

# WINQUIST AND THE ATKIN–SWINNERTON–DYER PARTITION CONGRUENCES FOR MODULUS 11

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## Abstract

We derive the Atkin–Swinerton–Dyer partition congruences for the modulus 11 by use of Winquist’s identity.

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## 1. Introduction

In 1969 Winquist [5] published a new identity and made one dramatic application of it — he gave a new and simple proof of Ramanujan’s partition congruence for the modulus 11, namely

$$p(11n + 6) \equiv 0 \pmod{11}.$$

( $p(n)$  denotes the number of partitions of  $n$ .)

Proofs of this congruence had previously been given by Ramanujan [4] and Atkin and Swinnerton–Dyer [1]. For a more recent, completely different, elementary proof, see [3].

In their paper, Atkin and Swinnerton–Dyer proved the startling fact that for the three values of  $m$ ,  $m = 5, 7, 11$  and every value of  $r$ ,  $r = 0, 1, \dots, m - 1$  the generating function

$$\sum_{n \geq 0} p(mn + r)q^n$$

is congruent modulo  $m$  to a simple infinite product.

The object of this note is to make use of Winquist’s identity to prove the full set of eleven Atkin–Swinerton–Dyer congruences for the modulus 11.

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If we employ the standard notation

$$\begin{aligned} (a; q)_\infty &= (1-a)(1-aq)(1-aq^2) \cdots, \\ (a_1, a_2, \dots, a_n; q)_\infty &= (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty, \\ (q)_\infty &= (q; q)_\infty, \end{aligned}$$

we can write the eleven congruences we are going to prove as one, namely, modulo 11,

$$\begin{aligned} \sum_{n \geq 0} p(n)q^n &\equiv (q^{22}, q^{33}, q^{44}, q^{55}, q^{66}, q^{77}, q^{88}, q^{99}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ q(q^{11}, q^{44}, q^{55}, q^{55}, q^{66}, q^{66}, q^{77}, q^{110}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ 2q^2(q^{22}, q^{33}, q^{33}, q^{55}, q^{66}, q^{88}, q^{88}, q^{99}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ 3q^3(q^{22}, q^{22}, q^{44}, q^{55}, q^{66}, q^{77}, q^{99}, q^{99}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ 5q^4(q^{11}, q^{33}, q^{44}, q^{55}, q^{66}, q^{77}, q^{88}, q^{110}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ 7q^5(q^{11}, q^{33}, q^{44}, q^{44}, q^{77}, q^{77}, q^{88}, q^{110}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ 4q^7(q^{11}, q^{22}, q^{44}, q^{55}, q^{66}, q^{77}, q^{99}, q^{110}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ 6q^{19}(q^{11}, q^{11}, q^{22}, q^{33}, q^{88}, q^{99}, q^{110}, q^{110}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ 8q^9(q^{11}, q^{22}, q^{33}, q^{55}, q^{66}, q^{88}, q^{99}, q^{110}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty \\ &+ 9q^{10}(q^{11}, q^{22}, q^{33}, q^{44}, q^{77}, q^{88}, q^{99}, q^{110}, q^{121}, q^{121}, q^{121})_\infty / (q^{11})_\infty. \end{aligned} \tag{1}$$

(Note that there are no powers of  $q$  congruent to 6 modulo 11 on the right of (1), which proves Ramanujan's congruence.)

## 2. Proof of the congruence (1)

We shall require Winquist's identity, in the form [2,5]

$$\begin{aligned} (a, a^{-1}q, b, b^{-1}q, ab, a^{-1}b^{-1}q, ab^{-1}, a^{-1}bq, q, q; q)_\infty \\ = (a^3, a^{-3}q^3, q^3; q^3)_\infty \{ (b^3q, b^{-3}q^2, q^3; q^3)_\infty - b(b^3q^2, b^{-3}q, q^3; q^3)_\infty \} \\ - ab^{-1}(b^3, b^{-3}q^3, q^3; q^3)_\infty \{ (a^3q, a^{-3}q^2, q^3; q^3)_\infty - a(a^3q^2, a^{-3}q, q^3; q^3)_\infty \} \end{aligned} \tag{2}$$

as well as the identity which can be deduced from (2) [2,5]

$$48(q)_\infty^{10} = \sum_{m, n = -\infty}^{\infty} (-1)^{m+n} ((6m+3)^3(6n+1) - (6m+3)(6n+1)^3) q^{(3m^2+3m+3n^2+n)/2}. \tag{3}$$

Modulo 11, (3) becomes

$$(q)_\infty^{10} \equiv 3 \left( \sum_{-\infty}^{\infty} (-1)^n (6n+3)^3 q^{(3n^2+3n)/2} \sum_{-\infty}^{\infty} (-1)^n (6n+1) q^{(3n^2+n)/2} - \sum_{-\infty}^{\infty} (-1)^n (6n+3) q^{(3n^2+3n)/2} \sum_{-\infty}^{\infty} (-1)^n (6n+1)^3 q^{(3n^2+n)/2} \right). \quad (4)$$

