ON THE EVALUATION OF CERTAIN IMPROPER INTEGRALS ARISING IN WAVEGUIDE THEORY

M. D Hirschhorn and W. D. McKee

School of Mathematics, University of New South Wales

1 Introduction

In discussing a problem arising from the presence of a thin barrier in a waveguide, John Miles [1] introduced the family of integrals

$$I(n,m) = I(m,n) = \frac{2b}{m\pi^2}\int_0^\pi \int_0^\pi \frac{\sin my}{\sin y}\,(1-\cos\theta)\,\frac{(1+\cos\phi)\cos n\eta}{\cos\phi - \cos\theta}\,d\theta\,d\phi\ ,$$

where n and m are non-negative integers (not both zero), $\cos y = b\cos\theta + 1 - b$, $\cos\eta = b\cos\phi + 1 - b$ and $0 < b < 1$. An equivalent form is

$$I(n,m) = I(m,n) = \frac{2b}{m\pi^2}\int_0^\pi \int_0^\pi U_{m-1}(y)\,(1-\cos\theta)\,\frac{(1+\cos\phi)T_n(\eta)}{\cos\phi - \cos\theta}\,d\theta\,d\phi\ , \quad (1)$$

where T_k and U_k are the Chebyshev polynomials of the first and second kind of order k respectively. The Cauchy Principal Value is taken in all cases and $I(m,n)$ turns out to be a polynomial in b of degree $n+m$. Miles gave the $I(m,n)$ for $0 \le n,m \le 3$. There appears to be no simple recurrence formula and evaluation of further members of the family by hand becomes more and more complicated as m and n increase. These later members of the family would be needed for accurate waveguide calculations; for example when the frequency is such that a large number of modes are propagating rather than evanescent. The computer algebra system Maple can be coerced into finding the $I(m,n)$ using the Chebyshev polynomial formulation but the computations consume more and more computer time and memory as n and m increase and soon become impractical. It is the purpose of this note to find an explicit form for $I(m,n)$. Without loss of generality we will assume that $m > 0$.

2 The Expansion

We start with the following fact. Let $u = 1 - \cos\theta$. Then $\cos y = 1 - bu$ and

$$\frac{\sin my}{\sin y} = \sum_{k=0}^{m-1}(-1)^k 2^k \binom{m+k}{2k+1} b^k u^k. \quad (2)$$

Both sides of (2) are readily shown to satisfy the recurrence

$$Q_m = 2\cos y\, Q_{m-1} - Q_{m-2}$$

with $Q_1 = 1$ and $Q_2 = 2\cos y$.

Let $v = 1 - \cos\phi$. Then $\cos\eta = 1 - bv$ and

$$\begin{aligned}
\cos n\eta &= \frac{\sin(n+1)\eta - \sin(n-1)\eta}{2\sin\eta} \\
&= \frac{1}{2}\left(\frac{\sin(n+1)\eta}{\sin\eta} - \frac{\sin(n-1)\eta}{\sin\eta}\right) \\
&= \frac{1}{2}\left(\sum_{l=0}^{n}(-1)^l 2^l \binom{n+1+l}{2l+1} b^l v^l - \sum_{l=0}^{n-2}(-1)^l 2^l \binom{n-1+l}{2l+1} b^l v^l\right) \\
&= \sum_{l=0}^{n}(-1)^l 2^{l-1}\left\{\binom{n+1+l}{2l+1} - \binom{n-1+l}{2l+1}\right\} b^l v^l.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\sin my}{\sin y}(1-\cos\theta)(1+\cos\phi)\cos n\eta \\
&= \sum_{k=0}^{m-1}(-1)^m 2^m \binom{m+k}{2k+1} b^k u^{k+1} \sum_{l=0}^{n}(-1)^l 2^{l-1}\left\{\binom{n+1+l}{2l+1} - \binom{n-1+l}{2l+1}\right\} b^l (2-v) v^l \\
&= \sum_{r=0}^{m+n-1}(-1)^r 2^{r-1} b^r \sum_{k+l=r}\binom{m+k}{2k+1}\left\{\binom{n+1+l}{2l+1} - \binom{n-1+l}{2l+1}\right\} u^{k+1}(2-v)v^l \\
&= \sum_{r=0}^{m+n-1}(-1)^r 2^{r-1} b^r \sum_{l=0}^{r}\binom{m+r-l}{2r-2l+1}\left\{\binom{n+1+l}{2l+1} - \binom{n-1+l}{2l+1}\right\} \\
&\qquad \times (1-\cos\theta)^{r-l+1}\left[2(1-\cos\phi)^l - (1-\cos\phi)^{l+1}\right] \\
&= \sum_{r=0}^{m+n-1}(-1)^r 2^{r-1} b^r \sum_{l=0}^{r}\binom{m+r-l}{2r-2l+1}\left\{\binom{n+1+l}{2l+1} - \binom{n-1+l}{2l+1}\right\} \\
&\qquad \times \sum_{s=0}^{r-l+1}(-1)^s\binom{r-l+1}{s}\cos^s\theta \times \sum_{t=0}^{l+1}(-1)^t\left\{2\binom{l}{t} - \binom{l+1}{t}\right\}\cos^t\phi.
\end{aligned}$$

