fact, all results of this work can be applied to any particular convention regarding the protection and fee legs. It is only required that all cash flows occur after the inception date T and the knock out feature, that is, the property that the contract becomes void if default occurs prior to or at T. Formally, it is sufficient to assume that the discounted payoff at time $t \in [0,T]$ of the examined contract has the form

$$D_t = P_t - \kappa A_t, \tag{7}$$

where P_t and A_t are discounted payoffs of certain T-*survival claims*, meaning that, for every $t \in [0,T]$,

$$P_t = \mathbb{1}_{\{T<\tau\}} P_t, \quad A_t = \mathbb{1}_{\{T<\tau\}} A_t. \tag{8}$$

To conclude, the specific formulae for processes P_t and A_t are not relevant and thus the results presented in the sequel can be applied to alternative variants of credit default swaps or other contracts with similar features.

In what follows, we only require that the inequality $G_t > 0$ holds for every $t \in [0,T_1]$, so that, in particular, we have that $G_{T_1} = \mathbb{Q}(\tau > T_1 \,|\, \mathcal{F}_{T_1}) > 0$.

Lemma 1.2 *The price at time $t \in [s,T]$ of the forward CDS issued at time $s \in [0,T]$ satisfies*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} G_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(D_t \,|\, \mathcal{F}_t) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa), \tag{9}$$

where the pre-default price satisfies $\widetilde{S}_t(\kappa) = \widetilde{P}_t - \kappa \widetilde{A}_t$, where in turn

$$\widetilde{P}_t = G_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(P_t \,|\, \mathcal{F}_t), \quad \widetilde{A}_t = G_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(A_t \,|\, \mathcal{F}_t). \tag{10}$$

If the discounted payoffs P_t and A_t are given by (4) and (5), respectively, then

$$\widetilde{P}_t = G_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} Z_\tau B_\tau^{-1} \mathbb{1}_{\{T_{j-1} < \tau \le T_j\}} \,\Big|\, \mathcal{F}_t \Big) \tag{11}$$

and

$$\widetilde{A}_t = G_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} \alpha_j B_{T_j}^{-1} \mathbb{1}_{\{T_j < \tau\}} + B_\tau^{-1} (\tau - T_{\beta(\tau)-1}) \mathbb{1}_{\{T_0 < \tau \le T_J\}} \,\Big|\, \mathcal{F}_t \Big). \tag{12}$$

Proof. Recall that by definition

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}}(D_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(P_t \,|\, \mathcal{G}_t) - \kappa \mathbb{E}_{\mathbb{Q}}(A_t \,|\, \mathcal{G}_t).$$

By combining (2) with (7) and (8), we thus obtain

$$\mathbb{E}_{\mathbb{Q}}(P_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} G_t^{-1}\, \mathbb{E}_{\mathbb{Q}}(P_t \,|\, \mathcal{F}_t) = \mathbb{1}_{\{t<\tau\}} \widetilde{P}_t$$

and

$$\mathbb{E}_{\mathbb{Q}}(A_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} G_t^{-1}\, \mathbb{E}_{\mathbb{Q}}(A_t \,|\, \mathcal{F}_t) = \mathbb{1}_{\{t<\tau\}} \widetilde{A}_t,$$

so that the proof of (10) is completed. To establish equalities (11) and (12), it suffices to make use of (11) and explicit representations (4) and (5) for P_t and A_t. □

The quantity $\widetilde{P}_t$ is the pre-default value at time $t \in [s, T]$ of the protection leg per unit of the nominal, whereas $\widetilde{A}_t$ represents the pre-default value at time $t \in [0, T]$ of the fee leg per one basis point of the spread. The latter is frequently referred to as the (pre-default) *present value of a basis point* of the CDS, but it is also known as the *risky PVBP* or the *CDS annuity.* It is worth noting that neither $\widetilde{P}_t$ nor $\widetilde{A}_t$ depend on the initiation date s of a forward CDS.

1.3 Pre-default Fair Forward CDS Spread

Since the forward CDS contract is terminated at default with no payments, the fair (or par) forward CDS spread is only defined prior to default. It is thus natural to introduce the concept of the *pre-default* fair forward CDS spread, rather than the fair forward CDS spread.

Definition 1.2 The *pre-default fair forward CDS spread* at time $t \in [0, T]$ is the $\mathcal{F}_t$-measurable random variable κ_t such that $\widetilde{S}_t(\kappa_t) = 0$.

The following result is a simple consequence of Lemma 1.2.

Lemma 1.3 *The pre-default fair forward CDS spread satisfies, for any $t \in [0, T]$,*

$$\kappa_t = \frac{\widetilde{P}_t}{\widetilde{A}_t} = \frac{\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^J Z_\tau B_\tau^{-1} \mathbb{1}_{\{T_{j-1}<\tau\le T_j\}} \,\Big|\, \mathcal{F}_t \Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^J \alpha_j B_{T_j}^{-1} \mathbb{1}_{\{T_j<\tau\}} + B_\tau^{-1}(\tau - T_{\beta(\tau)-1}) \mathbb{1}_{\{T_0<\tau\le T_J\}} \,\Big|\, \mathcal{F}_t \Big)}, \tag{13}$$

where the second equality holds if the discounted payoffs P_t and A_t are given by (4) and (5), respectively. The price of the forward CDS issued at $s \in [0, T]$ with an $\mathcal{F}_s$-measurable spread κ admits the following representation, for every $t \in [s, T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{A}_t(\kappa_t - \kappa). \tag{14}$$

Proof. Since κ_t is $\mathcal{F}_t$-measurable, from Lemma 1.2, we obtain that $\widetilde{S}_t(\kappa) = \widetilde{P}_t - \kappa_t \widetilde{A}_t$. It is thus clear that the first equality in (13) holds, provided that $\widetilde{A}_t > 0$. Under the standing assumption that $G_{T_1} > 0$, it can be deduced easily from (12) that $\widetilde{A}_t > 0$ for every $t \in [0, T]$ (recall that $\widetilde{A}_t$ does not depend on s). Indeed, using (12), we obtain

$$\begin{aligned}
\widetilde{A}_t &\ge G_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}}\big(\alpha_1 B_{T_1}^{-1} \mathbb{1}_{\{T_1<\tau\}} \,\big|\, \mathcal{F}_t\big) = G_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}}\big(\alpha_1 B_{T_1}^{-1} \mathbb{Q}(\tau > T_1 \,|\, \mathcal{F}_{T_1}) \,\big|\, \mathcal{F}_t\big) \\
&= G_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}}\big(\alpha_1 B_{T_1}^{-1} G_{T_1} \,\big|\, \mathcal{F}_t\big) > 0.
\end{aligned}$$

For the second equality in (13), we make use of (11) and (12). To derive (14), it is enough to observe that

$$\widetilde{S}_t(\kappa) = \widetilde{S}_t(\kappa) - \widetilde{S}_t(\kappa_t) = \widetilde{P}_t - \kappa \widetilde{A}_t - (\widetilde{P}_t - \kappa_t \widetilde{A}_t) = \widetilde{A}_t(\kappa_t - \kappa),$$

where we have used the equality $\widetilde{S}_t(\kappa_t) = 0$ (cf. Definition 1.2). □

It is worth noting that the pre-default fair forward CDS spread depends on the current date, the tenor structure of the forward CDS, the term structure of interest rates, the survival process G and the recovery rate δ. It is well defined for any market conventions regarding the payments structures for the protection and fee legs of a forward CDS. It is only required that the quantity $\widetilde{A}_t$ is non-zero; in fact, it is typically a strictly positive process. In particular, the first equality in (13) is universal and thus all foregoing general results are valid for any convention regarding the timing and amounts of cash flows of a forward CDS. In fact, after a minor, but essential, modification it will be also valid for the forward credit default index swap. We will show that it suffices to replace the default time τ with the moment of the last default in a reference credit portfolio (for details, see Section 2.3).

In practical implementations of the pricing formula (14), one needs to compute the quantity $\widetilde{A}_t$, since the market quote for this term is not readily available. The computation of $\widetilde{A}_t$ hinges on the concept of the *implied risk-neutral default probabilities*, which for a single-name case are obtained from market quotes for CDS spreads for different maturities, i.e., from the current CDS spread curve.

1.4 Credit Default Swaptions

Let us now focus on plain-vanilla options related to a forward credit default swap. A *credit default swaption* gives its holder, who pays an upfront fee, the right – but not the obligation – to buy (or sell) the protection on a prearranged single-name CDS. It thus can be seen as a call (or put) option with strike zero written on the market value of the underlying CDS at the option's expiry date.

An important feature of a credit default swaption is the *knock out* feature. For a single-name swaption, if the sole reference entity of the underlying CDS defaults at time τ before the option is exercised then the option is knocked out. This means that the credit default swaption is nullified and thus terminates with zero value. We will later see that this is different to the case of a *credit default index swaption* where, if several (but not all) of the many reference entities default, the swaption will continue until its maturity.

