RESULTS OF HURWITZ TYPE FOR THREE SQUARES

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1. Introduction and statement of results

Let $r_k(n)$ denote the number of representations of n as a sum of k squares. Hurwitz [3] gave eleven cases in which the generating function of $r_3(an+b)$ is a simple infinite product. We prove these eleven results, another twelve of the same sort and eighty infinite families of similar results. We also state as conjectures another eighty–two infinite families.

Hurwitz's results are our identities (1.3), (1.4), (1.6)–(1.10), (1.13), (1.15), (1.16) and (1.18).

Before stating our results, we need some definitions. Let

$$\phi(q)=\sum_{-\infty}^{\infty}q^{n^2},\quad \psi(q)=\sum_{n\ge 0}q^{(n^2+n)/2},\quad X(q)=\sum_{-\infty}^{\infty}q^{3n^2+2n},\quad P(q)=\sum_{-\infty}^{\infty}q^{(3n^2+n)/2},$$

$$A(q)=\sum_{-\infty}^{\infty}q^{9n^2+2n},\quad B(q)=\sum_{-\infty}^{\infty}q^{9n^2+4n},\quad C(q)=\sum_{-\infty}^{\infty}q^{9n^2+8n},\quad D(q)=\sum_{-\infty}^{\infty}q^{5n^2+2n},$$

$$E(q)=\sum_{-\infty}^{\infty}q^{5n^2+4n},\quad F(q)=\sum_{-\infty}^{\infty}q^{4n^2+n},\quad G(q)=\sum_{-\infty}^{\infty}q^{4n^2+3n},\quad H(q)=\sum_{-\infty}^{\infty}q^{(9n^2+n)/2},$$

$$I(q)=\sum_{-\infty}^{\infty}q^{(9n^2+5n)/2},\quad J(q)=\sum_{-\infty}^{\infty}q^{(9n^2+7n)/2},\quad K(q)=\sum_{-\infty}^{\infty}q^{(5n^2+n)/2},$$

$$L(q)=\sum_{-\infty}^{\infty}q^{(5n^2+3n)/2},\quad a(q)=\sum q^{m^2+mn+n^2},\quad b(q)=\sum \omega^{m-n}q^{m^2+mn+n^2}=\frac{(q)_\infty^3}{(q^3)_\infty},$$

$$c(q)=\sum q^{m^2+mn+n^2+m+n}=3\frac{(q^3)_\infty^3}{(q)_\infty}.$$

By Jacobi's triple product identity, each of $\phi(q)$, $\psi(q)$, $X(q)$, $P(q)$, $A(q)$, $B(q)$, $C(q)$, $D(q)$, $E(q)$, $F(q)$, $G(q)$, $H(q)$, $I(q)$, $J(q)$, $K(q)$ and $L(q)$ is a product.

We prove the following results.

Theorem 1:

$$\sum_{n\geq 0} r_3(3n)q^n = \phi(q)^4/\phi(q^3), \tag{1.1}$$

$$\sum_{n\geq 0} r_3(4n)q^n = \phi(q)^3, \tag{1.2}$$

$$\sum_{n\geq 0} r_3(4n+1)q^n = 6\phi(q)^2\psi(q^2), \tag{1.3}$$

$$\sum_{n\geq 0} r_3(4n+2)q^n = 12\phi(q)\psi(q^2)^2, \tag{1.4}$$

$$\sum_{n\geq 0} r_3(4n+3)q^n = 8\psi(q^2)^3, \tag{1.5}$$

$$\sum_{n\geq 0} r_3(8n+1)q^n = 6\phi(q)^2\psi(q), \tag{1.6}$$

$$\sum_{n\geq 0} r_3(8n+2)q^n = 12\phi(q^2)\psi(q)^2, \tag{1.7}$$

$$\sum_{n\geq 0} r_3(8n+3)q^n = 8\psi(q)^3, \tag{1.8}$$

$$\sum_{n\geq 0} r_3(8n+5)q^n = 24\psi(q)\psi(q^2)^2, \tag{1.9}$$

$$\sum_{n\geq 0} r_3(8n+6)q^n = 24\psi(q)^2\psi(q^4), \tag{1.10}$$

$$\sum_{n\geq 0} r_3(8n+7)q^n = 0. \tag{1.11}$$

$$\sum_{n\geq 0} r_3(12n+3)q^n = 8\psi(q^2)^4/\psi(q^6), \tag{1.12}$$

$$\sum_{n\geq 0} r_3(16n+2)q^n = 12\phi(q)\phi(q^2)\psi(q), \tag{1.13}$$

$$\sum_{n\geq 0} r_3(16n+5)q^n = 24\psi(q)^2F(q), \tag{1.14}$$

$$\sum_{n\geq 0} r_3(16n+6)q^n = 24\phi(q^2)\psi(q)\psi(q^2), \tag{1.15}$$

$$\sum_{n\geq 0} r_3(16n+10)q^n = 24\phi(q)\psi(q)\psi(q^4), \tag{1.16}$$

$$\sum_{n\geq 0} r_3(16n+13)q^n = 24\psi(q)^2 G(q), \tag{1.17}$$

$$\sum_{n\geq 0} r_3(16n+14)q^n = 48\psi(q)\psi(q^2)\psi(q^4), \tag{1.18}$$

$$\sum_{n\geq 0} r_3(24n+3)q^n = 8\psi(q)^4/\psi(q^3), \tag{1.19}$$

$$\sum_{n\geq 0} r_3(32n+6)q^n = 24\phi(q)\psi(q)F(q), \tag{1.20}$$

$$\sum_{n\geq 0} r_3(32n+14)q^n = 48\psi(q)\psi(q^2)F(q), \tag{1.21}$$

$$\sum_{n\geq 0} r_3(32n+22)q^n = 24\phi(q)\psi(q)G(q), \tag{1.22}$$

$$\sum_{n\geq 0} r_3(32n+30)q^n = 48\psi(q)\psi(q^2)G(q), \tag{1.23}$$

Theorem 2:

