

PARITY RESULTS FOR CERTAIN PARTITION FUNCTIONS

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ABSTRACT. Blecksmith, Brillhart and Gerst proved four congruences modulo 2 involving partition generating functions of the following sort.

$$1 / \prod_{n \in S} (1 - q^n) \equiv 1 + \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{3n^2} \pmod{2}$$

where $S = \{n > 0 : n \equiv \pm(1, 2, 3, 4) \text{ or } 6 \pmod{12}\}$.

We give simple and uniform proofs of their congruences and of several others of the same sort.

Each of these congruences yields a theorem on partitions. Thus the above congruence says that

The number of partitions of n into parts not congruent to 0 or $\pm 5 \pmod{12}$ is odd if and only if n is a square or three times a square.

1. Introduction

In an investigation of the parity of $p(n)$, Blecksmith et al.[1] were led to discover congruences modulo 2 involving certain partition generating functions. They found that [1, Theorem 2a]

$$1 / \prod_{n \in S_1} (1 - q^n) \equiv 1 + \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{2n^2} \pmod{2} \tag{1}$$

where $S_1 = \{n > 0 : n \equiv \pm(1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13 \text{ or } 15) \pmod{32}\}$, [1, Theorem 2b]

$$1 / \prod_{n \in S_2} (1 - q^n) \equiv \sum_{n \geq 1} q^{n^2-1} + \sum_{n \geq 1} q^{2n^2-1} \pmod{2} \tag{2}$$

where $S_2 = \{n > 0 : n \equiv \pm(1, 2, 3, 5, 7, 8, 9, 11, 12, 13, 14 \text{ or } 15) \pmod{32}\}$,

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[1, Theorem 4a]

$$1 / \prod_{n \in S_3} (1 - q^n) \equiv 1 + \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{3n^2} \pmod{2} \quad (3)$$

where $S_3 = \{n > 0 : n \equiv \pm(1, 2, 3, 4) \text{ or } 6 \pmod{12}\}$

and

[1, Theorem 4b]

$$1 / \prod_{n \in S_4} (1 - q^n) \equiv \sum_{n \geq 1} q^{n^2-1} + \sum_{n \geq 1} q^{3n^2-1} \pmod{2} \quad (4)$$

where $S_4 = \{n > 0 : n \equiv \pm(2, 3, 4, 5) \text{ or } 6 \pmod{12}\}$.

We give simple and uniform proofs of these and of another six results of the same sort, namely

$$1 / \prod_{n \in S_5} (1 - q^n) \equiv 1 + \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{5n^2} \pmod{2} \quad (5)$$

where $S_5 = \{n > 0 : n \equiv \pm(1, 2, 8, 9) \text{ or } 10 \pmod{20}\}$,

$$1 / \prod_{n \in S_6} (1 - q^n) \equiv \sum_{n \geq 1} q^{n^2-1} + \sum_{n \geq 1} q^{5n^2-1} \pmod{2} \quad (6)$$

where $S_6 = \{n > 0 : n \equiv \pm(3, 4, 6, 7) \text{ or } 10 \pmod{20}\}$,

$$1 / \prod_{n \in S_7} (1 - q^n) \equiv 1 + \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{2n^2} + \sum_{n \geq 1} q^{3n^2} + \sum_{n \geq 1} q^{6n^2} \pmod{2} \quad (7)$$

where $S_7 = \{n > 0 : n \equiv \pm(1, 5, 6, 7, 8, 11, 12, 13, 17, 18, 19, 23) \pmod{48}\}$,

$$1 / \prod_{n \in S_8} (1 - q^n) \equiv \sum_{n \geq 1} q^{n^2-1} + \sum_{n \geq 1} q^{2n^2-1} + \sum_{n \geq 1} q^{3n^2-1} + \sum_{n \geq 1} q^{6n^2-1} \pmod{2} \quad (8)$$

where $S_8 = \{n > 0 : n \equiv \pm(1, 4, 5, 6, 7, 8, 11) \text{ or } 12 \pmod{24}\}$,

$$1 / \prod_{n \in S_9} (1 - q^n) \equiv 1 + \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{2n^2} + \sum_{n \geq 1} q^{5n^2} + \sum_{n \geq 1} q^{10n^2} \pmod{2} \quad (9)$$

where $S_9 = \{n > 0 : n \equiv \pm(1, 3, 4, 7, 9, 11, 12, 13, 17, 19) \text{ or } 20 \pmod{40}\}$,

