

ON SOME SUM–TO–PRODUCT IDENTITIES

SHAUN COOPER AND MICHAEL HIRSCHHORN

We give new proofs of some sum–to–product identities due to Blecksmith, Brillhart and Gerst, as well as some other such identities found recently by us.

1. INTRODUCTION AND STATEMENT OF RESULTS

In the first two of a sequence of papers, Blecksmith, Brillhart and Gerst [2,3] give five pairs of simple and beautiful sum–to–product identities, Theorems 2–6 below. We give another such in Theorem 1. We do several new things — we prove Theorem 1 (§2), we show that Theorem 2 follows from Theorem 1 (§3), we present new proofs of Theorems 3 and 4 (§4) and we show that Theorems 5 and 6 follow from Theorems 3 and 4 (§5).

In order to state our results, we must explain some by–now fairly standard notation.

$$\begin{aligned} (a; q)_\infty &= (1-a)(1-aq)(1-aq^2) \cdots, \\ (a_1, a_2, \dots, a_n; q)_\infty &= (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty, \\ \left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_1, b_2, \dots, b_n \end{matrix}; q \right)_\infty &= \frac{(a_1, a_2, \dots, a_m; q)_\infty}{(b_1, b_2, \dots, b_n; q)_\infty}, \\ (q)_\infty &= (q; q)_\infty. \end{aligned}$$

We make use of Jacobi’s triple product identity [1, (2.2.10)]

$$(aq, a^{-1}, q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{(n^2+n)/2}$$

and the quintuple product identity [4,5]

$$\left(\begin{matrix} a^2q, a^{-2}, q \\ aq, a^{-1} \end{matrix}; q \right)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n a^{3n} q^{(3n^2+n)/2} + \sum_{n=-\infty}^{\infty} (-1)^n a^{3n-1} q^{(3n^2-n)/2}.$$

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The theorems referred to above are

Theorem 1.

$$\begin{aligned}\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{2n^2} &= 2 \left(q^3, q^5, q^8; q^8 \right)_{\infty}, \\ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{2n^2} &= 2q \left(q, q^7, q^8; q^8 \right)_{\infty}.\end{aligned}$$

Theorem 2. [2, Theorem 1]

$$\begin{aligned}\sum_{-\infty}^{\infty} (-1)^n q^{2n^2} + \sum_{-\infty}^{\infty} (-1)^n q^{n^2} &= 2(q)_{\infty} / (q^4, q^6, q^8, q^{10}, q^{22}, q^{24}, q^{26}, q^{28}; q^{32})_{\infty}, \\ \sum_{-\infty}^{\infty} (-1)^n q^{2n^2} - \sum_{-\infty}^{\infty} (-1)^n q^{n^2} &= 2q(q)_{\infty} / (q^2, q^8, q^{12}, q^{14}, q^{18}, q^{20}, q^{24}, q^{30}; q^{32})_{\infty}.\end{aligned}$$

Theorem 3. [2, Theorem 3]

$$\begin{aligned}\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{3n^2} &= 2 \left(q^2, q^6, q^{10}, q^{12}; q^{12} \right)_{\infty}, \\ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{3n^2} &= 2q \left(q^2, q^6, q^{10}, q^{12}; q^{12} \right)_{\infty}.\end{aligned}$$

Theorem 4. [3, Theorem 3]

$$\begin{aligned}\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{5n^2} &= 2 \left(q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20}; q^{20} \right)_{\infty}, \\ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{5n^2} &= 2q \left(q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20}; q^{20} \right)_{\infty}.\end{aligned}$$

Theorem 5. [3, Theorem 1]

$$\begin{aligned}\sum_{-\infty}^{\infty} q^{2n(n+1)} + \sum_{-\infty}^{\infty} q^{6n(n+1)+1} &= 2 \left(q^2, q^5, q^7, q^{12}, q^{17}, q^{19}, q^{22}, q^{24}; q^{24} \right)_{\infty}, \\ \sum_{-\infty}^{\infty} q^{2n(n+1)} - \sum_{-\infty}^{\infty} q^{6n(n+1)+1} &= 2 \left(q, q^{10}, q^{11}, q^{12}, q^{13}, q^{14}, q^{23}, q^{24}; q^{24} \right)_{\infty}.\end{aligned}$$