It follows that

$$
\begin{aligned}
I(m,n) &= \sum_{r=0}^{m+n-1} (-1)^r 2^r b^{r+1} \sum_{l=0}^{r} \frac{1}{m} \binom{m+r-l}{2r-2l+1} \left\{ \binom{n+1+l}{2l+1} - \binom{n-1+l}{2l+1} \right\} \\
&\quad \times \sum_{s=0}^{r-l+1} \sum_{t=0}^{l+1} (-1)^{s+t} \binom{r-l+1}{s} \left\{ 2\binom{l}{t} - \binom{l+1}{t} \right\} \times K(s,t)/\pi^2 \qquad (3)
\end{aligned}
$$

where

$$
K(s,t) = \int_0^\pi \int_0^\pi \frac{\cos^s\theta \cos^t\phi}{\cos\phi - \cos\theta} d\theta d\phi.
$$

Now

$$
\begin{aligned}
K(s,t) &= \int_0^\pi \int_0^\phi \frac{\cos^s\theta \cos^t\phi}{\cos\phi - \cos\theta}\, d\theta d\phi + \int_0^\pi \int_0^\theta \frac{\cos^s\theta \cos^t\phi}{\cos\phi - \cos\theta}\, d\phi d\theta \\
&= \int_0^\pi \int_0^\phi \frac{\cos^s\theta \cos^t\phi}{\cos\phi - \cos\theta} d\theta d\phi + \int_0^\pi \int_0^\phi \frac{\cos^s\phi \cos^t\theta}{\cos\theta - \cos\phi} d\theta d\phi \\
&= \int_0^\pi \int_0^\phi \frac{cos^s\theta \cos^t\phi - \cos^t\theta \cos^s\phi}{\cos\phi - \cos\theta}\, d\theta d\phi \\
&= \frac{1}{2} \left\{ \int_0^\pi \int_0^\phi \frac{\cos^s\theta \cos^t\phi - \cos^t\theta \cos^s\phi}{\cos\phi - \cos\theta}\, d\theta d\phi \right. \\
&\qquad \left. + \int_0^\pi \int_0^\phi \frac{\cos^s\theta \cos^t\phi - \cos^t\theta \cos^s\phi}{\cos\phi - \cos\theta}\, d\theta d\phi \right\} \\
&= \frac{1}{2} \left\{ \int_0^\pi \int_0^\phi \frac{\cos^s\theta \cos^t\phi - \cos^t\theta \cos^s\phi}{\cos\phi - \cos\theta}\, d\theta d\phi \right. \\
&\qquad \left. + \int_0^\pi \int_0^\theta \frac{\cos^s\phi \cos^t\theta - \cos^t\phi \cos^s\theta}{\cos\theta - \cos\phi}\, d\phi d\theta \right\} \\
&= \frac{1}{2} \left\{ \int_0^\pi \int_0^\phi \frac{\cos^s\theta \cos^t\phi - \cos^t\theta \cos^s\phi}{\cos\phi - \cos\theta}\, d\theta d\phi \right. \\
&\qquad \left. + \int_0^\pi \int_0^\theta \frac{\cos^s\theta \cos^t\phi - \cos^t\theta \cos^s\phi}{\cos\phi - \cos\theta}\, d\phi d\theta \right\} \\
&= \frac{1}{2} \int_0^\pi \int_0^\pi \frac{\cos^s\theta \cos^t\phi - \cos^t\theta \cos^s\phi}{\cos\phi - \cos\theta}\, d\theta d\phi.
\end{aligned}
$$

