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TWO OR THREE IDENTITIES OF RAMANUJAN

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1. Introduction. In his third Notebook [5, pp. 385-6], Ramanujan writes

"If $a/b = c/d$, then

$$\begin{aligned}&(a+b+c)^4+(b+c+d)^4+(a-d)^4\\&\quad=(c+d+a)^4+(d+a+b)^4+(b-c)^4\end{aligned}$$

and 4 may be replaced by 2 also."
and

$$\begin{aligned}&\text{“}64\{(a+b+c)^6+(b+c+d)^6-(c+d+a)^6\\&\qquad-(d+a+b)^6+(a-d)^6-(b-c)^6\}\\&\quad\times\{(a+b+c)^{10}+(b+c+d)^{10}-(c+d+a)^{10}\\&\qquad-(d+a+b)^{10}+(a-d)^{10}-(b-c)^{10}\}\\&\quad=45\{(a+b+c)^8+(b+c+d)^8-(c+d+a)^8\\&\qquad-(d+a+b)^8+(a-d)^8-(b-c)^8\}^2.\text{”}\end{aligned}\tag{$*$}$$

Berndt ([1], p. 3) describes this last identity as "an amazing identity" and on p. 102 as "one of the most fascinating identities we have ever seen". He reproduces a proof due to Berndt and Bhargava [2], and refers to proofs by Bhargava [3] and by Nanjundiah [4].

It is possible to verify such identities by using a software package. Thus, for example, we can factor the difference between the left and right sides of ($*$), and obtain

$$(ad-bc)F$$

where F is a polynomial in a, b, c, d, homogeneous of degree 14, with many terms. This verifies ($*$), but sheds no light on *why* it is true.

The identities given by Ramanujan are disguised by the fact that they involve *even* powers. The key to understanding the identities is realising that they do not really concern the quantities

$$a+b+c,\quad b+c+d,\quad a-d,\quad c+d+a,\quad d+a+b,\quad b-c$$

but rather the quantities

$$\alpha=a+b+c,\quad \beta=-b-c-d,\quad \gamma=-a+d,$$
$$\alpha'=-c-d-a,\quad \beta'=d+a+b,\quad \gamma'=-b+c$$

which satisfy

$$\alpha+\beta+\gamma=0,\quad \alpha'+\beta'+\gamma'=0.$$

Thus, Ramanujan's identities are, subject to the condition $ad=bc$,

$$\alpha^2+\beta^2+\gamma^2={\alpha'}^2+{\beta'}^2+{\gamma'}^2,$$
$$\alpha^4+\beta^4+\gamma^4={\alpha'}^4+{\beta'}^4+{\gamma'}^4$$

and

$$64(\alpha^6+\beta^6+\gamma^6-{\alpha'}^6-{\beta'}^6-{\gamma'}^6)(\alpha^{10}+\beta^{10}+\gamma^{10}-{\alpha'}^{10}-{\beta'}^{10}-{\gamma'}^{10})$$
$$=45(\alpha^8+\beta^8+\gamma^8-{\alpha'}^8-{\beta'}^8-{\gamma'}^8)^2.$$

If, further, we write

$$F_n=\alpha^n+\beta^n+\gamma^n-{\alpha'}^n-{\beta'}^n-{\gamma'}^n$$

the identities become

$$F_2=0,\quad F_4=0\ \text{ and }\ 64F_6F_{10}=45F_8^2.$$

It is also true that

$$25F_3F_7=21F_5^2$$

which, since it involves *odd* powers, does not conform to the pattern of Ramanujan's identities, but is

$$\begin{aligned}
25\{&(a+b+c)^3-(b+c+d)^3+(c+d+a)^3\\
&-(d+a+b)^3-(a-d)^3+(b-c)^3\}\\
\times\{&(a+b+c)^7-(b+c+d)^7+(c+d+a)^7\\
&-(d+a+b)^7-(a-d)^7+(b-c)^7\}\\
=21\{&(a+b+c)^5-(b+c+d)^5+(c+d+a)^5\\
&-(d+a+b)^5-(a-d)^5+(b-c)^5\}^2.
\end{aligned}$$

We will prove all these identities. In doing so we can forgo the assumption $ad = bc$ and obtain generalisations of them all. Indeed we will show that with

$$s = 2a^2 + 2b^2 + 2c^2 + 2d^2 + 2ab + 2ac + 2bd + 2cd + ad + bc,$$

$$p = (a + b + c)(b + c + d)(a - d) \text{ and } P = (c + d + a)(d + a + b)(b - c)$$

the following hold:

$$F_2 = 6(bc - ad), \quad F_4 = 6(bc - ad)s, \quad 25F_3F_7 - 21F_5^2 = -4725(bc - ad)^2 pP.$$

We establish $(*)$ only for the case $ad = bc$, although more generally

$$\begin{aligned}
&64F_6F_{10} - 45F_8^2 \\
&\quad = 9(bc - ad)\left\{40(p^2 - P^2)s^4 + 15(s^6 + 24(p^2 + P^2)s^3 - 192p^2P^2)(bc - ad)\right. \\
&\qquad\qquad +1080(p^2 - P^2)s^2(bc - ad)^2 + 90s(5s^3 - 12(p^2 + P^2))(bc - ad)^3 \\
&\qquad\qquad \left. +3888(p^2 - P^2)(bc - ad)^4 + 567s^2(bc - ad)^5 + 2916(bc - ad)^7\right\}.
\end{aligned}$$

