HOW MANY WAYS CAN A KING CROSS THE BOARD?

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In the 1999 UNSW School Mathematics Competition, candidates were given the following problem.

On a 6×6 chessboard there is a chesspiece (a king) placed in the lower left hand corner. In how many different ways can he reach the top right hand corner if he is never allowed to move "backwards", that is, he can move to a neighbouring square either upwards or to the right or diagonally upwards to the right?

We can easily solve this problem, as follows. Put a number in each square of the chessboard equal to the number of ways of getting to that square. It is equal to the sum of the numbers in the squares to the left, below, and diagonally below to the left. Start by putting 1 in the bottom left square. In this way we find that the number of routes to the top right square is 1683.

More generally, we can solve the problem on an $(n+1) \times (n+1)$ board, as follows. (Note that if the king is only allowed to move upwards or right, the solution is $\binom{2n}{n}$). Let $r_{x,y}$ be the number of routes to the square (x, y), and set

$$F(X,Y) = \sum_{x,y \geq 1} r_{x,y} X^x Y^y.$$

Then

$$F(X,Y) = XY + (X + Y + XY)F(X,Y),$$

so

$$\begin{aligned} F(X,Y) &= \frac{XY}{1 - X - Y - XY} \\ &= \frac{XY}{(1-Y) - (1+Y)X} \end{aligned}$$

$$
\begin{aligned}
&= \frac{XY}{1-Y} \cdot \frac{1}{1 - \dfrac{1+Y}{1-Y} X} \\
&= \frac{XY}{1-Y} \sum_{n \geq 0} \left(\frac{1+Y}{1-Y} \right)^n X^n \\
&= \sum_{n \geq 0} X^{n+1} Y (1+Y)^n (1-Y)^{-(n+1)} \\
&= \sum_{n \geq 0} X^{n+1} Y \sum_{l \geq 0} \binom{n}{l} Y^l \sum_{m \geq 0} \binom{n+m}{n} Y^m \\
&= \sum_{n \geq 0} X^{n+1} \sum_{l,m \geq 0} \binom{n}{l} \binom{n+m}{n} Y^{l+m+1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
r_{n+1,n+1} &= \sum_{l+m=n} \binom{n}{l} \binom{n+m}{n} \\
&= \sum_{l=0}^{n} \binom{n}{l} \binom{2n-l}{n} \qquad (1)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n \geq 0} r_{n+1,n+1} q^n &= \sum_{n \geq 0} \sum_{l=0}^{n} \binom{n}{l} \binom{2n-l}{n} q^n \\
&= \sum_{l \geq 0} \sum_{n \geq l} \binom{n}{l} \binom{2n-l}{n} q^n \\
&= \sum_{l \geq 0} \sum_{m \geq 0} \binom{l+m}{l} \binom{l+2m}{l+m} q^{l+m} \\
&= \sum_{l,m \geq 0} \frac{(l+2m)!}{l!m!m!} q^{l+m} \\
&= \sum_{l,m \geq 0} \binom{2m}{m} \binom{2m+l}{2m} q^{l+m} \\
&= \sum_{m \geq 0} \binom{2m}{m} q^m (1-q)^{-(2m+1)}
\end{aligned}
$$

$$
\begin{aligned}
&= (1-q)^{-1} \sum_{m\ge 0} \binom{2m}{m} \left(\frac{q}{(1-q)^2}\right)^m \\
&= (1-q)^{-1} \left(1 - \frac{4q}{(1-q)^2}\right)^{-\frac{1}{2}} \\
&= \Big((1-q)^2 - 4q\Big)^{-\frac{1}{2}} \\
&= (1 - 6q + q^2)^{-\frac{1}{2}}. \qquad (2)
\end{aligned}
$$