It is postulated throughout that the underlying contract is the forward CDS issued at time $s \in [0, T]$ with an $\mathcal{F}_s$-measurable spread κ, as specified by Definition 1.1. We assume that the exercise date of the swaption is $U \leq T$, that is, the swaption expires either before or at the start date $T_0 = T$ of the underlying forward CDS. Of course, we make the obvious assumption that $s \in [0, U]$.

Definition 1.3 The *credit default swaption* to enter a forward CDS with an $\mathcal{F}_s$-measurable spread κ at a future date $U \leq T$ has the payoff at maturity equal to $C_U = (S_U(\kappa))^+$.

It is worth noting that the credit default swaption is knocked out if default occurs prior to or at maturity U. This feature is already encoded in the payoff C_U, since we have that

$$C_U = (S_U(\kappa))^+ = \big(\mathbb{1}_{\{U<\tau\}} S_U(\kappa)\big)^+ = \mathbb{1}_{\{U<\tau\}} (S_U(\kappa))^+. \tag{15}$$

The price at time $t \in [s, U]$ of the European claim C_U is given by the risk-neutral valuation formula

$$C_t = B_t \, \mathbb{E}_{\mathbb{Q}} \Big(B_U^{-1} \big(S_U(\kappa)\big)^+ \,\Big|\, \mathcal{G}_t \Big) = B_t \, \mathbb{E}_{\mathbb{Q}} \Big(\mathbb{1}_{\{U<\tau\}} B_U^{-1} \widetilde{A}_U (\kappa_U - \kappa)^+ \,\Big|\, \mathcal{G}_t \Big), \tag{16}$$

where we used (14) in the second equality. The next lemma furnishes a representation for the price of a credit default swaption in terms of the reference filtration $\mathbb{F}$.

Lemma 1.4 *The price at time $t \in [s, U]$ of the credit default swaption equals*

$$C_t = \mathbb{1}_{\{t<\tau\}} B_t G_t^{-1} \, \mathbb{E}_{\mathbb{Q}} \left(G_U B_U^{-1} \widetilde{A}_U (\kappa_U - \kappa)^+ \,\Big|\, \mathcal{F}_t \right). \tag{17}$$

Proof. The random variable $Y = B_U^{-1}\widetilde{A}_U(\kappa_U - \kappa)^+$ is manifestly $\mathcal{F}_U$-measurable. Hence the equality is an immediate consequence of formulae (2) and (16). $\square$

The pricing formula (17) can be further simplified by a suitable change of a probability measure. Generally speaking, we follow here the ideas of Jamshidian [15], Brigo [7] and Brigo and Morini [18]. It is worth stressing, however, that neither the so-called hypothesis (H) nor any additional assumptions on the hazard process G of τ are required in the derivation of pricing formulae for the credit default swaps and swaptions. The only assumption we make is that the survival process G is strictly positive, but possibly discontinuous.

Let us define an equivalent probability measure $\widetilde{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_U)$ by postulating that the Radon-Nikodým density of $\widetilde{\mathbb{Q}}$ with respect to $\mathbb{Q}$ equals

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = cG_U B_U^{-1}\widetilde{A}_U =: \eta_U, \quad \mathbb{Q}\text{-a.s.} \tag{18}$$

Observe that the process $\eta_t = cG_t B_t^{-1}\widetilde{A}_t$, $t \in [s, U]$, is a strictly positive $\mathbb{F}$-martingale under $\mathbb{Q}$, since

$$\eta_t = cG_t B_t^{-1}\widetilde{A}_t = c\,\mathbb{E}_{\mathbb{Q}}(B_t^{-1}A_t \,|\, \mathcal{F}_t) = c\,\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{F}_t),$$

where the random variable $X = B_t^{-1}A_t$ is independent of $t \in [s, U]$ (cf. (5)). Therefore, for every $t \in [s, U]$,

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|\mathcal{F}_t = \mathbb{E}_{\mathbb{Q}}(\eta_U \,|\, \mathcal{F}_t) = \eta_t, \quad \mathbb{Q}\text{-a.s.}$$

The quantity $c = \big(\mathbb{E}_{\mathbb{Q}}(G_U B_U^{-1}\widetilde{A}_U)\big)^{-1}$ is simply the normalizing constant, which ensures that $\mathbb{E}_{\mathbb{Q}}(\eta_U) = 1$, so that $\widetilde{\mathbb{Q}}$ given by (18) is indeed a probability measure on $(\Omega, \mathcal{F}_U)$.

Lemma 1.5 *The price at time $t \in [s, U]$ of a credit default swaption satisfies $C_t = \mathbb{1}_{\{t<\tau\}}\widetilde{C}_t$, where the pre-default price of the swaption equals*

$$\widetilde{C}_t = \widetilde{A}_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big((\kappa_U - \kappa)^+ \,\big|\, \mathcal{F}_t\big). \tag{19}$$

Proof. Using (17), we obtain

$$\begin{aligned} C_t &= \mathbb{1}_{\{t<\tau\}} B_t G_t^{-1}\,\mathbb{E}_{\mathbb{Q}}\Big(G_U B_U^{-1}\widetilde{A}_U(\kappa_U - \kappa)^+ \,\Big|\, \mathcal{F}_t\Big) = \mathbb{1}_{\{t<\tau\}}\widetilde{A}_t \eta_t^{-1}\,\mathbb{E}_{\mathbb{Q}}\big(\eta_U(\kappa_U - \kappa)^+ \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{t<\tau\}}\widetilde{A}_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big((\kappa_U - \kappa)^+ \,\big|\, \mathcal{F}_t\big), \end{aligned}$$

where the last equality is an immediate consequence of the abstract Bayes formula. $\square$

The next lemma shows the change of the probability measure from $\mathbb{Q}$ to $\widetilde{\mathbb{Q}}$ is crucial. It shows that the drift term in the dynamics of the fair forward CDS spread κ under $\mathbb{Q}$ will not appear in the pricing formula for the credit default swaption.

Lemma 1.6 *The pre-default fair forward CDS spread κ_t, $t \in [0, U]$ is a strictly positive $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$.*

Proof. The product $\kappa\eta$ is manifestly an $\mathbb{F}$-martingale under $\mathbb{Q}$, since it satisfies, for every $t \in [0, U]$,

$$\kappa_t \eta_t = c\kappa_t G_t B_t^{-1}\widetilde{A}_t = cG_t B_t^{-1}\widetilde{P}_t = c\,\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} Z_\tau B_\tau^{-1}\mathbb{1}_{\{T_{j-1}<\tau\leq T_j\}} \Big|\, \mathcal{F}_t\Big).$$

By the well-known result, this implies that κ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. $\square$

1.5 Black Formula for Credit Default Swaptions

Let us assume that $\mathbb{F}$ is the Brownian filtration, specifically, that it is generated by some Brownian motion W defined on the underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. It is not essential for our purposes to assume that W is a Brownian motion with respect to the filtration $\mathbb{G}$. In other words, we need not postulate that the hypothesis (H) is satisfied by filtrations $\mathbb{F}$ and $\mathbb{G}$.

Recall that the process κ_t, $t \in [0, U]$ is a strictly positive, $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. Since W is a Brownian motion under $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ is equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{F}_U)$, the standard arguments can be used to show that κ admits the following integral representation

$$\kappa_t = \kappa_0 + \int_0^t \sigma_u \kappa_u \, d\widetilde{W}_u, \quad \forall \, t \in [0, U], \tag{20}$$

where $\widetilde{W}$ is a Brownian motion under $\widetilde{\mathbb{Q}}$ and σ is some $\mathbb{F}$-predictable process.