It is easy to show by considering  $n \equiv -5, -4, \dots, 5 \pmod{11}$  that

$$\sum_{-\infty}^{\infty} (-1)^n (6n+3) q^{(3n^2+3n)/2} \equiv -5P_3 + 4q^3 P_9 - 3q^9 P_{15} + 2q^{18} P_{21} - q^{30} P_{27}, \quad (5)$$

$$\sum_{-\infty}^{\infty} (-1)^n (6n+3)^3 q^{(3n^2+3n)/2} \equiv -P_3 + 5q^3 P_9 - 4q^9 P_{15} + 2q^{18} P_{21} - 3q^{30} P_{27}, \quad (6)$$

$$\begin{aligned} \sum_{-\infty}^{\infty} (-1)^n (6n+1) q^{(3n^2+n)/2} &\equiv P_1 + 5q P_5 + 4q^2 P_7 + 2q^7 P_{13} - 5q^{12} P_{17} + 3q^{15} P_{19} \\ &\quad - q^{22} P_{23} + 3q^{26} P_{25} - 4q^{35} P_{29} + 2q^{40} P_{31} \end{aligned} \quad (7)$$

and

$$\begin{aligned} \sum_{-\infty}^{\infty} (-1)^n (6n+1)^3 q^{(3n^2+n)/2} &\equiv P_1 + 4q P_5 - 2q^2 P_7 - 3q^7 P_{13} - 4q^{12} P_{17} + 5q^{15} P_{19} \\ &\quad - q^{22} P_{23} + 5q^{26} P_{25} + 2q^{35} P_{29} - 3q^{40} P_{31} \end{aligned} \quad (8)$$

where

$$P_k = \sum_{-\infty}^{\infty} (-1)^n q^{(363n^2+11kn)/2} = (q^{(363-11k)/2}, q^{(363+11k)/2}, q^{363}; q^{363})_\infty.$$

If we substitute (5), (6), (7) and (8) into (4) and rearrange, we obtain

$$\begin{aligned} (q)_\infty^{10} &\equiv (P_1 P_3 - q^{22} P_3 P_{23} - q^{33} P_{19} P_{21} - q^{44} P_{21} P_{25}) \\ &\quad + q(P_3 P_5 - q^{11} P_3 P_{17} - q^{44} P_{19} P_{27} - q^{55} P_{25} P_{27}) \\ &\quad + 2q^2(P_3 P_7 - q^{22} P_{15} P_{19} - q^{33} P_{15} P_{25} - q^{33} P_3 P_{29}) \\ &\quad + 3q^3(P_1 P_9 - q^{22} P_9 P_{23} - q^{22} P_{13} P_{21} - q^{55} P_{21} P_{31}) \end{aligned} \quad (9)$$

$$\begin{aligned}
& + 5q^4(P_5P_9 - q^{11}P_9P_{17} - q^{33}P_{13}P_{27} - q^{66}P_{27}P_{31}) \\
& + 7q^5(P_7P_9 - q^{11}P_{13}P_{15} - q^{33}P_9P_{29} - q^{44}P_{15}P_{31}) \\
& + 4q^7(P_3P_{13} - q^{11}P_9P_{19} - q^{22}P_9P_{25} + q^{33}P_3P_{31}) \\
& + 6q^{19}(P_5P_{21} - q^{11}P_1P_{27} - q^{11}P_{17}P_{21} + q^{33}P_{23}P_{27}) \\
& + 8q^9(P_1P_{15} - q^{11}P_7P_{21} - q^{22}P_{15}P_{23} + q^{44}P_{21}P_{29}) \\
& + 9q^{10}(P_5P_{15} - q^{11}P_{15}P_{17} - q^{22}P_7P_{27} + q^{55}P_{27}P_{29}).
\end{aligned}$$

Every bracketed term on the right of (9) can, remarkably, be expressed as a single product.

For example, the first term is

$$\begin{aligned}
& P_1P_3 - q^{22}P_3P_{23} - q^{33}P_{19}P_{21} - q^{44}P_{21}P_{25} \\
& = P_3(P_1 - q^{22}P_{23}) - q^{33}P_{21}(P_{19} + q^{11}P_{25}) \\
& = (q^{165}, q^{198}, q^{363}; q^{363})_\infty \\
& \quad \times ((q^{176}, q^{187}, q^{363}; q^{363})_\infty - q^{22}(q^{55}, q^{308}, q^{363}; q^{363})_\infty) \\
& - q^{33}(q^{66}, q^{297}, q^{363}; q^{363})_\infty \\
& \quad \times ((q^{77}, q^{286}, q^{363}; q^{363})_\infty + q^{11}(q^{44}, q^{319}, q^{363}; q^{363})_\infty) \\
& = (q^{22}, q^{33}, q^{44}, q^{55}, q^{66}, q^{77}, q^{88}, q^{99}, q^{121}, q^{121}; q^{121})_\infty,
\end{aligned}$$

where we have used (2) with  $q$  replaced by  $q^{121}$  and  $a = q^{55}$ ,  $b = q^{22}$ .

For the remaining terms, the relevant substitutions in (2) (after replacing  $q$  by  $q^{121}$ ) are respectively  $(a, b) =$

$$(q^{55}, q^{11}), (q^{55}, q^{33}), (q^{44}, q^{22}), (q^{44}, q^{11}), (q^{44}, q^{33}), (q^{55}, q^{44}), (q^{22}, q^{11}), (q^{33}, q^{22}), (q^{33}, q^{11}).$$

(1) then follows from the fact that

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(q)_\infty} = \frac{(q)_\infty^{10}}{(q)_\infty^{11}} \equiv \frac{(q)_\infty^{10}}{(q^{11})_\infty}. \quad \blacksquare$$

## References

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