(2) can be expanded to give a large number of the $r_{n+1,n+1}$. Thus

$$
\begin{aligned}
&\sum_{n\ge 0} r_{n+1,n+1} q^n = (1 - 6q + q^2)^{-\frac{1}{2}} \\
&= 1 + 3q + 13q^2 + 63q^3 + 321q^4 + 1683q^5 + 8989q^6 + 48639q^7 + 265729q^8 + \ \cdots \ .
\end{aligned}
$$

From (2) we can obtain an asymptotic formula for $r_{n+1,n+1}$, as follows.
With $\alpha = 3 + \sqrt{8} = (\sqrt{2}+1)^2$, $\beta = 3 - \sqrt{8} = (\sqrt{2}-1)^2$,

$$
\begin{aligned}
&\sum_{n\ge 0} r_{n+1,n+1} q^n = (1-\alpha q)^{-\frac{1}{2}} (1-\beta q)^{-\frac{1}{2}} \\
&= \sum_{l\ge 0} \binom{2l}{l} \left(\frac{\alpha}{4}\right)^l q^l \sum_{m\ge 0} \binom{2m}{m} \left(\frac{\beta}{4}\right)^m q^m.
\end{aligned}
$$

It follows that (we make use of the fact that $\alpha >> \beta$)

$$
\begin{aligned}
r_{n+1,n+1} &= \sum_{m=0}^{n} \binom{2n-2m}{n-m} \binom{2m}{m} \left(\frac{\alpha}{4}\right)^{n-m} \left(\frac{\beta}{4}\right)^m \\
&= \binom{2n}{n} \left(\frac{\alpha}{4}\right)^n \sum_{m=0}^{n} \binom{2m}{m} \binom{2n-2m}{n-m} \Big/ \binom{2n}{n} \left(\frac{1}{\alpha^2}\right)^m \\
&= \binom{2n}{n} \left(\frac{\alpha}{4}\right)^n {}_2F_1\left(\begin{matrix} \frac{1}{2}, -n \\ -n + \frac{1}{2} \end{matrix} ; \frac{1}{\alpha^2} \right). \qquad (3)
\end{aligned}
$$

Now, it can be shown that

$$
{}_2F_1\left(\begin{matrix} \frac{1}{2}, -n \\ -n + \frac{1}{2} \end{matrix} ; x \right) = (1-x)^{-\frac{1}{2}} + \frac{x}{4}(1-x)^{-\frac{3}{2}} \cdot \frac{1}{n} + O\Big(\frac{1}{n^2}\Big) \text{ as } n \to \infty. \qquad (4)
$$

It follows that

$$r_{n+1,n+1} = \binom{2n}{n} \frac{1}{2^{\frac{1}{4}}} \left(\frac{\sqrt{2}+1}{2} \right)^{2n+1} \left(1 + \frac{3\sqrt{2}-4}{32} \cdot \frac{1}{n} + O\Big(\frac{1}{n^2}\Big) \right). \tag{5}$$

From (2) we can get a recurrence for the $r_{n,n}$ as follows. If we let

$$f(q) = (1-6q+q^2)^{-\frac{1}{2}}$$

then

$$(1-6q+q^2) f'(q) = (3-q) f(q).$$

That is,

$$(1-6q+q^2) \sum_{n \geq 0} (n+1) r_{n+2,n+2} q^n = (3-q) \sum_{n \geq 0} r_{n+1,n+1} q^n.$$

It follows that

$$(n+1) r_{n+2,n+2} - 6n r_{n+1,n+1} + (n-1) r_{n,n} = 3 r_{n+1,n+1} - r_{n,n},$$

$$(n+1) r_{n+2,n+2} = (6n+3) r_{n+1,n+1} - n r_{n,n},$$

or, finally,

$$r_{n+2,n+2} = \frac{6n+3}{n+1} r_{n+1,n+1} - \frac{n}{n+1} r_{n,n}. \tag{6}$$

Together with $r_{1,1} = 1$, $r_{2,2} = 3$, we obtain $r_{3,3} = 13$, $r_{4,4} = 63$ and so on. It is remarkable on the face of it that the recurrence (6) yields integers.

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