Proposition 1.1 *Assume that the volatility σ of the pre-default fair forward CDS spread is a positive function. Then the pre-default price of the credit default swaption with an $\mathcal{F}_s$-measurable strike κ equals, for every $t \in [s, U]$,*

$$\widetilde{C}_t = \widetilde{A}_t \Big(\kappa_t N\big(d_+(\kappa_t, t, U)\big) - \kappa N\big(d_-(\kappa_t, t, U)\big) \Big) \tag{21}$$

or, equivalently,

$$\widetilde{C}_t = \widetilde{P}_t \, N\big(d_+(\kappa_t, t, U)\big) - \kappa \widetilde{A}_t \, N\big(d_-(\kappa_t, t, U)\big), \tag{22}$$

where

$$d_\pm(\kappa_t, t, U) = \frac{\ln(\kappa_t/\kappa) \pm \frac{1}{2} \int_t^U \sigma^2(u) \, du}{\big(\int_t^U \sigma^2(u) \, du \big)^{1/2}}.$$

We also have that

$$d(\widetilde{C}_t / \widetilde{A}_t) = N\big(d_+(\kappa_t, t, U)\big) \, d\kappa_t. \tag{23}$$

Proof. In view of (19), we obtain

$$\widetilde{C}_t = \widetilde{A}_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big((\kappa_U - \kappa)^+ \,\big|\, \mathcal{F}_t \big) = \widetilde{A}_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big((\kappa_U - \kappa)^+ \,\big|\, \kappa_t \big) = \widetilde{A}_t \Big(\kappa_t N\big(d_+(\kappa_t, t, U)\big) - \kappa N\big(d_-(\kappa_t, t, U)\big) \Big)$$

where the last equality follows by standard computations. Equality (23) follows by an application of Itô's formula to (21). □

1.6 Hedging of Credit Default Swaptions

To get the simplest hedging strategy for a credit default swaption, we assume that a forward CDS issued at time $v \in [0, s]$ with an $\mathcal{F}_v$-measurable rate $\widehat{\kappa}$ is traded. In other words, we assume that the traded forward CDS and the forward CDS underlying the swaption have the same covenants, but they are issued at (possibly) different dates and thus they have different spreads, in general. Of course, the most natural candidate for the forward CDS used for hedging the swaption is the underlying forward CDS, but this particular choice is not necessary.

It is apparent from (10) that $\widetilde{A}_t$ can be seen as the pre-default value at time $t \in [0, T]$ of a particular portfolio of defaultable bonds with zero recovery, referred to hereafter as the *swap portfolio*. We assume that the swap portfolio is traded. Note that if default occurs at some date $t \in [0, T]$, the wealth of this portfolio falls to zero. Of course, the same property holds also for the CDS and the credit default swaption. Hence in what follows it suffices to focus on the dynamics of the pre-default value of the swaption and the pre-default wealth of a hedging portfolio.

Let A_t be the price process of the swap portfolio at time $t \in [0, T]$. Formally, we set $A_t = \mathbb{1}_{\{t < \tau\}} \widetilde{A}_t$. Recall also that $S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} \widetilde{S}_t(\kappa)$.

Let $\phi = (\phi^1, \phi^2)$ be *trading strategy* on $[s, U]$, where ϕ^1 and ϕ^2 are $\mathbb{G}$-predictable processes. The wealth of ϕ equals, for any $t \in [s, U]$,

$$V_t(\phi) = \phi^1_t S_t(\widehat{\kappa}) + \phi^2_t A_t$$

and thus the pre-default wealth satisfies, for any $t \in [s, U]$,

$$\widetilde{V}_t(\phi) = \phi^1_t \widetilde{S}_t(\widehat{\kappa}) + \phi^2_t \widetilde{A}_t.$$

Of course, the equality $V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)$ holds for any $t \in [s, U]$. Therefore, it suffices to examine a hedging strategy on the interval $[s, s \vee \tau \wedge U]$. Put another way, we postulate that trading is stopped either at default or maturity, whichever comes first. In view of Lemma 1.1, it thus suffices to consider $\mathbb{F}$-predictable processes ϕ^1 and ϕ^2 representing the pre-default values of the corresponding $\mathbb{G}$-predictable processes, in the sense of Lemma 1.1.

A strategy ϕ is required to be *self-financing*, in the sense that

$$d\widetilde{V}_t(\phi) = \phi^1_t \, d\widetilde{S}_t(\widehat{\kappa}) + \phi^2_t \, d\widetilde{A}_t.$$

It is easy to show by Itô's formula that the relative pre-default wealth satisfies

$$d(\widetilde{V}_t(\phi)/\widetilde{A}_t) = \phi^1_t \, d(\widetilde{S}_t(\widehat{\kappa})/\widetilde{A}_t). \tag{24}$$

Proposition 1.2 *The replicating strategy ϕ for the credit default swaption with the terminal payoff $C_U = (S_U(\kappa))^+$ is given by, for any $t \in [s, s \vee \tau \wedge U]$,*

$$\phi^1_t = N\big(d_+(\kappa_t, t, U)\big), \quad \widetilde{C}_t = \phi^1_t S_t(\widehat{\kappa}) + \phi^2_t \widetilde{A}_t. \tag{25}$$

Proof. On the one hand, we have (cf. (23))

$$d(\widetilde{C}_t/\widetilde{A}_t) = N\big(d_+(\kappa_t, t, U)\big) \, d\kappa_t.$$

On the other hand, since $\widetilde{S}_t(\widehat{\kappa}) = \widetilde{P}_t - \widehat{\kappa}\widetilde{A}_t$ and $\widehat{\kappa}$ is a fixed random variable, we obtain

$$d(\widetilde{S}_t(\widehat{\kappa})/\widetilde{A}_t) = d(\kappa_t - \widehat{\kappa}) = d\kappa_t$$

and thus, in view of (24),

$$d(\widetilde{V}_t(\phi)/\widetilde{A}_t) = \phi^1_t \, d(\widetilde{S}_t(\widehat{\kappa})/\widetilde{A}_t) = \phi^1_t \, d\kappa_t.$$

It thus is apparent that the strategy ϕ given by (25) is self-financing and its pre-default wealth satisfies $\widetilde{V}_t(\phi) = \widetilde{C}_t$ for any $t \in [s, U]$. As already mentioned above, it default occurs prior to or at maturity U then the wealth of the hedging portfolio falls to zero and the same feature is enjoyed by the value of the credit default swaption. □

In view of (24), we also have that, for any $t \in [s, U]$,

$$C_t/\widetilde{A}_t = C_s/\widetilde{A}_s + \int_s^t \phi^1_u \, d(\widetilde{S}_u(\widehat{\kappa})/\widetilde{A}_u) = C_s/\widetilde{A}_s + \int_s^t \phi^1_u \, d\kappa_u. \tag{26}$$

Hence hedging can also be interpreted in terms of the forward CDS.

In practice, hedging can also be achieved by taking positions at any date t in the *market CDS*, that is, the just-issued CDS with the spread κ_t. However, an explicit representation for this hedging strategy is rather cumbersome in the continuous-time set-up, since one needs to deal with a continuum of traded assets.

2 Credit Default Index Swaps and Swaptions

A *credit default index swap* (CDIS) is a standardized contract that is based upon a fixed portfolio of reference entities. The ever increasing trade of index credit derivatives has been estimated at being upwards of US\$90 billion annually. The two main indices to which CDSs are referenced are CDX, referring to companies within North America and iTraxx, which refers to companies within Europe and Asia. We look at the case of the CDX, although iTraxx and other indices have very similar characteristics. At its conception, the CDX is referenced to $n = 125$ fixed companies that are chosen by market makers. These 125 reference entities are specified to have equal weights within the CDX. If we assume each has a nominal value of one then, because of the equal weighting, the total notional would be 125. In effect, one CDX provides on average the same protection to that of 125 single-name CDSs upon the same reference entities.

By contrast to a standard single-name CDS, the 'buyer' of the CDX provides protection to the market makers. In other words, by purchasing a CDX from market makers the investor is not receiving protection, rather they are providing it to the market makers. In exchange for the protection the investor is providing, the market makers pay the investor a periodic fixed premium, otherwise known as the *credit default index spread.* Such standardized contracts promote liquidity within the derivatives market.

Typically, the recovery rate $\delta \in [0, 1]$ is predetermined and constant for all reference entities in the index. By purchasing the index the investor is agreeing to pay the market makers $1 - \delta$ for any default that occurs before maturity. That is, following a default the investor has to cover the loss incurred, which is achieved by paying to the market maker the amount of $1 - \delta$. Following this, the nominal value of the CDX is reduced by one. Once a removal has taken place there is no replacement of the defaulted firm. This process repeats after every default and the CDX continues on until maturity.

The standard maturities of a CDX are five and ten years with payments occurring quarterly. However, in more recent times, three and seven-year products have been introduced. New CDXs are defined semi-annually and the fixed rate, reference entities and maturities are reconfigured by the market makers according to current market conditions. Such changes do not alter pre-existing contracts. iTraxx and other credit default index swaps operate analogously with the CDX, with the only distinctions being in the contract details – the premium, the number and choice of reference entities and reconfiguration procedures.

2.1 Default Times and Reference Filtration

As we are now working within the multi-name case, the appropriate notation needs to be introduced. Let $\mathbb{G}$ be the filtration generated by the reference filtration $\mathbb{F}$ and filtrations $\mathbb{H}^1, \ldots, \mathbb{H}^n$, where $\mathbb{H}^i$ is the filtration generated by the default indicator $H^i_t = \mathbb{1}_{\{\tau_i \le t\}}$ of the ith credit name. We may now define $\mathbb{H} = \mathbb{H}^1 \vee \ldots \vee \mathbb{H}^n$ and we may represent $\mathbb{G}$ as follows $\mathbb{G} = \mathbb{H} \vee \mathbb{F} = \mathbb{H}^1 \vee \ldots \vee \mathbb{H}^n \vee \mathbb{F}$. This decomposition of the filtration $\mathbb{G}$ is not suitable for efficient computations based on a standard reduced-form approach summarized in Section 1.1, however.