and

$$1 / \prod_{n \in S_{10}} (1 - q^n) \equiv \sum_{n \geq 1} q^{n^2-1} + \sum_{n \geq 1} q^{2n^2-1} + \sum_{n \geq 1} q^{5n^2-1} + \sum_{n \geq 1} q^{10n^2-1} \pmod{2} \quad (10)$$

where $S_{10} = \{n > 0 : n \equiv \pm(1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19) \text{ or } 20 \pmod{40}\}$. We also give a further three results of the same sort which follow from those already stated, namely

$$1 / \prod_{n \in S_{11}} (1 - q^n) \equiv \sum_{n \geq 1} q^{(n^2-n)/2} \pmod{2} \quad (11)$$

where $S_{11} = \{n > 0 : n \equiv \pm 1 \text{ or } 2 \pmod{4}\}$ (this is also easily proved directly),

$$1 / \prod_{n \in S_{12}} (1 - q^n) \equiv \sum_{n \geq 1} q^{2n^2-2n} + \sum_{n \geq 1} q^{6n^2-6n+1} \pmod{2} \quad (12)$$

where $S_{12} = \{n > 0 : n \equiv \pm(1, 2, 5, 6, 7, 10, 11) \text{ or } 12 \pmod{24}\}$

and

$$1 / \prod_{n \in S_{13}} (1 - q^n) \equiv \sum_{n \geq 1} q^{n^2-n} + \sum_{n \geq 1} q^{5n^2-5n+1} \pmod{2} \quad (13)$$

where $S_{13} = \{n > 0 : n \equiv \pm(1, 3, 7, 9) \text{ or } 10 \pmod{20}\}$.

Each of (1)–(13) has a corresponding interpretation in terms of partitions. Thus, for example, (3) says that

The number of partitions of n into parts $\pm(1, 2, 3, 4)$ or $6 \pmod{12}$ is odd unless n is a square or three times a square.

Our main tool in proving the results (1)–(10) is a pair of identities given in Lemma 1, which we prove in Section 3.

Lemma 1

$$\begin{aligned} & \frac{1}{2} \left\{ \frac{(-aq; q^2)_\infty (-a^{-1}q; q^2)_\infty}{(aq; q^2)_\infty (a^{-1}q; q^2)_\infty} + \frac{(-bq; q^2)_\infty (-b^{-1}q; q^2)_\infty}{(bq; q^2)_\infty (b^{-1}q; q^2)_\infty} \right\} \\ &= \frac{(abq^2; q^4)_\infty (a^{-1}b^{-1}q^2; q^4)_\infty (ab^{-1}q^2; q^4)_\infty (a^{-1}bq^2; q^4)_\infty (q^4; q^4)_\infty^2}{(aq; q^2)_\infty (a^{-1}q; q^2)_\infty (bq; q^2)_\infty (b^{-1}q; q^2)_\infty (q^2; q^2)_\infty^2} \end{aligned}$$

and

$$\frac{1}{2} \left\{ \frac{(-aq; q^2)_\infty (-a^{-1}q; q^2)_\infty}{(aq; q^2)_\infty (a^{-1}q; q^2)_\infty} - \frac{(-bq; q^2)_\infty (-b^{-1}q; q^2)_\infty}{(bq; q^2)_\infty (b^{-1}q; q^2)_\infty} \right\}$$

$$= a^{-1}q \frac{(ab; q^4)_\infty (a^{-1}b^{-1}q^4; q^4)_\infty (ab^{-1}; q^4)_\infty (a^{-1}bq^4; q^4)_\infty (q^4; q^4)_\infty^2}{(aq; q^2)_\infty (a^{-1}q; q^2)_\infty (bq; q^2)_\infty (b^{-1}q; q^2)_\infty (q^2; q^2)_\infty^2}.$$

We also require the fact that, modulo 2,

$$\sum_{n \geq 1} q^{n^2} \equiv \sum_{n \geq 1} d(n)q^n = \sum_{n \geq 1} \frac{q^n}{1 - q^n}$$

where $d(n)$ is the number of divisors of n ,

and we use without comment the fact that, modulo 2,

$$(q; q)_\infty (q; q^2)_\infty \equiv (-q; q^2)_\infty (q; q^2)_\infty = \frac{(q^2; q^2)_\infty}{(q; q)_\infty} (q; q^2)_\infty = 1.$$

2. Proofs of the congruences

We shall prove only a few of our results. Proofs of the remaining results differ only slightly.