2. The Proofs. Let

$$\alpha = a + b + c, \ \beta = -b - c - d, \ \gamma = -a + d,$$

$$\alpha' = -c - d - a, \ \beta' = d + a + b, \ \gamma' = -b + c.$$

Then

$$\alpha + \beta + \gamma = 0, \quad \alpha' + \beta' + \gamma' = 0.$$

We find

$$\alpha^2 + \beta^2 + \gamma^2 = 2q, \quad {\alpha'}^2 + {\beta'}^2 + {\gamma'}^2 = 2Q$$

where

$$\begin{aligned}
q &= a^2 + b^2 + c^2 + d^2 + ab + ac + bd + cd + 2bc - ad, \\
Q &= a^2 + b^2 + c^2 + d^2 + ab + ac + bd + cd + 2ad - bc.
\end{aligned}$$

Note that

$$q - Q = 3(bc - ad),$$

so if $ad = bc$, $q = Q$.

Since $\alpha+\beta+\gamma=0$ and $\alpha^2+\beta^2+\gamma^2=2q$, we have

$$\alpha\beta+\beta\gamma+\gamma\alpha=-q.$$

Similarly,

$$\alpha'\beta'+\beta'\gamma'+\gamma'\alpha'=-Q.$$

Now define

$$p=\alpha\beta\gamma,\quad P=\alpha'\beta'\gamma'$$

and

$$F_n=\alpha^n+\beta^n+\gamma^n-\alpha'^n-\beta'^n-\gamma'^n.$$

Then

$$\begin{aligned}
\sum_{n\geq 0}F_nt^n &= \left(\frac{1}{1-\alpha t}+\frac{1}{1-\beta t}+\frac{1}{1-\gamma t}\right)-\left(\frac{1}{1-\alpha' t}+\frac{1}{1-\beta' t}+\frac{1}{1-\gamma' t}\right)\\
&= \frac{3-2(\alpha+\beta+\gamma)t+(\alpha\beta+\beta\gamma+\gamma\alpha)t^2}{1-(\alpha+\beta+\gamma)t+(\alpha\beta+\beta\gamma+\gamma\alpha)t^2-\alpha\beta\gamma t^3}\\
&\quad -\frac{3-2(\alpha'+\beta'+\gamma')t+(\alpha'\beta'+\beta'\gamma'+\gamma'\alpha')t^2}{1-(\alpha'+\beta'+\gamma')t+(\alpha'\beta'+\beta'\gamma'+\gamma'\alpha')t^2-\alpha'\beta'\gamma' t^3}\\
&= \frac{3-qt^2}{1-qt^2-pt^3}-\frac{3-Qt^2}{1-Qt^2-Pt^3}\\
&= 2(q-Q)t^2+3(p-P)t^3+2(q^2-Q^2)t^4+5(pq-PQ)t^5\\
&\quad +\{3(p^2-P^2)+2(q^3-Q^3)\}t^6+7(pq^2-PQ^2)t^7\\
&\quad +\{8(p^2q-P^2Q)+2(q^4-Q^4)\}t^8+\{3(p^3-P^3)+9(pq^3-PQ^3)\}t^9\\
&\quad +\{15(p^2q^2-P^2Q^2)+2(q^5-Q^5)\}t^{10}+\ \cdots .
\end{aligned}$$

Thus we have

$$\begin{aligned}
F_2&=2(q-Q),\quad F_3=3(p-P),\ F_4=2(q^2-Q^2),\\
F_5&=5(pq-PQ),\quad F_6=3(p^2-P^2)+2(q^3-Q^3),\\
F_7&=7(pq^2-PQ^2),\quad F_8=8(p^2q-P^2Q)+2(q^4-Q^4),\\
F_9&=3(p^3-P^3)+9(pq^3-PQ^3),\\
F_{10}&=15(p^2q^2-P^2Q^2)+2(q^5-Q^5)
\end{aligned}$$

and so on.

In particular

$$F_2 = 2(q - Q) = 6(bc - ad),$$

and

$$F_4 = 2(q - Q)(q + Q) = 6(bc - ad)s.$$

If $ad = bc$, $q = Q$,

$$F_3 = 3(p - P), \quad F_5 = 5q(p - P), \quad F_7 = 7q^2(p - P), \quad \text{and} \quad 25F_3F_7 = 21F_5^2.$$

More generally,

$$25F_3F_7 - 21F_5^2 = -525pP(q - Q)^2 = -4725(bc - ad)^2pP.$$

Again, if $ad = bc$,

$$F_6 = 3(p^2 - P^2),\ F_8 = 8q(p^2 - P^2),\ F_{10} = 15q^2(p^2 - P^2) \text{ and } 64F_6F_{10} = 45F_8^2,$$

which is $(*)$. The author has further established the last equation of Section 1, but the somewhat lengthy calculations seemed best omitted in a short note.

REFERENCES

[1] B. C. Berndt, *Ramanujan's Notebooks*, Part IV, Springer-Verlag,New York, 1994.

[2] B. C. Berndt and S. Bhargava, A remarkable identity found in Ramanujan's third notebook, *Glasgow Math. J.* 34 (1992), 341–345.

[3] T. S. Nanjundiah, A note on an identity of Ramanujan, *Amer. Math. Monthly* 100 (1993), 485–487.

[4] S. Ramanujan, *Notebooks*, Tata Institute of Fundamental Research, Bombay, 1957, Vol. 2.