$$\sum_{n\geq 0} r_3(9^\lambda(3n+1))q^n = 6(2\times 3^\lambda - 1)\phi(q^3)^2 X(q), \tag{2.1}$$

$$\sum_{n\geq 0} r_3(9^\lambda(3n+2))q^n = 4\times 3^{\lambda+1}\phi(q)\left\{c(q^2)/3\right\}, \tag{2.2}$$

$$\sum_{n\geq 0} r_3(9^\lambda(6n+2))q^n = 4\times 3^{\lambda+1}\phi(q^2)\left\{c(q)/3\right\}, \tag{2.3}$$

$$\sum_{n\geq 0} r_3(9^\lambda(6n+5))q^n = 8\times 3^{\lambda+1}\psi(q^4)\left\{c(q)/3\right\}, \tag{2.4}$$

$$\sum_{n\geq 0} r_3(9^\lambda(9n+1))q^n = 6(2\times 3^\lambda - 1)\phi(q)^2 A(q), \tag{2.5}$$

$$\sum_{n\geq 0} r_3(9^\lambda(9n+4))q^n = 6(2\times 3^\lambda - 1)\phi(q)^2 B(q), \tag{2.6}$$

$$\sum_{n\geq 0} r_3(9^\lambda(9n+6))q^n = 12(3^{\lambda+1} - 1)\phi(q^3)\left\{c(q^2)/3\right\}. \tag{2.7}$$

$$\sum_{n\geq 0} r_3(9^\lambda(9n+7))q^n = 6(2\times 3^\lambda - 1)q\phi(q)^2 C(q), \tag{2.8}$$

$$\sum_{n\geq 0} r_3(9^\lambda(12n+1))q^n = 6(2\times 3^\lambda - 1)\phi(q)^2 P(q^2), \tag{2.9}$$

$$\sum_{n\geq 0} r_3(9^\lambda(12n+2))q^n = 4\times 3^{\lambda+1}\phi(q)\psi(q)P(q), \tag{2.10}$$

$$\sum_{n\geq 0} r_3(9^\lambda(12n+5))q^n = 8\times 3^{\lambda+1}\psi(q^2)\psi(q)P(q), \tag{2.11}$$

$$\sum_{n\geq 0} r_3(9^\lambda(12n+7))q^n = 24(2\times 3^\lambda - 1)q\psi(q^2)\left\{c(q^4)/3\right\}, \tag{2.12}$$

$$\sum_{n\geq 0} r_3(9^\lambda(12n+10))q^n = 24(2\times 3^\lambda - 1)\psi(q^2)^2 X(q), \tag{2.13}$$

$$\sum_{n\geq 0} r_3(9^\lambda(12n+11))q^n = 8\times 3^{\lambda+1}\psi(q^2)\left\{c(q^2)/3\right\}, \tag{2.14}$$

$$\sum_{n\geq 0} r_3(9^\lambda(18n+6))q^n = 12(3^{\lambda+1} - 1)\phi(q^6)\left\{c(q)/3\right\}, \tag{2.15}$$

$$\sum_{n\geq 0} r_3(9^\lambda(18n+15))q^n = 24(3^{\lambda+1} - 1)q\psi(q^{12})\left\{c(q)/3\right\}, \tag{2.16}$$

$$\sum_{n\geq 0} r_3(9^\lambda(24n+1))q^n = 6(2\times 3^\lambda - 1)P(q)\phi(q)^2, \tag{2.17}$$

$$\sum_{n\geq 0} r_3(9^\lambda(24n+5))q^n = 8\times 3^{\lambda+1}\phi(q)\psi(q)P(q^2), \tag{2.18}$$

$$\sum_{n\geq 0} r_3(9^\lambda(24n+10))q^n = 24(2\times 3^\lambda - 1)\psi(q)^2 P(q^4), \tag{2.19}$$

$$\sum_{n\geq 0} r_3(9^\lambda(24n+11))q^n = 8\times 3^{\lambda+1}\psi(q)\left\{c(q)/3\right\}, \tag{2.20}$$

$$\sum_{n\geq 0} r_3(9^\lambda(24n+13))q^n = 24(2\times 3^\lambda - 1)P(q)\psi(q^2)^2, \tag{2.21}$$

$$\sum_{n\geq 0} r_3(9^\lambda(24n+17))q^n = 16\times 3^{\lambda+1}\psi(q)\psi(q^2)X(q), \tag{2.22}$$

$$\sum_{n\geq 0} r_3(9^\lambda(24n+19))q^n = 24(2\times 3^\lambda - 1)\psi(q)\left\{c(q^2)/3\right\}, \tag{2.23}$$