As a more viable alternative, we first introduce the sequence $\tau_{(1)} \le \ldots \le \tau_{(n)}$ of ordered default times associated with the original sequence of default times $\tau_1, \ldots, \tau_n$. In fact, since we will only deal with underlying models in which simultaneous defaults are excluded, we may assume, without loss of generality, that the ordering above is strict.

We thus have $\mathbb{G} = \mathbb{H}^{(n)} \vee \widehat{\mathbb{F}}$, where $\mathbb{H}^{(n)}$ is the filtration generated by the indicator process $H^{(n)}_t = \mathbb{1}_{\{\tau_{(n)} \le t\}}$ of the last default and $\widehat{\mathbb{F}} = \mathbb{F} \vee \mathbb{H}^{(1)} \vee \ldots \vee \mathbb{H}^{(n-1)}$. For brevity, in what follows we will write $\widehat{\tau} = \tau_{(n)}$ to denote the random time when all firms in a given portfolio are in default. The usefulness of this particular decomposition of the full filtration in the context of valuation of credit default index swaptions was noted independently by Armstrong and Rutkowski [2] and Morini and Brigo [18].

We will be interested in events of the form $\{\widehat{\tau} \leq t\}$ and $\{\widehat{\tau} > t\}$ for a fixed t. Morini and Brigo [18] refer to these events as the *armageddon* and the *no-armageddon* events, respectively. We decided to use instead the terms *collapse* event and the *pre-collapse* event, respectively. The event $\{\widehat{\tau} \leq t\}$ corresponds to the total collapse of the reference portfolio, in the sense that all underlying credit names default either prior to or at time t.

Similarly as in Section 1.1 (cf. formula (1)), we start by defining the auxiliary process $\widehat{F}$, which is now given by the following expression, for every $t \in \mathbb{R}_+$,

$$\widehat{F}_t = \mathbb{Q}(\widehat{\tau} \leq t \,|\, \widehat{\mathcal{F}}_t).$$

Let us denote by $\widehat{G}_t = 1 - \widehat{F}_t = \mathbb{Q}(\widehat{\tau} > t \,|\, \widehat{\mathcal{F}}_t)$ the corresponding survival process with respect to the filtration $\widehat{\mathbb{F}}$ and let us temporarily assume that the inequality $\widehat{G}_t > 0$ holds for every $t \in \mathbb{R}_+$. Then for any $\mathbb{Q}$-integrable and $\widehat{\mathcal{F}}_T$-measurable random variable Y we have that (cf. (2))

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{T<\widehat{\tau}\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\widehat{\tau}\}} \, \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(\widehat{G}_T Y \,|\, \widehat{\mathcal{F}}_t). \tag{27}$$

For the reader's convenience, let us state the following immediate consequence of Lemma 1.1.

Lemma 2.1 *Assume that Y is some $\mathbb{G}$-adapted stochastic process. Then there exists a unique $\widehat{\mathbb{F}}$-adapted process $\widehat{Y}$ such that, for every $t \in [0,T]$,*

$$Y_t = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{Y}_t. \tag{28}$$

The process $\widehat{Y}$ is termed the pre-collapse value *of the process Y.*

2.2 Forward-Start Credit Default Index Swap

We write $T_0 = T < T_1 < \cdots < T_J$ to denote the *tenor structure* of the forward-start CDIS, where:

- $T_0 = T$ is the inception date;
- T_J is the maturity date;
- T_j is the jth fee payment date for $j = 1, 2, \ldots, J$.

Let $\alpha_j = T_j - T_{j-1}$ for every $j = 1, 2, \ldots, J$. As before, B is an $\mathbb{F}$-adapted (or, at least, $\widehat{\mathbb{F}}$-adapted) and strictly positive process representing the price of the savings account (or any other strictly positive numeraire) and the underlying probability measure $\mathbb{Q}$ is interpreted as a martingale measure associated with the choice of B as the numeraire asset.

Definition 2.1 The discounted cash flows for the seller (that is, for the protection buyer) of the *forward CDIS* issued at time $s \in [0,T]$ with an $\mathcal{F}_s$-measurable spread κ are, for every $t \in [s,T]$,

$$D^n_t = P^n_t - \kappa A^n_t, \tag{29}$$

where

$$P^n_t = (1-\delta) B_t \sum_{i=1}^{n} B^{-1}_{\tau_i} \mathbb{1}_{\{T<\tau_i \leq T_J\}} \tag{30}$$

and

$$A^n_t = B_t \sum_{j=1}^{J} \alpha_j B^{-1}_{T_j} \sum_{i=1}^{n} \big(1 - \mathbb{1}_{\{T_j \geq \tau_i\}}\big) \tag{31}$$

are discounted payoffs of the protection leg and the fee leg per one basis point, respectively. The *fair price* at time $t \in [s,T]$ of a forward credit default index swap for the protection buyer equals

$$S^n_t(\kappa) = \mathbb{E}_{\mathbb{Q}}(D^n_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(P^n_t \,|\, \mathcal{G}_t) - \kappa \mathbb{E}_{\mathbb{Q}}(A^n_t \,|\, \mathcal{G}_t).$$

Of course, for $s = T$, i.e. when the issuance date coincides with the inception date, the forward CDIS becomes the (spot) CDIS. From (30) and (31), we obtain the following simple – but crucial – properties of P_t^n and A_t^n:

$$P_t^n = \mathbb{1}_{\{T<\widehat{\tau}\}} P_t^n, \quad A_t^n = \mathbb{1}_{\{T<\widehat{\tau}\}} A_t^n. \tag{32}$$

Let us note that the quantities P_t^n and A_t^n are well defined for any $t \in [0, T]$ and they do not depend on the issuance date s of the forward CDIS under consideration.

Remarks 2.1 Let us make few comments on the scope of the method presented in what follows. As in the single name case, only the equalities (32) are essential for our method to work, as opposed to the exact specification of the payoffs P_t^n and A_t^n, which is of a minor importance. Therefore, the same approach can be applied to other conventions regarding multi-name credit default swaps and not only to the standard forward CDIS, as specified in Definition 2.1.

The cash flows of the protection leg of the forward credit default index swap are somewhat similar to summing the cash flows of n individual single-name forward CDS cash flows. Indeed, the right-hand side in (30) is simply the discounted sum of constant protection payouts $(1-\delta)$ for all the reference entities which have defaulted during the lifetime of the forward CDIS, that is, between the inception date T and the maturity date T_J.

The fee leg is somewhat different, however. The premium payment of the forward CDIS decreases following every default. This is because the constant premium is only paid on the nominal value of the remaining entities for which the investor is being provided protection. As the nominal value reduces after every default, so does the investor's premium payment which decreases in proportion to the change in nominal value. In the case of a portfolio of single-name CDSs, the total nominal also decreases after each default, but the total premium paid after defaults depends also on the identities of defaulted names, since spreads of individual CDSs are typically different.

For brevity, we will write J_t to denote the *reduced nominal* at time $t \in [s, T]$, as given by the formula

$$J_t = \sum_{i=1}^{n} \big(1 - \mathbb{1}_{\{t \geq \tau_i\}}\big). \tag{33}$$

In what follows, we only require that the inequality $\widehat{G}_t > 0$ holds for every $t \in [s, T_1]$, so that, in particular, $\widehat{G}_{T_1} = \mathbb{Q}(\widehat{\tau} > T_1 \,|\, \widehat{\mathcal{F}}_{T_1}) > 0$.

The proof of the following pricing result is exactly the same as the proof of Lemma 1.2. It is based on formulae (27) and (32).

Lemma 2.2 *The price at time $t \in [s, T]$ of the forward CDIS satisfies*

$$S_t^n(\kappa) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(D_t^n \,|\, \widehat{\mathcal{F}}_t) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{S}_t^n(\kappa), \tag{34}$$

where the pre-collapse price *of the forward CDIS satisfies $\widehat{S}_t^n(\kappa) = \widehat{P}_t^n - \kappa \widehat{A}_t^n$, where in turn*

$$\widehat{P}_t^n = \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(P_t^n \,|\, \widehat{\mathcal{F}}_t), \quad \widehat{A}_t^n = \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(A_t^n \,|\, \widehat{\mathcal{F}}_t) \tag{35}$$

or, more explicitly,

$$\widehat{P}_t^n = (1-\delta) \widehat{G}_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^{n} B_{\tau_i}^{-1} \mathbb{1}_{\{T<\tau_i \leq T_J\}} \Big| \, \widehat{\mathcal{F}}_t \Big) \tag{36}$$

and

$$\widehat{A}_t^n = \widehat{G}_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} \alpha_j B_{T_j}^{-1} J_{T_j} \, \Big| \, \widehat{\mathcal{F}}_t \Big). \tag{37}$$

The process $\widehat{A}_t^n$ may be thought of as the pre-collapse present value of receiving risky one basis point on the forward CDIS payment dates T_j on the residual nominal value J_{T_j}. Similarly, the process $\widehat{P}_t^n$ represents the pre-collapse present value of the protection leg of the contract.