Proof of (1) Modulo 2,

$$\begin{aligned} & 1 + \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{2n^2} \\ & \equiv 1 + \sum_{n \geq 1} \frac{q^n}{1 - q^n} + \sum_{n \geq 1} \frac{q^{2n}}{1 - q^{2n}} \\ & \equiv 1 + \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{q^n}{1 - q^n} \\ & = 1 + \sum_{n \geq 0} \left(\frac{q^{8n+1}}{1 - q^{8n+1}} + \frac{q^{8n+3}}{1 - q^{8n+3}} + \frac{q^{8n+5}}{1 - q^{8n+5}} + \frac{q^{8n+7}}{1 - q^{8n+7}} \right) \\ & = \frac{1}{2} \left\{ 1 + 2 \sum_{n \geq 0} \left(\frac{q^{8n+1}}{1 - q^{8n+1}} + \frac{q^{8n+7}}{1 - q^{8n+7}} \right) + 1 + 2 \sum_{n \geq 0} \left(\frac{q^{8n+3}}{1 - q^{8n+3}} + \frac{q^{8n+5}}{1 - q^{8n+5}} \right) \right\} \\ & \equiv \frac{1}{2} \left\{ \prod_{n \geq 0} \left(1 + \frac{2q^{8n+1}}{1 - q^{8n+1}} \right) \left(1 + \frac{2q^{8n+7}}{1 - q^{8n+7}} \right) + \prod_{n \geq 0} \left(1 + \frac{2q^{8n+3}}{1 - q^{8n+3}} \right) \left(1 + \frac{2q^{8n+5}}{1 - q^{8n+5}} \right) \right\} \\ & = \frac{1}{2} \left\{ \frac{(-q; q^8)_\infty (-q^7; q^8)_\infty}{(q; q^8)_\infty (q^7; q^8)_\infty} + \frac{(-q^3; q^8)_\infty (-q^5; q^8)_\infty}{(q^3; q^8)_\infty (q^5; q^8)_\infty} \right\} \end{aligned}$$

$$= \frac{(q^4; q^{16})_\infty (q^6; q^{16})_\infty (q^{10}; q^{16})_\infty (q^{12}; q^{16})_\infty (q^{16}; q^{16})_\infty^2}{(q; q^8)_\infty (q^3; q^8)_\infty (q^5; q^8)_\infty (q^7; q^8)_\infty (q^8; q^8)_\infty^2}$$

by Lemma 1 with q^4 for q , $a = q^3$, $b = q$.

$$\begin{aligned} &= \frac{(q^2; q^2)_\infty (q^{16}; q^{16})_\infty}{(q; q^2)_\infty (q^2; q^{16})_\infty (q^8; q^{16})_\infty (q^{14}; q^{16})_\infty (q^8; q^8)_\infty^2} \\ &\equiv 1/\{(q; q^2)_\infty (q^2; q^{16})_\infty (q^8; q^{16})_\infty (q^{14}; q^{16})_\infty (q^2; q^4)_\infty\} \\ &= 1/\{(q; q^2)_\infty (q^2; q^{16})_\infty^2 (q^6; q^{16})_\infty (q^8; q^{16})_\infty (q^8; q^{16})_\infty (q^{10}; q^{16})_\infty (q^{14}; q^{16})_\infty^2\} \\ &\equiv 1/\{(q; q^2)_\infty (q^4; q^{32})_\infty (q^6; q^{16})_\infty (q^8; q^{16})_\infty (q^{10}; q^{16})_\infty (q^{28}; q^{32})_\infty\} \\ &= 1/\prod_{n \in S_1} (1 - q^n) \end{aligned}$$

where $S_1 = \{n > 0 : n \equiv \pm(1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13 \text{ or } 15) \pmod{32}\}$, as claimed.