$$\sum_{n\geq 0} r_3(9^\lambda(24n+22))q^n = 24(2\times 3^\lambda - 1)\psi(q)^2 X(q^2), \tag{2.24}$$

$$\sum_{n\geq 0} r_3(9^\lambda(36n+6))q^n = 12(3^{\lambda+1} - 1)P(q^2)^2 \phi(q), \tag{2.25}$$

$$\sum_{n\geq 0} r_3(9^\lambda(36n+7))q^n = 24(2\times 3^\lambda - 1)q\psi(q^2)^2 I(q^2), \tag{2.26}$$

$$\sum_{n\geq 0} r_3(9^\lambda(36n+15))q^n = 24(3^{\lambda+1} - 1)q\psi(q^6)\left\{c(q^2)/3\right\}, \tag{2.27}$$

$$\sum_{n\geq 0} r_3(9^\lambda(36n+19))q^n = 24(2\times 3^\lambda - 1)\psi(q^2)^2 H(q^2), \tag{2.28}$$

$$\sum_{n\geq 0} r_3(9^\lambda(36n+21))q^n = 24(3^{\lambda+1} - 1)\psi(q)\psi(q^3)P(q^2), \tag{2.29}$$

$$\sum_{n\geq 0} r_3(9^\lambda(36n+30))q^n = 24(3^{\lambda+1} - 1)X(q^2)P(q^4)\phi(q), \tag{2.30}$$

$$\sum_{n\geq 0} r_3(9^\lambda(36n+31))q^n = 24(2\times 3^\lambda - 1)q\psi(q^2)^2 J(q^2), \tag{2.31}$$

$$\sum_{n\geq 0} r_3(9^\lambda(36n+33))q^n = 24(3^{\lambda+1} - 1)\psi(q^2)X(q)^2, \tag{2.32}$$

$$\sum_{n\geq 0} r_3(9^\lambda(48n+10))q^n = 24(2\times 3^\lambda - 1)\phi(q^2)P(q^2)\psi(q), \tag{2.33}$$

$$\sum_{n\geq 0} r_3(9^\lambda(48n+19))q^n = 24(2\times 3^\lambda - 1)\left\{c(q)/3\right\}F(q), \tag{2.34}$$

$$\sum_{n\geq 0} r_3(9^\lambda(48n+22))q^n = 24(2\times 3^\lambda - 1)\phi(q^2)\psi(q)X(q), \tag{2.35}$$

$$\sum_{n\geq 0} r_3(9^\lambda(48n+34))q^n = 48(2\times 3^\lambda - 1)P(q^2)\psi(q^4)\psi(q), \tag{2.36}$$

$$\sum_{n\geq 0} r_3(9^\lambda(48n+43))q^n = 24(2\times 3^\lambda - 1)\left\{c(q)/3\right\}G(q), \tag{2.37}$$

$$\sum_{n\geq 0} r_3(9^\lambda(48n+46))q^n = 48(2\times 3^\lambda - 1)\psi(q^4)\psi(q)X(q), \tag{2.38}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+6))q^n = 12(3^{\lambda+1} - 1)\phi(q^2)P(q)^2, \tag{2.39}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+19))q^n = 24(2\times 3^\lambda - 1)\psi(q)^2 H(q), \tag{2.40}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+21)q^n = 24(3^{\lambda+1} - 1)P(q^2)^2 \psi(q), \tag{2.41}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+30))q^n = 24(3^{\lambda+1} - 1)\phi(q^2)P(q^2)X(q), \tag{2.42}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+33))q^n = 24(3^{\lambda+1} - 1)\phi(q^3)\psi(q)X(q), \tag{2.43}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+42))q^n = 24(3^{\lambda+1} - 1)\psi(q^4)P(q)^2, \tag{2.44}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+43))q^n = 24(2\times 3^\lambda - 1)\psi(q)^2 I(q), \tag{2.45}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+51))q^n = 24(3^{\lambda+1} - 1)\psi(q^3)\left\{c(q)/3\right\}, \tag{2.46}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+57))q^n = 24(3^{\lambda+1} - 1)\phi(q)\psi(q^6)P(q), \tag{2.47}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+66))q^n = 48(3^{\lambda+1} - 1)\psi(q^4)P(q^2)X(q) \tag{2.48}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+67))q^n = 24(2\times 3^\lambda - 1)\psi(q)^2 J(q), \tag{2.49}$$

$$\sum_{n\geq 0} r_3(9^\lambda(72n+69))q^n = 48(3^{\lambda+1} - 1)X(q^2)P(q^4)\psi(q), \tag{2.50}$$

$$\sum_{n\geq 0} r_3(9^\lambda(96n+10))q^n = 24(2\times 3^\lambda - 1)\phi(q)P(q)F(q), \tag{2.51}$$

$$\sum_{n\geq 0} r_3(9^\lambda(96n+34))q^n = 48(2\times 3^\lambda - 1)\psi(q^2)P(q)F(q), \tag{2.52}$$

$$\sum_{n\geq 0} r_3(9^\lambda(96n+58))q^n = 24(2\times 3^\lambda - 1)\phi(q)P(q)G(q), \tag{2.53}$$

$$\sum_{n\geq 0} r_3(9^\lambda(96n+82))q^n = 48(2\times 3^\lambda - 1)\psi(q^2)P(q)G(q), \tag{2.54}$$