and

Theorem 6. [3, Theorem 4]

$$\begin{aligned} \sum_{-\infty}^{\infty} q^{n(n+1)} + \sum_{-\infty}^{\infty} q^{5n(n+1)+1} &= 2 \left(\begin{matrix} q^3, q^7, q^{10}, q^{13}, q^{17}, q^{20} \\ q, q^6, q^9, q^{11}, q^{14}, q^{19} \end{matrix}; q^{20} \right)_{\infty}, \\ \sum_{-\infty}^{\infty} q^{n(n+1)} - \sum_{-\infty}^{\infty} q^{5n(n+1)+1} &= 2 \left(\begin{matrix} q, q^9, q^{10}, q^{11}, q^{19}, q^{20} \\ q^2, q^3, q^7, q^{13}, q^{17}, q^{18} \end{matrix}; q^{20} \right)_{\infty}. \end{aligned}$$

2. PROOF OF THEOREM 1

We shall start by obtaining the 2-dissections

$$(1) \quad (q^3, q^5, q^8; q^8)_{\infty}^2 = \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v},$$

$$(q, q^7, q^8; q^8)_{\infty}^2 = \sum_{u,v=-\infty}^{\infty} q^{8u^2-6u+8v^2} - q \sum_{u,v=-\infty}^{\infty} q^{8u^2+2u+8v^2+8v}.$$

We have

$$\begin{aligned} (q^3, q^5, q^8; q^8)_{\infty}^2 &= \left\{ \sum_{-\infty}^{\infty} (-1)^n q^{4n^2-n} \right\}^2 \\ &= \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{4m^2-m+4n^2-n} \\ &= \sum_{m \equiv n \pmod{2}} q^{4m^2-m+4n^2-n} - \sum_{m \not\equiv n \pmod{2}} q^{4m^2-m+4n^2-n} \\ &= \sum_{u,v=-\infty}^{\infty} q^{4(u+v)^2-(u+v)+4(u-v)^2-4(u-v)} \\ &\quad - \sum_{u,v=-\infty}^{\infty} q^{4(u+v+1)^2-4(u+v+1)+4(u-v)^2-(u-v)} \\ &= \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v} \end{aligned}$$

as claimed. The other identity is proved similarly.

It follows that

$$(2) \quad (q^3, q^5, q^8; q^8)_\infty^2 + q(q, q^7, q^8; q^8)_\infty^2 = \frac{(q^2)_\infty^4}{(q)_\infty (q^4)_\infty},$$

$$(q^3, q^5, q^8; q^8)_\infty^2 - q(q, q^7, q^8; q^8)_\infty^2 = \frac{(q)_\infty (q^4)_\infty^6}{(q^2)_\infty^3 (q^8)_\infty^2}.$$

For,

$$\begin{aligned} & (q^3, q^5, q^8; q^8)_\infty^2 + q(q, q^7, q^8; q^8)_\infty^2 \\ &= \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} - q^2 \sum_{u,v=-\infty}^{\infty} q^{8u^2+2u+8v^2+8v} \\ & \quad + q \sum_{u,v=-\infty}^{\infty} q^{8u^2-6u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v} \\ &= \sum_{-\infty}^{\infty} q^{8u^2-2u} \left\{ \sum_{-\infty}^{\infty} q^{8v^2} - \sum_{-\infty}^{\infty} q^{8v^2+8v+2} \right\} \\ & \quad + q \sum_{-\infty}^{\infty} q^{8u^2-6u} \left\{ \sum_{-\infty}^{\infty} q^{8v^2} - \sum_{-\infty}^{\infty} q^{8v^2+8v+2} \right\} \\ &= \sum_{-\infty}^{\infty} (-1)^v q^{2v^2} \left\{ \sum_{-\infty}^{\infty} q^{8u^2-2u} + \sum_{-\infty}^{\infty} q^{8u^2-6u+1} \right\} \\ &= \sum_{-\infty}^{\infty} (-1)^v q^{2v^2} \sum_{-\infty}^{\infty} q^{2u^2-u} \\ &= \prod_{n \geq 1} \frac{1 - q^{2n}}{1 + q^{2n}} \prod_{n \geq 1} \frac{(1 - q^{2n})^2}{1 - q^n} \\ &= \frac{(q^2)_\infty^4}{(q)_\infty (q^4)_\infty} \end{aligned}$$

as claimed, and the other identity is proved similarly.