2.3 Pre-Collapse Fair CDIS Spread

Since the forward CDIS is terminated at the moment of the nth default with no further payments, it makes sense to define the forward CDS spread only prior to $\widehat{\tau}$. It is thus natural to introduce the concept of the *pre-collapse* fair forward CDIS spread, rather than the fair forward CDS spread.

Definition 2.2 The *pre-collapse fair forward CDIS spread* at time $t \in [0,T]$ is the $\widehat{\mathcal{F}}_t$-measurable random variable κ_t^n such that $\widehat{S}_t^n(\kappa_t^n) = 0$.

The following result, which is a counterpart of Lemma 1.3, is a straightforward consequence of Lemma 2.2. It is worth noting that the quantity κ_t^n is well defined for every $t \in [0,T]$ and, manifestly, it does not depend on the issuance date s.

Lemma 2.3 *Assume that* $\widehat{G}_{T_1} = \mathbb{Q}(\widehat{\tau} > T_1 \,|\, \widehat{\mathcal{F}}_{T_1}) > 0$. *Then the pre-collapse fair forward CDIS spread satisfies, for every* $t \in [0,T]$,

$$\kappa_t^n = \frac{\widehat{P}_t^n}{\widehat{A}_t^n} = \frac{(1-\delta)\,\mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^n B_{\tau_i}^{-1} \mathbb{1}_{\{T<\tau_i \leq T_J\}} \,\Big|\, \widehat{\mathcal{F}}_t \Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^J \alpha_j B_{T_j}^{-1} J_{T_j} \,\Big|\, \widehat{\mathcal{F}}_t \Big)}. \tag{38}$$

The price of the forward CDIS admits the following representation, for every $t \in [0,T]$,

$$S_t^n(\kappa) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{A}_t^n (\kappa_t^n - \kappa). \tag{39}$$

Proof. The proof is essentially the same as the proof of Lemma 1.3. We first note that, since κ_t^n is $\widehat{\mathcal{F}}_t$-measurable, we have that $\widehat{S}_t^n(\kappa) = \widehat{P}_t^n - \kappa_t^n \widehat{A}_t^n$. Moreover, under the standing assumption that $\widehat{G}_{T_1} > 0$, it can be deduced easily from (37) that $\widehat{A}_t > 0$ for every $t \in [0,T]$. Indeed, (37) yields, for every $t \in [0,T]$,

$$\begin{aligned} \widehat{A}_t^n &\geq \widehat{G}_t^{-1} B_t \,\mathbb{E}_{\mathbb{Q}}\big(\alpha_1 B_{T_1}^{-1} J_{T_1} \,\big|\, \widehat{\mathcal{F}}_t\big) \geq \widehat{G}_t^{-1} B_t \,\mathbb{E}_{\mathbb{Q}}\big(\alpha_1 B_{T_1}^{-1} \mathbb{1}_{\{T_1<\widehat{\tau}\}} \,\big|\, \widehat{\mathcal{F}}_t\big) \\ &= \widehat{G}_t^{-1} B_t \,\mathbb{E}_{\mathbb{Q}}\big(\alpha_1 B_{T_1}^{-1} \mathbb{Q}(\widehat{\tau} > T_1 \,|\, \widehat{\mathcal{F}}_{T_1}) \,\big|\, \widehat{\mathcal{F}}_t\big) = \widehat{G}_t^{-1} B_t \,\mathbb{E}_{\mathbb{Q}}\big(\alpha_1 B_{T_1}^{-1} \widehat{G}_{T_1} \,\big|\, \widehat{\mathcal{F}}_t\big) > 0. \end{aligned}$$

It is thus clear that the first equality in (38) is valid. For the second equality, we make use of (36) and (37). To derive (39), it is enough to observe that

$$\widehat{S}_t^n(\kappa) = \widehat{S}_t^n(\kappa) - \widehat{S}_t^n(\kappa_t) = \widehat{P}_t^n - \kappa \widehat{A}_t^n - (\widehat{P}_t^n - \kappa_t^n \widehat{A}_t^n) = \widehat{A}_t^n (\kappa_t^n - \kappa),$$

where we have used the equality $\widehat{S}_t^n(\kappa_t^n) = 0$ (cf. Definition 1.2). ☐

2.4 Market Convention for Valuing a CDIS

Unfortunately, a market quote for the quantity $\widehat{A}_t^n$, which is essential in marking-to-market of a CDIS, is not directly available. The market convention for approximation of the value of A_t^n hinges on the following bold postulates:

- all firms are identical from time t onwards (homogeneous portfolio); therefore, we just deal with a single-name case, so that either all firms default or none;
- the implied risk-neutral default probabilities are computed using a flat single-name CDS curve with a constant spread equal to κ_t^n.

Under this set of conventional postulates, the right-hand side in (37) is approximated using the market convention that

$$\widehat{A}_t^n \approx J_t PV_t(\kappa_t^n),$$

where $PV_t(\kappa_t)$ is the risky present value of receiving one basis point at all CDIS payment dates calibrated to a flat CDS curve with spread equal to κ_t^n, where κ_t^n is the quoted CDIS spread at time t. Consequently, the conventional market formula for the value of the CDIS with fixed spread κ reads, on the pre-collapse event $\{t < \widehat{\tau}\}$,

$$\widehat{S}_t(\kappa) = J_t PV_t(\kappa_t^n)(\kappa_t^n - \kappa). \tag{40}$$

In particular, if the credit default index swap was issued at time 0 with the spread κ_0^n then its marked-to-market value at time t equals

$$\widehat{S}_t(\kappa_0^n) = \mathbb{1}_{\{t<\widehat{\tau}\}} PV_t\big(\kappa_t^n\big) J_t(\kappa_t^n - \kappa_0^n). \tag{41}$$

Let us stress that the quantity $PV_t(\kappa_t^n)$ is computed as if it was a single-name case, not a multiname. For this very reason, we underline the importance of this step in the market convention for the CDIS value as well as for the credit default index swaption examined in the foregoing section. As we shall see in that follows, from the theoretical viewpoint, it is much easier to work with formula (39), rather than with the conventional expression (40).

2.5 Market Payoff of a Credit Default Index Swaption

In Section 1.4, we have examined the valuation of options on a single-name credit default swap. This is now extended to options on a credit default index swap, referred to as *credit default index swaptions*. Credit default index swaptions are European options and thus can only be exercised at expiry at the preset exercise spread κ. Standard contracts have maturities of either three or six-months. For example, in a standard CDX swaption contract the specifics would be: the underlying CDX, the expiry date, the strike level κ and the type (payer or receiver).

Let us first describe the market convention regarding the payoff of the payer credit default index swaption. We refer to Pedersen [19] for more details and comments. It is assumed here that the credit default index swap was issued at time 0, with the constant spread κ_0^n and κ_U^n represents the corresponding market quote at time U.

Definition 2.3 The conventional market formula for the payoff at maturity $U \leq T$ of the *payer credit default index swaption* with strike level κ reads

$$C_U = \Big(\mathbb{1}_{\{U<\widehat{\tau}\}} PV_U\big(\kappa_U^n\big) J_U(\kappa_U^n - \kappa_0^n) - \mathbb{1}_{\{U<\widehat{\tau}\}} PV_U(\kappa) n(\kappa - \kappa_0^n) + L_U\Big)^+, \tag{42}$$

where the reduced nominal J is given by (33) and L stands for the loss process for our portfolio so that, for every $t \in \mathbb{R}_+$,

$$L_t = (1-\delta)\sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \leq t\}}.$$

Let us make some comments regarding the swaption's payoff. Note first that in Definition 2.3 we set our underlying forward CDIS to have inception date $T_0 = T$ and maturity T_J. However, the losses from the portfolio are computed from time 0 onwards. Hence the holder of the swaption has the right to enter the underlying forward CDIS at time U and, if this option is exercised, he also gains protection against losses from the portfolio between time 0 and time U. The key difference between this cash flow and the credit default swaption cash flow is that here we no longer deal with the knock out feature, even after the nth default. This lack of knock out proves to be difficult in the valuation and hedging of index swaptions. No longer may the standard Black formula be used to price the credit default index swaptions as this formula only works for options that do knock out at default (such as single-name credit default swaptions).