Proof of (8) Modulo 2,

$$\begin{aligned} &\sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{2n^2} + \sum_{n \geq 1} q^{3n^2} + \sum_{n \geq 1} q^{6n^2} \\ &\equiv \sum_{n \geq 1} \frac{q^n}{1 - q^n} + \sum_{n \geq 1} \frac{q^{2n}}{1 - q^{2n}} + \sum_{n \geq 1} \frac{q^{3n}}{1 - q^{3n}} + \sum_{n \geq 1} \frac{q^{6n}}{1 - q^{6n}} \\ &= \sum_{\substack{n > 0 \\ n \equiv 0, \pm 1, \pm 2, 3 \pmod{6}}} \frac{q^n}{1 - q^n} + \sum_{\substack{n > 0 \\ n \equiv 0, \pm 2 \pmod{6}}} \frac{q^n}{1 - q^n} + \sum_{\substack{n > 0 \\ n \equiv 0, \pm 3 \pmod{6}}} \frac{q^n}{1 - q^n} \\ &\quad + \sum_{\substack{n > 0 \\ n \equiv 0 \pmod{6}}} \frac{q^n}{1 - q^n} \\ &\equiv \sum_{\substack{n > 0 \\ n \equiv \pm 1 \pmod{6}}} \frac{q^n}{1 - q^n} \\ &= \sum_{n \geq 0} \left(\frac{q^{12n+1}}{1 - q^{12n+1}} + \frac{q^{12n+5}}{1 - q^{12n+5}} + \frac{q^{12n+7}}{1 - q^{12n+7}} + \frac{q^{12n+11}}{1 - q^{12n+11}} \right) \\ &\equiv \frac{1}{2} \left\{ 1 + 2 \sum_{n \geq 0} \left(\frac{q^{12n+1}}{1 - q^{12n+1}} + \frac{q^{12n+11}}{1 - q^{12n+11}} \right) - 1 - 2 \sum_{n \geq 0} \left(\frac{q^{12n+5}}{1 - q^{12n+5}} + \frac{q^{12n+7}}{1 - q^{12n+7}} \right) \right\} \\ &\equiv \frac{1}{2} \left\{ \prod_{n \geq 0} \left(1 + \frac{2q^{12n+1}}{1 - q^{12n+1}} \right) \left(1 + \frac{2q^{12n+11}}{1 - q^{12n+11}} \right) - \prod_{n \geq 0} \left(1 + \frac{2q^{12n+5}}{1 - q^{12n+5}} \right) \left(1 + \frac{2q^{12n+7}}{1 - q^{12n+7}} \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left\{ \frac{(-q; q^{12})_\infty (-q^{11}; q^{12})_\infty}{(q; q^{12})_\infty (q^{11}; q^{12})_\infty} - \frac{(-q^5; q^{12})_\infty (-q^7; q^{12})_\infty}{(q^5; q^{12})_\infty (q^7; q^{12})_\infty} \right\} \\
&= q \frac{(q^4; q^{24})_\infty (q^6; q^{24})_\infty (q^{18}; q^{24})_\infty (q^{20}; q^{24})_\infty (q^{24}; q^{24})_\infty^2}{(q; q^{12})_\infty (q^5; q^{12})_\infty (q^7; q^{12})_\infty (q^{11}; q^{12})_\infty (q^{12}; q^{12})_\infty^2} \\
&\equiv q \frac{(q^4; q^{24})_\infty (q^{20}; q^{24})_\infty (q^{24}; q^{24})_\infty}{(q; q^6)_\infty (q^5; q^6)_\infty (q^6; q^6)_\infty} \\
&= q \frac{(q^4; q^4)_\infty}{(q; q^6)_\infty (q^5; q^6)_\infty (q^6; q^6)_\infty (q^8; q^{24})_\infty (q^{12}; q^{24})_\infty (q^{16}; q^{24})_\infty} \\
&\equiv q \frac{(q^4; q^4)_\infty}{(q; q^6)_\infty (q^5; q^6)_\infty (q^6; q^{12})_\infty (q^8; q^{24})_\infty (q^{16}; q^{24})_\infty} \\
&\equiv q / \{(q; q^6)_\infty (q^5; q^6)_\infty (q^6; q^{12})_\infty (q^8; q^{24})_\infty (q^{16}; q^{24})_\infty (q^4; q^8)_\infty\} \\
&= q / \prod_{n \in S_8} (1 - q^n)
\end{aligned}$$

where $S_8 = \{n > 0 : n \equiv \pm(1, 4, 5, 6, 7, 8, 11), 12 \pmod{24}\}$.

Results (2)–(7),(9),(10) can similarly be proved using Lemma 1 and manipulating the infinite products which arise.