$$\sum_{n\geq 0} r_3(9^\lambda(144n+21))q^n = 24(3^{\lambda+1} - 1)P(q)^2 F(q), \tag{2.55}$$

$$\sum_{n\geq 0} r_3(9^\lambda(144n+30))q^n = 24(3^{\lambda+1}-1)\phi(q)P(q)P(q^4), \tag{2.56}$$

$$\sum_{n\geq 0} r_3(9^\lambda(144n+66))q^n = 48(3^{\lambda+1}-1)\psi(q^2)P(q)P(q^4), \tag{2.57}$$

$$\sum_{n\geq 0} r_3(9^\lambda(144n+69))q^n = 48(3^{\lambda+1}-1)X(q)P(q^2)F(q), \tag{2.58}$$

$$\sum_{n\geq 0} r_3(9^\lambda(144n+93))q^n = 24(3^{\lambda+1}-1)P(q)^2G(q), \tag{2.59}$$

$$\sum_{n\geq 0} r_3(9^\lambda(144n+102))q^n = 24(3^{\lambda+1}-1)\phi(q)P(q)X(q^2), \tag{2.60}$$

$$\sum_{n\geq 0} r_3(9^\lambda(144n+138))q^n = 48(3^{\lambda+1}-1)\psi(q^2)P(q)X(q^2), \tag{2.61}$$

$$\sum_{n\geq 0} r_3(9^\lambda(144n+141))q^n = 48(3^{\lambda+1}-1)X(q)P(q^2)G(q), \tag{2.62}$$

Theorem 3:

$$\sum_{n\geq 0} r_3(25^\lambda(5n+1))q^n = 6\times 5^\lambda\phi(q)^2D(q), \tag{3.1}$$

$$\sum_{n\geq 0} r_3(25^\lambda(5n+4))q^n = 6\times 5^\lambda\phi(q)^2E(q), \tag{3.2}$$

$$\sum_{n\geq 0} r_3(25^\lambda(20n+11))q^n = 24\times 5^\lambda\psi(q^2)^2K(q^2), \tag{3.3}$$

$$\sum_{n\geq 0} r_3(25^\lambda(20n+19))q^n = 24\times 5^\lambda\psi(q^2)^2L(q^2), \tag{3.4}$$

$$\sum_{n\geq 0} r_3(25^\lambda(25n+5))q^n = 6(5^{\lambda+1}-1)\phi(q)\phi(q^5)D(q), \tag{3.5}$$

$$\sum_{n\geq 0} r_3(25^\lambda(25n+10))q^n = 6(5^{\lambda+1}-1)\phi(q)D(q)^2, \tag{3.6}$$

$$\sum_{n\geq 0} r_3(25^\lambda(25n+15))q^n = 6(5^{\lambda+1}-1)q\phi(q)E(q)^2, \tag{3.7}$$

$$\sum_{n\geq 0} r_3(25^\lambda(25n+20))q^n = 6(5^{\lambda+1}-1)\phi(q)\phi(q^5)E(q). \tag{3.8}$$

$$\sum_{n\geq 0} r_3(25^\lambda(40n+11))q^n = 24\times 5^\lambda\psi(q)^2K(q), \tag{3.9}$$

$$\sum_{n\geq 0} r_3(25^\lambda(40n+19))q^n = 24\times 5^\lambda \psi(q)^2 L(q), \tag{3.10}$$

$$\sum_{n\geq 0} r_3(25^\lambda(100n+15))q^n = (68\times 5^\lambda - 20)q\psi(q^2)L(q^2)^2, \tag{3.11}$$

$$\sum_{n\geq 0} r_3(25^\lambda(100n+35))q^n = (68\times 5^\lambda - 20)\psi(q^2)K(q^2)^2, \tag{3.12}$$

$$\sum_{n\geq 0} r_3(25^\lambda(100n+55))q^n = (136\times 5^\lambda - 40)q\psi(q^2)\psi(q^{10})K(q^2), \tag{3.13}$$

$$\sum_{n\geq 0} r_3(25^\lambda(100n+95))q^n = (136\times 5^\lambda - 40)q\psi(q^2)\psi(q^{10})L(q^2), \tag{3.14}$$

$$\sum_{n\geq 0} r_3(25^\lambda(200n+35))q^n = (68\times 5^\lambda - 20)\psi(q)K(q)^2, \tag{3.15}$$

$$\sum_{n\geq 0} r_3(25^\lambda(200n+115))q^n = (68\times 5^\lambda - 20)\psi(q)L(q)^2, \tag{3.16}$$

$$\sum_{n\geq 0} r_3(25^\lambda(200n+155))q^n = (136\times 5^\lambda - 40)\psi(q)\psi(q^5)K(q), \tag{3.17}$$

$$\sum_{n\geq 0} r_3(25^\lambda(200n+195))q^n = (136\times 5^\lambda - 40)\psi(q)\psi(q^5)L(q). \tag{3.18}$$

We further conjecture that for the following values of a, b and p, and for no other (non-trivial) a, b with $0 \leq b < a < 200$, $\sum_{n\geq 0} r_3(p^{2\lambda}(an+b))q^n$ is a simple infinite product, that is,

$$\sum_{n\geq 0} r_3(p^{2\lambda}(an+b))q^n = Cq^k \prod_{i=1}^{m} (q^{\alpha_i};q^{\beta_i})_\infty^{k_i},$$

where C, m, α_i, β_i are positive integers, k is a non–negative integer and k_i are integers.

a	b	p
15	7, 13	3, 5
30	7, 13, 22, 28	3, 5
40	13, 17, 33, 37	5
45	21, 39	3, 5
60	7, 13, 22, 37, 43, 58	3, 5
63	15, 30, 39, 51, 57, 60	3, 7
75	55, 70	3, 5
80	17, 33, 57, 73	5
120	13, 37, 43, 67, 73, 97	3, 5
147	7, 28, 112	3, 7
150	55, 70, 130, 145	3, 5
180	39, 111	3, 5.