The market convention (42) is due to the fact that the swaption has physical settlement and the CDIS with spread κ is not traded. If the swaption is exercised, its holder takes a long position in the on-the-run index and is compensated for the difference between the value of the on-the-run index and the value of the (non-traded) index with spread κ, as well as for defaults that occurred in the interval $[0, U]$.

Recall that $PV_U(\kappa)$ is the risky present value at time U of receiving one basis point at all CDIS payment dates calibrated to a flat single-name CDS curve with spread equal to κ. It is worth observing that $PV_U(\kappa)$ is random only in the interest rates (typically, forward LIBORs), whereas $PV_U(\kappa^n_U)$ is random in both interest rates and the index spread κ^n_U. In order to make $PV_U(\kappa)$ completely deterministic, one may use a common assumption that the interest rate for some future date lies on the current forward curve for that same date. Under this additional assumption, the quantity $PV_U(\kappa^n_U)$ will only be random via its dependence on the index spread.

2.6 Put-Call Parity for Credit Default Index Swaptions

For the sake of brevity, let us denote, for any fixed $\kappa > 0$,

$$f(\kappa, L_U) = L_U - \mathbb{1}_{\{U<\widehat{\tau}\}} PV_U(\kappa) n(\kappa - \kappa^n_0).$$

Then the payoff of the payer credit default index swaption entered at time 0 and maturing at U equals

$$C_U = \Big(\mathbb{1}_{\{U<\widehat{\tau}\}} PV_U\big(\kappa^n_U\big) J_U(\kappa^n_U - \kappa^n_0) + f(\kappa, L_U)\Big)^+,$$

whereas the payoff of the corresponding *receiver credit default index swaption* satisfies

$$P_U = \Big(\mathbb{1}_{\{U<\widehat{\tau}\}} PV_U\big(\kappa^n_U\big) J_U(\kappa^n_0 - \kappa^n_U) - f(\kappa, L_U)\Big)^+.$$

This leads to the following equality, which holds at maturity date U

$$C_U - P_U = \mathbb{1}_{\{U<\widehat{\tau}\}} PV_U\big(\kappa^n_U\big) J_U(\kappa^n_U - \kappa^n_0) + f(\kappa, L_U).$$

2.7 Model Payoff of a Credit Default Index Swaption

The actual payoff (42) of a credit default index swaption is rather difficult to handle analytically, in general. The advantage of this formula is that it is based on market data and it is easy to implement. The major drawback is that it is internally inconsistent since the quantities $PV_U\big(\kappa^n_U\big)$ and $PV_U(\kappa)$ are computed on the basis of a single-name case and thus are not consistent with any model for default times $\tau_1, \dots, \tau_n$. For this reason, we will consider in what follows the simplified version of the swaption's payoff.

Definition 2.4 The *model payoff* of the payer credit default index swaption entered at time 0 with maturity date U and strike level κ equals

$$C_U = (S^n_U(\kappa) + L_U)^+ \tag{43}$$

or, more explicitly (cf. (39))

$$C_U = \Big(\mathbb{1}_{\{U<\widehat{\tau}\}} \widehat{A}^n_U(\kappa_U - \kappa) + L_U\Big)^+. \tag{44}$$

To formally derive (44) from (42), it suffices to postulate that

$$PV_U(\kappa)n \approx PV_U\big(\kappa_U\big) J_U \approx \widehat{A}^n_U.$$

We will first use representation (43) to establish, in Section 2.10, the pricing formula for a credit default index swaption, which was proposed recently by Morini and Brigo [18]. Subsequently, in Section 2.12, we will use (44) to justify the market pricing formula for a credit default index swaption. The crucial difference between the two approaches is that the market pricing formula (58) refers to the fair CDIS spread κ^n, whereas the model formula (55) of Morini and Brigo [18], as well as the related Pedersen's [19] formula, are based on the loss-adjusted fair CDIS spread κ^a, which will be introduced in Section 2.9 below.

2.8 Loss-Adjusted CDIS

Since $L_U \geq 0$ and, obviously, $L_U = \mathbb{1}_{\{U<\widehat{\tau}\}} L_U + \mathbb{1}_{\{U\geq\widehat{\tau}\}} L_U$, the payoff (44) can also be represented as follows

$$C_U = (S_U^n(\kappa) + \mathbb{1}_{\{U<\widehat{\tau}\}} L_U)^+ + \mathbb{1}_{\{U\geq\widehat{\tau}\}} L_U = (S_U^a(\kappa))^+ + C_U^L, \tag{45}$$

where we denote

$$S_U^a(\kappa) = S_U^n(\kappa) + \mathbb{1}_{\{U<\widehat{\tau}\}} L_U, \quad C_U^L = \mathbb{1}_{\{U\geq\widehat{\tau}\}} L_U.$$

The quantity $S_U^a(\kappa)$ represents the payoff at time U of the loss-adjusted forward CDIS, which is formally defined as follows.

Definition 2.5 The discounted cash flows for the seller of the *loss-adjusted forward CDIS* (that is, for the buyer of the protection) are, for every $t \in [0, U]$,

$$D_t^a = P_t^a - \kappa A_t^n,$$

where

$$P_t^a = P_t^n + B_t B_U^{-1} \mathbb{1}_{\{U<\widehat{\tau}\}} L_U.$$

It is essential to observe that the payoff D_U^a is the U-survival claim, in the sense that

$$D_U^a = \mathbb{1}_{\{U<\widehat{\tau}\}} D_U^a.$$

Let us note that any other adjustments to the payoff P_t^n (or A_t^n) of the CDIS is also admissible, provided that the property $P_U^a = \mathbb{1}_{\{U<\widehat{\tau}\}} P_U^a$ (or $A_U^a = \mathbb{1}_{\{U<\widehat{\tau}\}} A_U^a$) holds. Therefore, if we wish to define a particular adjustment of the fair CDIS spread for any date $t \in [0, U]$, we only need to ensure that the modified protection and fee payoffs are U-survival claims,

Lemma 2.4 *The price of the loss-adjusted forward CDIS equals, for every $t \in [0, U]$,*

$$S_t^a(\kappa) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{G}_t^{-1}\, \mathbb{E}_{\mathbb{Q}}(D_t^a \,|\, \widehat{\mathcal{F}}_t) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{S}_t^a(\kappa),$$

where the pre-collapse price satisfies $\widehat{S}_t^a(\kappa) = \widehat{P}_t^a - \kappa \widehat{A}_t^n$, where in turn

$$\widehat{P}_t^a = \widehat{G}_t^{-1}\, \mathbb{E}_{\mathbb{Q}}(P_t^a \,|\, \widehat{\mathcal{F}}_t), \quad \widehat{A}_t^n = \widehat{G}_t^{-1}\, \mathbb{E}_{\mathbb{Q}}(A_t^n \,|\, \widehat{\mathcal{F}}_t)$$

or, more explicitly,

$$\widehat{P}_t^a = \widehat{G}_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}}\Big((1-\delta) \sum_{i=1}^n B_{\tau_i}^{-1} \mathbb{1}_{\{T<\tau_i\leq T_J\}} + \mathbb{1}_{\{U<\widehat{\tau}\}} B_U^{-1} L_U \Big| \widehat{\mathcal{F}}_t\Big)$$

and

$$\widehat{A}_t^n = \widehat{G}_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^J \alpha_j B_{T_j}^{-1} J_{T_j} \,\Big|\, \widehat{\mathcal{F}}_t\Big).$$

2.9 Pre-Collapse Loss-Adjusted Fair CDIS Spread

We are in a position to define the fair loss-adjusted forward CDIS spread.

Definition 2.6 The *pre-collapse loss-adjusted fair forward CDIS spread* at time $t \in [0, U]$ is the $\widehat{\mathcal{F}}_t$-measurable random variable κ_t^a such that $\widehat{S}_t^a(\kappa_t^a) = 0$.

Then we have the following result, which corresponds to Lemma 2.3.