If we write (2)

$$q / \prod_{n \in S_2} (1 - q^n) \equiv \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{2n^2} \quad (2')$$

and multiply (1) and (2'), we obtain

$$\begin{aligned}
\sum_{\substack{n \geq 1 \\ n \text{ odd}}} q^{n^2} &\equiv \sum_{n \geq 1} q^{n^2} + \sum_{n \geq 1} q^{4n^2} \\
&\equiv q \frac{(q^2; q^2)_\infty (q^{16}; q^{32})_\infty (q^{32}; q^{32})_\infty}{(q; q)_\infty^2 (q^8; q^{32})_\infty (q^{24}; q^{32})_\infty} \\
&= q \frac{(q^2; q^2)_\infty (q^{16}; q^{16})_\infty}{(q; q)_\infty^2 (q^8; q^{16})_\infty} \\
&\equiv q / \{(q^8; q^{16})_\infty (q^{16}; q^{32})_\infty\} \\
&\equiv q / \prod_{\substack{n > 0 \\ n \equiv \pm 8, 16 \pmod{32}}} (1 - q^n).
\end{aligned}$$

If we now divide by q and replace q^8 by q , we obtain (11).

Similarly, (7) and (8) yield (12), (9) and (10) yield (13). For sake of completeness, we note that (3) and (4) yield (8), (5) and (6) yield (10).

3. Proof of Lemma 1

We shall first prove

Lemma 2

$$\begin{aligned} & (-aq; q^2)_\infty (-a^{-1}q; q^2)_\infty (-bq; q^2)_\infty (-b^{-1}q; q^2)_\infty (q^2; q^2)_\infty \\ &= (-abq^2; q^4)_\infty (-a^{-1}b^{-1}q^2; q^4)_\infty (-ab^{-1}q^2; q^4)_\infty (-a^{-1}bq^2; q^4)_\infty (q^4; q^4)_\infty^2 \\ &+ a^{-1}q(-ab; q^4)_\infty (-a^{-1}b^{-1}q^4; q^4)_\infty (-ab^{-1}; q^4)_\infty (-a^{-1}bq^4; q^4)_\infty (q^4; q^4)_\infty^2. \end{aligned}$$

If we put $-b$ for b , then $-a$ for a in Lemma 2 and add the two results, then divide by

$$2(aq; q^2)_\infty (a^{-1}q; q^2)_\infty (bq; q^2)_\infty (b^{-1}q; q^2)_\infty (q^2; q^2)_\infty^2,$$

we obtain the first identity of Lemma 1. If instead we subtract, we obtain the second identity of Lemma 1.

Proof of Lemma 2: We have

$$\begin{aligned} & (-a^2q^2; q^4)_\infty (-a^{-2}q^2; q^4)_\infty (-b^2q^2; q^4)_\infty (-b^{-2}q^2; q^4)_\infty (q^4; q^4)_\infty^2 \\ &= \sum_{r,s=-\infty}^{\infty} a^{2r}b^{2s}q^{2r^2+2s^2} \\ &= \sum_{r,s=-\infty}^{\infty} a^{2r}b^{2s}q^{(r+s)^2+(r-s)^2} \\ &= \sum_{\substack{u,v=-\infty \\ u \equiv v \pmod{2}}}^{\infty} a^{u+v}b^{u-v}q^{u^2+v^2} \\ &= \sum_{u,v \text{ even}} (ab)^u q^{u^2} (ab^{-1})^v q^{v^2} + \sum_{u,v \text{ odd}} (ab)^u q^{u^2} (ab^{-1})^v q^{v^2} \\ &= \sum_{m,n=-\infty}^{\infty} (ab)^{2m} q^{4m^2} (ab^{-1})^{2n} q^{4n^2} + \sum_{m,n=-\infty}^{\infty} (ab)^{2m-1} q^{4m^2-4m+1} (ab^{-1})^{2n-1} q^{4n^2-4n+1} \\ &= (-a^2b^2q^4; q^8)_\infty (-a^{-2}b^{-2}q^4; q^8)_\infty (-a^2b^{-2}q^4; q^8)_\infty (-a^{-2}b^2q^4; q^8)_\infty (q^8; q^8)_\infty^2 \\ &+ a^{-2}q^2(-a^2b^2; q^8)_\infty (-a^{-2}b^{-2}q^8; q^8)_\infty (-a^2b^{-2}; q^8)_\infty (-a^{-2}b^2q^8; q^8)_\infty (q^8; q^8)_\infty^2. \end{aligned}$$

Now put a for a^2 , b for b^2 and q for q^2 . ■

Reference

- [1] Richard Blecksmith, John Brillhart and Irving Gerst, Parity results for certain partition functions and identities similar to theta function identities, *Mathematics of Computation*, Vol. 48, No. 177 (1987), 29-38.