2. Preliminary lemmas

We shall require the following lemmas, where $\omega = e^{2\pi i/3}$.

(i)
$$\phi(q) = \phi(q^4) + 2q\psi(q^8)$$

(ii)
$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2$$

(iii)
$$(q)_\infty(\omega q)_\infty(\omega^2 q)_\infty = \frac{(q^3)_\infty^4}{(q^9)_\infty}$$

(iv)
$$\phi(q)\phi(\omega q)\phi(\omega^2 q) = \frac{\phi(q^3)^4}{\phi(q^9)}$$

(v)
$$\phi(q) = \phi(q^9) + 2qX(q^3)$$

(vi)
$$\phi(q^9)^3 + 8q^3X(q^3)^3 = \frac{\phi(q^3)^4}{\phi(q^9)}$$

(vii) $$\phi(q)^3 = \frac{\phi(q^3)^4}{\phi(q^9)} + 6q\phi(q^9)^2X(q^3) + 12q^2\phi(q^9)X(q^3)^2$$

(viii) $$\frac{\phi(q^3)^4}{\phi(q)} = \phi(q^9)^3 - 2q\phi(q^9)^2X(q^3) + 4q^2\phi(q^9)X(q^3)^2$$

(ix)

$$\frac{\phi(q)^4}{\phi(q^3)} = \left(4\phi(q^3)^3 - 3\frac{\phi(q^9)^4}{\phi(q^3)}\right) + 2q\left(\frac{\phi(q^3)^3}{\phi(q^9)} + 3\frac{\phi(q^9)^3}{\phi(q^3)}\right)X(q^3) + 24q^2\frac{\phi(q^9)^2}{\phi(q^3)}X(q^3)^2$$

(x) $$\phi(q) = \phi(q^{25}) + 2qD(q^5) + 2q^4E(q^5)$$

(xi) $$\phi(q^5)^2 + 4qD(q)E(q) = \phi(q)^2$$

(xii) $$\psi(q) = F(q^2) + qG(q^2)$$

(xiii) $$\psi(q)^2 = \phi(q^4)\psi(q^2) + 2q\psi(q^2)\psi(q^8)$$

(xiv) $$\psi(q) = P(q^3) + q\psi(q^9)$$

(xv) $$\psi(q)^3 = \frac{\psi(q^3)^4}{\psi(q^9)} + 3qP(q^3)^2\psi(q^9) + 3q^2P(q^3)\psi(q^9)^2$$

(xvi) $$\psi(q) = K(q^5) + qL(q^5) + q^3\psi(q^{25})$$

(xvii) $$q\psi(q^5)^2 + K(q)L(q) = \psi(q)^2$$

(xviii) $$X(q) = P(q^8) + qX(q^4)$$

(xix) $$X(q)^2 = \phi(q^6)X(q^2) + 2qX(q^4)P(q^8)$$

(xx) $$\psi(q)P(q) = \phi(q^2)P(q^4) + 2q\psi(q^4)X(q^2)$$

(xxi) $$X(q) = A(q^3) + qB(q^3) + q^5C(q^3)$$

(xxii) $$P(q) = H(q^3) + qI(q^3) + q^2J(q^3)$$

(xxiii) $$D(q) = K(q^8) + q^3E(q^4)$$

(xxiv) $$E(q) = D(q^4) + qL(q^8)$$

(xxv) $$a(q) = a(q^4) + 6q\psi(q^2)\psi(q^6)$$

(xxvi) $$b(q) = b(q^4) - 3q\frac{(q^2)_\infty^2(q^{12})_\infty^3}{(q^4)_\infty(q^6)_\infty^2},$$

(xxvii) $$\{c(q)/3\} = \psi(q^2)P(q^2) + q\left\{c(q^4)/3\right\}$$

(xxviii) $$\psi(q^2)^2 + q\psi(q^6)^2 = \frac{(q^3)_\infty^3(q^4)_\infty(q^{12})_\infty}{(q)_\infty(q^6)_\infty^2}$$

(xxix) $$\psi(q^2)^2 + 3q\psi(q^6)^2 = \frac{(q^2)_\infty^7(q^3)_\infty(q^{12})_\infty^2}{(q)_\infty^3(q^4)_\infty^2(q^6)_\infty^3}$$

(xxx) $$\phi(q^3)P(q^2) + 2qX(q)\psi(q^6) = P(q)^2$$

(xxxi) $$\psi(q^3)^2P(q^2) + q\psi(q^6)^2X(q) = \psi(q^2)^2X(q)$$

(xxxii) $$\phi(q^3)^2P(q^2) + 4q\psi(q^3)^2X(q) = \phi(q)^2P(q^2)$$

(xxxiii) $$\psi(q)\psi(q^3) = \phi(q^6)\psi(q^4) + q\phi(q^2)\psi(q^{12})$$

Proofs:

(i) is straightforward. Thus

$$\phi(q)=\sum_{-\infty}^{\infty}q^{n^2}=\sum_{n\text{ even}}q^{n^2}+\sum_{n\text{ odd}}q^{n^2}=\sum_{-\infty}^{\infty}q^{4n^2}+\sum_{-\infty}^{\infty}q^{4n^2+4n+1}=\phi(q^4)+2q\psi(q^8).$$

(v), (x), (xii), (xiv), (xvi), (xviii), (xxi), (xxii), (xxiii) and (xxiv) are similarly straightforward.