Lemma 2.5 *Assume that* $\widehat{G}_{T_1} = \mathbb{Q}(\widehat{\tau} > T_1 \,|\, \widehat{\mathcal{F}}_{T_1}) > 0$. *Then the pre-collapse loss-adjusted fair forward CDIS spread satisfies, for every* $t \in [0, U]$,

$$\kappa^a_t = \frac{\widehat{P}^a_t}{\widehat{A}^n_t} = \frac{\mathbb{E}_{\mathbb{Q}}\Big((1-\delta)\sum_{i=1}^n B_{\tau_i}^{-1}\mathbb{1}_{\{T<\tau_i\le T_J\}} + \mathbb{1}_{\{U<\widehat{\tau}\}}B_U^{-1}L_U \,\Big|\, \widehat{\mathcal{F}}_t\Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^J \alpha_j B_{T_j}^{-1} J_{T_j} \,\Big|\, \widehat{\mathcal{F}}_t\Big)}. \tag{46}$$

The price of the forward CDIS admits the following representation, for every $t \in [0, T]$,

$$S^a_t(\kappa) = \mathbb{1}_{\{t<\widehat{\tau}\}}\widehat{A}^n_t(\kappa^a_t - \kappa). \tag{47}$$

Proof. The proof of this result is essentially the same as the proof of Lemma 2.3 and thus it is omitted. □

2.10 Model Pricing Formula for Credit Default Index Swaptions

It is easy to check that the model payoff (44) of the credit default index swaption can be represented as follows

$$C_U = \mathbb{1}_{\{U<\widehat{\tau}\}}\widehat{A}^n_U(\kappa^a_U - \kappa)^+ + \mathbb{1}_{\{U\ge\widehat{\tau}\}}L_U. \tag{48}$$

This equality should be seen as the loss-adjusted simplification of formula (45). The price at time $t \in [0, U]$ of the claim C_U is thus given by the risk-neutral valuation formula

$$C_t = B_t\,\mathbb{E}_{\mathbb{Q}}\big(B_U^{-1}C_U \,|\, \mathcal{G}_t\big) = B_t\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{U<\widehat{\tau}\}}B_U^{-1}\widehat{A}^n_U(\kappa^a_U-\kappa)^+ \,|\, \mathcal{G}_t\big) + B_t\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{U\ge\widehat{\tau}\}}B_U^{-1}L_U \,|\, \mathcal{G}_t\big).$$

Using the filtration $\widehat{\mathbb{F}}$, we can obtain a more explicit representation for the first term in the formula above, as the following result shows.

Lemma 2.6 *The price at time* $t \in [0, U]$ *of the payer credit default index swaption equals*

$$C_t = \mathbb{1}_{\{t<\widehat{\tau}\}}B_t\widehat{G}_t^{-1}\,\mathbb{E}_{\mathbb{Q}}\left(\widehat{G}_U B_U^{-1}\widehat{A}^n_U(\kappa^a_U-\kappa)^+ \,\Big|\, \widehat{\mathcal{F}}_t\right) + B_t\,\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{U\ge\widehat{\tau}\}}B_U^{-1}L_U \,\Big|\, \mathcal{G}_t\right). \tag{49}$$

Proof. The random variable $Y = B_U^{-1}\widehat{A}^n_U(\kappa^a_U-\kappa)^+$ is manifestly $\widehat{\mathcal{F}}_U$-measurable and $Y = \mathbb{1}_{\{U<\widehat{\tau}\}}Y$. Hence the equality is an immediate consequence of formula (27). □

Let us first note that, on the collapse event $\{t \ge \widehat{\tau}\}$ we have that $\mathbb{1}_{\{U\ge\widehat{\tau}\}}B_U^{-1}L_U = B_U^{-1}n(1-\delta)$ and thus the pricing formula (49) reduces to

$$C_t = B_t\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{U\ge\widehat{\tau}\}}B_U^{-1}L_U \,|\, \mathcal{G}_t\big) = n(1-\delta)\mathbb{E}_{\mathbb{Q}}\left(B_U^{-1} \,|\, \mathcal{G}_t\right) = n(1-\delta)B(t,T), \tag{50}$$

where $B(t,T)$ is the price a t of U-maturity risk-free zero-coupon bond. This case is thus easy to handle, so that it will not be considered in what follows.

Let us thus concentrate on the pre-collapse event $\{t < \widehat{\tau}\}$. We now have $C_t = C^a_t + C^L_t$, where

$$C^a_t = B_t\widehat{G}_t^{-1}\,\mathbb{E}_{\mathbb{Q}}\left(\widehat{G}_U B_U^{-1}\widehat{A}^n_U(\kappa^a_U-\kappa)^+ \,\Big|\, \widehat{\mathcal{F}}_t\right) \tag{51}$$

and

$$C^L_t = B_t\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{U\ge\widehat{\tau}>t\}}B_U^{-1}L_U \,|\, \widehat{\mathcal{F}}_t\big),$$

where the last equality follows from the well known fact that on $\{t < \widehat{\tau}\}$ any $\mathcal{G}_t$-measurable event can be represented by an $\widehat{\mathcal{F}}_t$-measurable event, in the sense that for any event $A \in \mathcal{G}_t$ there exists an event $\widehat{A} \in \widehat{\mathcal{F}}_t$ such that $\mathbb{1}_{\{t<\widehat{\tau}\}}A = \mathbb{1}_{\{t<\widehat{\tau}\}}\widehat{A}$ (cf. Lemma 2.1).

The computation of C^L_t relies only on the knowledge of the risk-neutral conditional distribution of $\widehat{\tau}$ given $\widehat{\mathcal{F}}_t$ and the term structure of interest rates, since on the event $\{U \ge \widehat{\tau} > t\}$ we have that $B_U^{-1}L_U = B_U^{-1}n(1-\delta)$.

By contrast, the computation of C^a_t hinges on exactly the same arguments as in the single-name case. First, we define an equivalent probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \widehat{\mathcal{F}}_U)$ by postulating that the Radon-Nikodým density of $\widehat{\mathbb{Q}}$ with respect to $\mathbb{Q}$ equals

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = c\widehat{G}_U B_U^{-1} \widehat{A}^n_U, \quad \mathbb{Q}\text{-a.s.} \tag{52}$$

Let us note that the process $\widehat{\eta}_t = c\widehat{G}_t B_t^{-1} \widehat{A}^n_t$, $t \in [0, U]$, is a strictly positive $\widehat{\mathbb{F}}$-martingale under $\mathbb{Q}$, since

$$\widehat{\eta}_t = c\widehat{G}_t B_t^{-1} \widehat{A}^n_t = c\, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} \alpha_j B_{T_j}^{-1} J_{T_j} \,\Big|\, \widehat{\mathcal{F}}_t \Big)$$

and we have that $\mathbb{Q}(\tau > T_j \,|\, \widehat{\mathcal{F}}_{T_j}) = \widehat{G}_{T_j} > 0$ for every j. Therefore, for every $t \in [0, U]$,

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} \,|\, \widehat{\mathcal{F}}_t = \mathbb{E}_{\mathbb{Q}}(\widehat{\eta}_U \,|\, \widehat{\mathcal{F}}_t) = \widehat{\eta}_t, \quad \mathbb{Q}\text{-a.s.}$$

The quantity $c = \big(\mathbb{E}_{\mathbb{Q}}(\widehat{G}_U B_U^{-1} \widehat{A}^n_U)\big)^{-1}$ is the normalizing constant, which ensures that $\widehat{\mathbb{Q}}$ given by (52) is indeed a probability measure on $(\Omega, \widehat{\mathcal{F}}_U)$.

Lemma 2.7 *The price at time $t \in [0, U]$ of the payer credit default index swaption on the collapse event $\{t \geq \widehat{\tau}\}$ is given by (50). On the pre-collapse event $\{t < \widehat{\tau}\}$ it equals*

$$C_t = \widehat{A}^n_t \, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\kappa^a_U - \kappa)^+ \,\big|\, \widehat{\mathcal{F}}_t \big) + B_t \, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{U \geq \widehat{\tau} > t\}} B_U^{-1} L_U \,\big|\, \widehat{\mathcal{F}}_t \big). \tag{53}$$

Proof. It suffices to examine C^a_t. Using (51) and (52), we obtain

$$\begin{aligned} C^a_t &= B_t G_t^{-1} \, \mathbb{E}_{\mathbb{Q}} \Big(\widehat{G}_U B_U^{-1} \widehat{A}^n_U (\kappa^a_U - \kappa)^+ \,\Big|\, \widehat{\mathcal{F}}_t \Big) \\ &= \widehat{A}^n_t \widehat{\eta}_t^{-1} \, \mathbb{E}_{\mathbb{Q}} \Big(\widehat{\eta}_U (\kappa^a_U - \kappa)^+ \,\Big|\, \widehat{\mathcal{F}}_t \Big) \\ &= \widehat{A}^n_t \, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\kappa^a_U - \kappa)^+ \,\big|\, \widehat{\mathcal{F}}_t \big), \end{aligned}$$

where the last equality follows from the abstract Bayes formula. ☐

The next lemma establishes the martingale property of the process κ^a under $\widehat{\mathbb{Q}}$.