Clearly, $K(t,s) = -K(s,t)$ and in particular $K(s,t) = 0$ if $s = t$. Hence, it suffices to consider $t > s$. In that case

$$
K(s,t) = \frac{1}{2} \int_0^\pi \int_0^\pi \left\{ \cos^{t-1}\phi \cos^s\theta + \cos^{t-2}\phi \cos^{s+1}\theta + \cdots + \cos^s\phi \cos^{t-1}\theta \right\}\, d\theta d\phi\ ,
$$

We now use the fact that

$$\int_0^\pi \cos^n \xi d\xi = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{2^n}\binom{n}{n/2}\pi & \text{if } n \text{ is even} \end{cases}$$

If $s \equiv t \pmod 2$, then all the individual integrals are 0, and $K(s,t) = 0$.

If t is odd and s is even

$$\begin{aligned} K(s,t) &= \frac{1}{2}\left\{ \frac{1}{2^{t-1}}\binom{t-1}{(t-1)/2}\frac{1}{2^s}\binom{s}{s/2} + \frac{1}{2^{t-3}}\binom{t-3}{(t-3)/2}\frac{1}{2^{s+2}}\binom{s+2}{(s+2)/2} + \cdots \right. \\ &\qquad \left. \cdots + \frac{1}{2^s}\binom{s}{s/2}\frac{1}{2^{t-1}}\binom{t-1}{(t-1)/2} \right\}\pi^2 \\ &= \frac{1}{2^{s+t}}\sum_{u=0}^{(t-s-1)/2}\binom{t-1-2u}{(t-1)/2-u}\binom{s+2u}{s/2+u}\pi^2, \end{aligned}$$

while if t is even and s is odd,

$$K(s,t) = \frac{1}{2^{s+t}}\sum_{u=0}^{(t-s-3)/2}\binom{t-2-2u}{(t-2)/2-u}\binom{s+1+2u}{(s+1)/2+u}\pi^2.$$

This, together with (3), completes the evaluation of $I(m,n)$.

3 Discussion

The main advantage of having an explicit expression for $I(m,n)$ lies in the rapidity of computer evaluation. For example, on an 800MHz PC running Linux, Maple took 242 seconds of CPU and 616540768 bytes of memory to find $I(8,8)$ when required to use the form (1) and perform the symbolic integrations. Using the explicit expression took 0.83 seconds and 7282672 bytes. The form of $I(8,8)$ follows:

$$\begin{aligned} I(8,8) &= -\frac{41409225}{8}b^{16} + 41593266\,b^{15} - 151465743\,b^{14} + 330609510\,b^{13} - \frac{964134171}{2}b^{12} \\ &\quad +495381978\,b^{11} - 368859645\,b^{10} + 201489838\,b^9 - \frac{323388615}{4}b^8 + 23626878\,b^7 \\ &\quad -4935693\,b^6 + 714378\,b^5 - \frac{136479}{2}b^4 + 3990\,b^3 - 127\,b^2 + 2\,b \end{aligned}$$

References

[1] J. W. Miles “Lee waves in a stratified flow. Part I. Thin barrier.”, *J. Fluid Mech.* **32**, 549 – 567 (1968).