(ii). By (i),

$$\phi(q^4)^2-4q^2\psi(q^8)^2=\phi(q)\phi(-q)=\frac{(q^2)_\infty^5}{(q)_\infty^2(q^4)_\infty^2}\frac{(q)_\infty^2}{(q^2)_\infty}=\frac{(q^2)_\infty^4}{(q^4)_\infty^2}=\phi(-q^2)^2.$$

Replace q^2 by q, then q by $-q$.

(iii).

$$(q)_\infty(\omega q)_\infty(\omega^2 q)_\infty=\prod_{3\nmid n}(1-q^{3n})\prod_{3|n}(1-q^n)^3=\frac{\prod(1-q^{3n})^4}{\prod(1-q^{9n})}=\frac{(q^3)_\infty^4}{(q^9)_\infty}.$$

(iv) follows easily from the fact that $\phi(q)=\dfrac{(q^2)_\infty^5}{(q)_\infty^2(q^4)_\infty^2}$ together with (iii).

(vi). In (v), replace q by q, ωq and $\omega^2 q$, multiply the three results and use (iv).

(vii). Cube (v), then use (vi).

(viii). Write $\dfrac{1}{\phi(q)}=\dfrac{\phi(\omega q)\phi(\omega^2 q)}{\phi(q)\phi(\omega q)\phi(\omega^2 q)}$, use (iv) on the denominator and (v) in the numerator, then multiply by $\phi(q^3)^4$.

(ix). Take the fourth power of (v), use (vi) to replace $X(q^3)^3$ in two terms, then divide by $\phi(q^3)$.

(xi). We have

$$\phi(q^5)^2=\sum_{m,n=-\infty}^{\infty}q^{5m^2+5n^2}=\sum q^{(2m-n)^2+(m+2n)^2}=\sum_{2u+v\equiv 0\pmod 5}q^{u^2+v^2}$$
$$=\sum_{m,n=-\infty}^{\infty}q^{(5m)^2+(5n)^2}+\sum_{m,n=-\infty}^{\infty}q^{(5m+1)^2+(-5n-2)^2}$$

$$+\sum_{m,n=-\infty}^{\infty} q^{(5n+2)^2+(5m+1)^2} + \sum_{m,n=-\infty}^{\infty} q^{(-5m-1)^2+(5n+2)^2}$$
$$+\sum_{m,n=-\infty}^{\infty} q^{(-5n-2)^2+(-5m-1)^2}$$
$$=\sum_{m,n=-\infty}^{\infty} q^{25m^2+25n^2} + 4q^5 \sum_{m,n=-\infty}^{\infty} q^{25m^2+10m+25n^2+20n}$$
$$=\phi(q^{25})^2 + 4q^5 D(q^5)E(q^5).$$

(xiii). By (i),

$$\psi(q)^2 = \phi(q)\psi(q^2) = \psi(q^2)\left(\phi(q^4) + 2q\psi(q^8)\right) = \phi(q^4)\psi(q^2) + 2q\psi(q^2)\psi(q^8).$$

(xv) follows from (xiv) as (vii) follows from (v) and (vi).

(xvii).

$$4q^{10}\psi(q^{40})^2 = \sum_{m,n=-\infty}^{\infty} q^{20m^2+20m+20n^2+20n+10}$$
$$= \sum q^{5(2m+1)^2+5(2n+1)^2} = \sum q^{(4m-2n+1)^2+(2m+4n+3)^2}$$
$$= \sum_{\substack{u,v \text{ odd} \\ 2u+v\equiv 5 \pmod{10}}} q^{u^2+v^2}$$
$$= \sum_{m,n=-\infty}^{\infty} q^{(10m+5)^2+(10n+5)^2} + \sum_{m,n=-\infty}^{\infty} q^{(10m-3)^2+(10n+1)^2}$$
$$+ \sum_{m,n=-\infty}^{\infty} q^{(-10m-1)^2+(-10n-3)^2} + \sum_{m,n=-\infty}^{\infty} q^{(10m+1)^2+(10n+3)^2}$$
$$+ \sum_{m,n=-\infty}^{\infty} q^{(10m+3)^2+(-10n-1)^2}$$
$$= 4q^{50}\psi(q^{200})^2 + 4q^{10}K(q^{40})L(q^{40}).$$