Lemma 2.8 *The pre-collapse loss-adjusted fair forward CDIS spread κ^a_t, $t \in [0, U]$, is a strictly positive $\widehat{\mathbb{F}}$-martingale under $\widehat{\mathbb{Q}}$.*

Proof. Similarly as in the proof of Lemma 1.6, it suffices to observe that the product $\kappa^a \widehat{\eta}$ satisfies

$$\kappa^a_t \widehat{\eta}_t = c\kappa^a_t \widehat{G}_t B_t^{-1} \widehat{A}^n_t = c\widehat{G}_t B_t^{-1} \widehat{P}^a_t = c\, \mathbb{E}_{\mathbb{Q}}\Big((1 - \delta) \sum_{i=1}^{n} B_{\tau_i}^{-1} \mathbb{1}_{\{T < \tau_i \leq T_J\}} + \mathbb{1}_{\{U < \widehat{\tau}\}} B_U^{-1} L_U \,\Big|\, \widehat{\mathcal{F}}_t \Big)$$

so that $\kappa^a \widehat{\eta}$ is an $\widehat{\mathbb{F}}$-martingale under $\mathbb{Q}$. By the well known argument, we conclude that κ^a is an $\widehat{\mathbb{F}}$-martingale under $\widehat{\mathbb{Q}}$. ☐

2.11 Black Formula for Credit Default Index Swaptions

Our next goal is to establish a suitable version of the Black formula for the credit default index swaption. To this end, we postulate that the pre-collapse loss-adjusted fair forward CDIS spread satisfies

$$\kappa^a_t = \kappa^a_0 + \int_0^t \sigma_u \kappa^a_u \, d\widehat{W}_u, \quad \forall\, t \in [0, U], \tag{54}$$

where $\widehat{W}$ is the one-dimensional standard Brownian motion under $\widehat{\mathbb{Q}}$ with respect to $\widehat{\mathbb{F}}$ and σ is an $\widehat{\mathbb{F}}$-predictable process. Let us emphasize that the assumption that the filtration $\widehat{\mathbb{F}}$ is the Brownian filtration (cf. Section 1.5) would be too restrictive, since $\widehat{\mathbb{F}} = \mathbb{F} \vee \mathbb{H}^{(1)} \vee \ldots \vee \mathbb{H}^{(n-1)}$ and thus $\widehat{\mathbb{F}}$ will typically need to support also discontinuous martingales.

Proposition 2.1 *Assume that the volatility σ of the pre-collapse loss-adjusted fair forward CDIS spread is a positive function. Then the pre-default price of the payer credit default index swaption equals, for every $t \in [0, U]$ on the pre-collapse event $\{t < \widehat{\tau}\}$,*

$$C_t = \widehat{A}^n_t \Big(\kappa^a_t N\big(d_+(\kappa^a_t, t, U)\big) - \kappa N\big(d_-(\kappa^a_t, t, U)\big) \Big) + C^L_t \tag{55}$$

or, equivalently,

$$C_t = \widehat{P}^a_t \, N\big(d_+(\kappa^a_t, t, U)\big) - \kappa \widehat{A}^n_t \, N\big(d_-(\kappa^a_t, t, U)\big) + C^L_t, \tag{56}$$

where

$$d_\pm(\kappa^a_t, t, U) = \frac{\ln(\kappa^a_t/\kappa) \pm \frac{1}{2} \int_t^U \sigma^2(u)\, du}{\big(\int_t^U \sigma^2(u)\, du \big)^{1/2}} \, .$$

Proof. It suffices to focus on C^a_t. In view of (53), we obtain

$$C^a_t = \widehat{A}^n_t \, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\kappa^a_U - \kappa)^+ \,\big|\, \widehat{\mathcal{F}}_t \big) = \widehat{A}^n_t \, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\kappa^a_U - \kappa)^+ \,\big|\, \kappa^a_t \big) = \widehat{A}^n_t \Big(\kappa^a_t N\big(d_+(\kappa^a_t, t, U)\big) - \kappa N\big(d_-(\kappa^a_t, t, U)\big) \Big)$$

where the last equality follows by standard computations. ☐

A slightly different approach to valuation of a credit default index swaption was proposed by Pedersen [19]. The derivation of Pedersen's formula hinges on the simplification of the actual payoff of the swaption combined with the assumption that not every reference names will default prior to the swaption's maturity date $U \leq T$. Formally, it is enough to postulate that

$$L_U = \mathbb{1}_{\{U < \widehat{\tau}\}} L_U \tag{57}$$

so that $C^L_U = 0$. Under this assumption, the second term in pricing formulae (49), (53) and (55) will vanish and thus, for instance, expression (55) will reduce to the standard Black swaptions formula.

Under usual circumstances, the probability of all defaults occurring prior to U is expected to be very low, and thus assumption (57) seems to be reasonable. Morini and Brigo [18] argue, however, that this assumption is not always justified, in particular, it is not suitable for periods when the market conditions deteriorate. It is also worth mentioning that since we deal here with the risk-neutral probability measure, the probabilities of default events are known to drastically exceed statistically observed default probabilities, that is, probabilities of default events under the physical probability measure. Under assumption (57), the second term in (48), and thus also C^L_t, vanish and thus the pricing formula (55) reduces to a single term.

Let us finally mention that Jackson [14] proposed an alternative approach to valuation of a credit default index swaption by conditioning on the number of defaults prior to the swaption's maturity. He obtains, under rather stringent model assumptions, the pricing formula in the form of a weighted average of suitable Black's formulae. We do not discuss his method here; for a detailed analysis of assumptions underpinning Jackson's approach, which is based on conditioning on the number of defaults that occur prior to the swaption's expiry, the interested reader is referred to Section 3.5 in [1] or Section 4.4 in [2].

2.12 Market Pricing Formula for Credit Default Index Swaptions

Before concluding this paper, let us briefly examine one of the conventional market formulae for valuing credit default index swaption. Let us emphasize that pricing formula (58) refers to the quoted fair forward CDIS spread κ^n, rather than to its loss-adjusted (and thus not directly observed) version κ^a. To account for a potential loss prior to the swaption's maturity, a suitable (although somewhat ad hoc) adjustment to the strike level κ is introduced.

Proposition 2.2 *The price of a payer credit default index swaption can be approximated as follows*

$$C_U \approx \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{A}^n_t \Big(\kappa^n_t N\big(d_+(\kappa^n_t, t, U)\big) - (\kappa - \bar{L}_t) N\big(d_-(\kappa^n_t, t, U)\big)\Big), \tag{58}$$

where

$$d_\pm(\kappa^n_t, t, U) = \frac{\ln(\kappa^n_t/(\kappa - \bar{L}_t)) \pm \frac{1}{2}\int_t^U \sigma^2(u)\,du}{\big(\int_t^U \sigma^2(u)\,du\big)^{1/2}}$$

in which $\bar{L}_t = \mathbb{E}_{\widehat{\mathbb{Q}}}\big((A^n_U)^{-1} L_U \,|\, \widehat{\mathcal{F}}_t\big)$ *for* $t \in [0, U]$.

Proof. We merely sketch the proof. We start by approximating the model payoff, which is given by the expression (cf. (44))

$$C_U = \Big(\mathbb{1}_{\{U<\widehat{\tau}\}} \widehat{A}^n_U (\kappa^n_U - \kappa) + L_U \Big)^+,$$

in the following way

$$C_U \approx \mathbb{1}_{\{U<\widehat{\tau}\}} \widehat{A}^n_U \Big(\kappa^n_U - \kappa + \bar{L}_t \Big)^+,$$

where

$$\bar{L}_t = \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\widehat{A}^n_U)^{-1} L_U \,|\, \widehat{\mathcal{F}}_t\big).$$

This last equality may be written as

$$\widehat{C}_U \approx \mathbb{1}_{\{U<\widehat{\tau}\}} \widehat{A}^n_U \Big(\kappa^n_U - \big(\kappa - \widehat{L}_t\big) \Big)^+,$$

where the random variable $\widehat{L}_t$ is manifestly $\widehat{\mathcal{F}}_t$-measurable. By applying the risk-neutral valuation and proceeding as in the proof of Proposition 2.1, we obtain the stated formula. □

2.13 Concluding Remarks

As already observed in Remark 2.1, the approach developed in Section 2 can be applied to other variants of multi-name credit default swaps and not only to standard credit default index swaps and swaptions. In particular, the valuation of a single tranche of a synthetic collateralized debt obligation (CDO) and the related option can be done along the same lines. For a CDO tranche, the random time $\widehat{\tau} = \tau_{(n)}$ should be replaced by the random time $\tau_{(k)}$, where k stands for the minimal number of default events for which the percentage loss process, with constant jumps, crosses the detachment point of the tranche.

There is one important caveat here. The assumption that the tranche spread is a lognormally distributed martingale under the corresponding martingale measure, as defined by a suitable modification of formula (52), can only be justified for pricing of a tranche option on a stand alone basis. For the consistent simultaneous valuation of options on various CDO tranches, we would need to produce first a multidimensional arbitrage-free model for the tranche spreads with desired distributional properties. An important issue arising in this context is the specification of drift terms in the joint dynamics of tranche spreads under a common martingale probability, which would allow us to value also more complex credit correlation products.

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