(xix). By (xviii),

$$P(q^8)^2 - q^2X(q^4)^2 = X(q)X(-q)$$

$$= \prod_{n\geq 1}(1+q^{6n-5})(1+q^{6n-1})(1-q^{6n-5})(1-q^{6n-1})(1-q^{6n})^2$$
$$= \prod_{n\geq 1}(1-q^{12n-10})(1-q^{12n-6})^2(1-q^{12n-2})(1-q^{12n})^2,$$

so

$$P(q^8)^2 + q^2X(q^4)^2 = \prod_{n\geq 1}(1+q^{12n-10})(1+q^{12n-6})^2(1+q^{12n-2})(1-q^{12n})^2$$
$$= \phi(q^6)X(q^2).$$

It follows that

$$X(q)^2 = \big(P(q^8)+qX(q^4)\big)^2 = \big(P(q^8)^2+q^2X(q^4)^2\big) + 2qX(q^4)P(q^8))$$
$$= \phi(q^6)X(q^2) + 2qX(q^4)P(q^8).$$

(xx). Multiply (xix) by $\dfrac{(q^4)_\infty^2(q^6)_\infty}{(q^2)_\infty(q^{12})_\infty^2}$.

(xxv), (xxvi), (xxvii). See [1, (1.34), (1.35) and (1.36)] for elementary proofs.

(xxviii). Multiply (xxvii) by $\dfrac{(q^4)_\infty(q^{12})_\infty}{(q^6)_\infty^2}$.

(xxix). Put $-q$ for q in (xxvi), then multiply by $\dfrac{(q^4)_\infty(q^{12})_\infty}{(q^2)_\infty^2}$.

(xxx). By (v) and (xiv), $\phi(q)\psi(q^2) = \psi(q)^2$ becomes

$$\big(\phi(q^9)+2qX(q^3)\big)\big(P(q^6)+q^2\psi(q^{18})\big) = \big(P(q^3)+q\psi(q^9)\big)^2.$$

Extract powers congruent to 0 (mod 3) and replace q^3 by q.

(xxxi), (xxxii). Using (xxviii), (xxx) can be written

$$\phi(q^3)P(q^2) + 2qX(q)\psi(q^6) = \frac{(q^2)_\infty^2(q^3)_\infty}{(q)_\infty(q^4)_\infty(q^{12})_\infty}\left\{\psi(q^2)^2 + q\psi(q^6)^2\right\}.$$

Subtract $qX(q)\psi(q^6) = \dfrac{(q^2)_\infty^2(q^3)_\infty}{(q)_\infty(q^4)_\infty(q^{12})_\infty}q\psi(q^6)^2$, then multiply by $\psi(q^6)$ to obtain (xxxi); add twice the same quantity, use (xxix) and then multiply by $\phi(q^3)$ to obtain (xxxii).

(xxxiii).

$$q^4\psi(q^8)\psi(q^{24}) = \sum_{k,l=-\infty}^{\infty} q^{(4k+1)^2+3(4l+1)^2} = \sum_{k,l=-\infty}^{\infty} q^{4(k+3l+1)^2+12(k-l)^2}$$
$$= \sum_{u-v\equiv 1 \pmod 4} q^{4u^2+12v^2}.$$

Consider the two cases, v even, u even. For v even, let $v = 2k$, $u = 4l+1$ or $-4l-1$ according as k is even or odd, while for u even, let $u = 2k$, $v = 4l+1$ or $-4l-1$ according as k is odd or even. Then

$$q^4\psi(q^8)\psi(q^{24}) = \sum_{k,l=-\infty}^{\infty} q^{4(4l+1)^2+12(2k)^2} + \sum_{k,l=-\infty}^{\infty} q^{4(2k)^2+12(4l+1)^2}$$
$$= \sum_{k,l=-\infty}^{\infty} q^{48k^2+4(4l+1)^2} + \sum_{k,l=-\infty}^{\infty} q^{16k^2+12(4l+1)^2}$$
$$= q^4\phi(q^{48})\psi(q^{32}) + q^{12}\phi(q^{16})\psi(q^{96}).$$

3. Proof of Theorem 1

By (vii),

$$\sum_{n\geq 0} r_3(n)q^n = \phi(q)^3 = \frac{\phi(q^3)^4}{\phi(q^9)} + 6q\phi(q^9)^2X(q^3) + 12q^2\phi(q^9)X(q^3)^2.$$

(1.1) follows.

By (i),

$$\sum_{n\geq 0} r_3(n)q^n = \phi(q)^3 = \phi(q^4)^3 + 6q\phi(q^4)^2\psi(q^8) + 12q^2\phi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3.$$

(1.2) — (1.5) follow.

In the same way, (1.3) and (ii) lead to (1.6) and (1.9), (1.4) and (i) to (1.7) and (1.10), (1.8) and (1.11) follow directly from (1.5), (1.5) and (xv) yield (1.12), (1.7) and (xiii) give (1.13) and (1.16), (1.9) and (xii) lead to (1.14) and (1.17), (1.10) and (xiii) to (1.15) and (1.18), (1.19) follows directly from (1.12), (1.15) and (xii) yield (1.20) and (1.22), while (1.18) and (xii) give (1.21) and (1.23).

4. Proof of Theorem 2

Using (vii), (viii) and (ix), it is easy to prove by induction on λ that for $\lambda \geq 0$

$$\sum_{n\geq 0} r_3(3^{2\lambda}n)q^n = \frac{1}{2}(3^{\lambda+1}-1)\phi(q)^3 - \frac{1}{2}(3^{\lambda+1}-3)\frac{\phi(q^3)^4}{\phi(q)} \tag{4.1}$$

and

$$\sum_{n\geq 0} r_3(3^{2\lambda+1}n)q^n = \frac{1}{2}(3^{\lambda+1}-1)\frac{\phi(q)^4}{\phi(q^3)} - \frac{1}{2}(3^{\lambda+1}-3)\phi(q^3)^3. \tag{4.2}$$

¿From (4.1), (vii) and (viii) we obtain (2.1) and (2.2), from (2.2) and (i) we obtain (2.3) and (2.4), (2.1) and (xxi) yield (2.5), (2.6) and (2.8), from (4.2) and (ix) we deduce (2.7); (2.1), (i) and (xviii) yield (2.9) (after using (xxxii)), (2.12) and (2.13) (after using (xxxi)). (2.3) and (xxvii) yield (2.10), while (2.4) and (xxvii) yield (2.11) and (2.14). (2.7) and (i) give (2.15) and (2.16), (2.9) and (ii) give (2.17) and (2.21), (2.11) and (xx) give (2.18) and (2.22), (2.13) and (xviii) give (2.19) and (2.24), (2.20) follows directly from (2.14), (2.23) from (2.12). (2.15) and (xxvii) yield (2.25), (2.12) and (xxii) yield (2.26), (2.28) and (2.31) after noting that $\psi(q^2)\left\{c(q^4)/3\right\} = \psi(q^6)^2 P(q^2)$, and (2.16) together with (xxvii) gives (2.27) and (2.32).

¿From (4.2) and (ix), we deduce that

$$\sum_{n\geq 0} r_3(9^{\lambda}(9n+3))q^n = (3^{\lambda+1}-1)\left(\frac{\phi(q)^3}{\phi(q^3)} + 3\frac{\phi(q^3)^3}{\phi(q)}\right)X(q).$$

On the other hand, it was shown in [2] by elementary methods that

$$\sum_{n\geq 0} r_3(9n+3)q^n = 8X(q)a(q^2).$$

Thus we find the remarkable identity

$$\frac{\phi(q)^3}{\phi(q^3)} + 3\frac{\phi(q^3)^3}{\phi(q)} = 4a(q^2) \tag{4.3}$$

and it follows that

$$\sum_{n\geq 0} r_3(9^{\lambda}(9n+3))q^n = 4(3^{\lambda+1}-1)X(q)a(q^2). \tag{4.4}$$

¿From (4.4), (xviii) and (xxv), we obtain (2.29) and (2.30). (2.19) and (xiii) lead to (2.33) and (2.36), (2.23) and (xii) to (2.34) and (2.37), (2.24) and (xiii) to (2.35) and (2.38), (2.25) and (i) to (2.39) and (2.44). (2.40) follows directly from (2.28). (2.29) and (xxxiii) yield (2.41) and (2.47), (2.30) and (i) give (2.42) and (2.48), (2.32) and (xix) give (2.43) and (2.50). (2.45) follows directly from (2.26), (2.46) from (2.27), (2.49) from (2.31). ¿From (2.33) and (xii) we obtain (2.51) and (2.53), (2.36) and (xii) lead to (2.52) and (2.54), (2.41) and (xii) to (2.55) and (2.59), (2.42) and (xviii) to (2.56) and (2.60), (2.48) and (xviii) to (2.57) and (2.61), and (2.50) and (xii) lead to (2.58) and (2.62).

5. Proof of Theorem 3

Using (x) and (xi) it is easy to prove by induction on λ that for $\lambda \geq 0$,

$$\sum_{n\geq 0} r_3(5^{2\lambda}n)q^n = \frac{1}{2}(3\times 5^{\lambda}-1)\phi(q)^3 - \frac{3}{2}(5^{\lambda}-1)\phi(q^5)^2\phi(q), \tag{5.1}$$

and

$$\sum_{n\geq 0} r_3(5^{2\lambda+1}n)q^n = \frac{3}{2}(5^{\lambda+1}-1)\phi(q^5)\phi(q)^2 - \frac{5}{2}(3\times 5^{\lambda}-1)\phi(q^5)^3. \tag{5.2}$$

¿From (5.1), (x) and (xi) we obtain (3.1) and (3.2), (3.1), (i) and (xxiii) lead to (3.3), (3.2), (i) and (xxiv) to (3.4). ¿From (5.2) and (x) we obtain (3.5), (3.6), (3.7) and (3.8). (3.9) follows directly from (3.3), (3.10) from (3.4).

Using (1.5), (xvi) and (xvii) it is easy to prove by induction on λ that for $\lambda \geq 0$,

$$\sum_{n\geq 0} r_3(5^{2\lambda}(4n+3))q^n = (12\times 5^{\lambda}-4)\psi(q^2)^3 - (4\times 5^{\lambda}-4)q^2\psi(q^{10})^2\psi(q^2), \tag{5.3}$$

and

$$\sum_{n\geq 0} r_3(5^{2\lambda+1}(4n+3))q^n = (68\times 5^{\lambda}-20)q\psi(q^{10})\psi(q^2)^2 - (60\times 5^{\lambda}-20)q^3\psi(q^{10})^3. \tag{5.4}$$

¿From (5.4) and (xvi) we obtain (3.11), (3.12), (3.13) and (3.14). (3.15) follows directly from (3.11), (3.16) from (3.12), (3.17) from (3.13), (3.18) from (3.14).

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