If we divide (2) by

$$(q, q^3, q^4, q^5, q^7, q^8; q^8)_\infty = (q, q^3, q^4; q^4)_\infty = \frac{(q)_\infty (q^4)_\infty}{(q^2)_\infty}$$

we obtain

$$(3) \quad \left(\begin{matrix} q^3, q^5, q^8 \\ q, q^4, q^7 \end{matrix}; q^8 \right)_{\infty} + q \left(\begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5 \end{matrix}; q^8 \right)_{\infty} = \frac{(q^2)_{\infty}^5}{(q)_{\infty}^2 (q^4)_{\infty}^2} = \sum_{-\infty}^{\infty} q^{n^2}$$

and

$$\left(\begin{matrix} q^3, q^5, q^8 \\ q, q^4, q^7 \end{matrix}; q^8 \right)_{\infty} - q \left(\begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5 \end{matrix}; q^8 \right)_{\infty} = \frac{(q^4)_{\infty}^5}{(q^2)_{\infty}^2 (q^8)_{\infty}^2} = \sum_{-\infty}^{\infty} q^{2n^2}.$$

Theorem 1 follows. ■

3. PROOF OF THEOREM 2

We start by proving

Corollary to Theorem 1.

$$(4) \quad \begin{aligned} \left(\begin{matrix} q^3, q^5, q^8 \\ q, q^4, q^7 \end{matrix}; q^8 \right)_{\infty} &= \left(\begin{matrix} q^6, q^{10}, q^{16} \\ q^2, q^8, q^{14} \end{matrix}; q^{16} \right)_{\infty} + q \left(\begin{matrix} q^{16} \\ q^8 \end{matrix}; q^{16} \right)_{\infty}, \\ \left(\begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5 \end{matrix}; q^8 \right)_{\infty} &= \left(\begin{matrix} q^{16} \\ q^8 \end{matrix}; q^{16} \right)_{\infty} - q \left(\begin{matrix} q^2, q^{14}, q^{16} \\ q^6, q^8, q^{10} \end{matrix}; q^{16} \right)_{\infty}. \end{aligned}$$

Proof. We have

$$\begin{aligned} 2 \left(\begin{matrix} q^3, q^5, q^8 \\ q, q^4, q^7 \end{matrix}; q^8 \right)_{\infty} &= \sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{2n^2} \\ &= \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{(2n+1)^2} + \sum_{-\infty}^{\infty} q^{2n^2} \\ &= \left\{ \sum_{-\infty}^{\infty} q^{2n^2} + \sum_{-\infty}^{\infty} q^{4n^2} \right\} + 2q \sum_{n \geq 0} q^{4n^2+4n} \\ &= 2 \left(\begin{matrix} q^6, q^{10}, q^{16} \\ q^2, q^8, q^{14} \end{matrix}; q^{16} \right)_{\infty} + 2q \left(\begin{matrix} q^{16} \\ q^8 \end{matrix}; q^{16} \right)_{\infty} \end{aligned}$$

as claimed. The other identity is proved similarly.

Next we show that

$$(5) \quad (q)_{\infty} = (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_{\infty} - q(q^4, q^6, q^{10}, q^{16}, q^{22}, q^{26}, q^{28}, q^{32}; q^{32})_{\infty}.$$

We have

$$\begin{aligned}
(q)_\infty &= \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2} \\
&= \sum_{-\infty}^{\infty} q^{(3(4n)^2-(4n))/2} - \sum_{-\infty}^{\infty} q^{(3(4n-1)^2-(4n-1))/2} \\
&\quad - \sum_{-\infty}^{\infty} q^{(3(4n+1)^2-(4n+1))/2} + \sum_{-\infty}^{\infty} q^{(3(4n+2)^2-(4n+2))/2} \\
&= \left\{ \sum_{-\infty}^{\infty} q^{24n^2-2n} - \sum_{-\infty}^{\infty} q^{24n^2-14n+2} \right\} - \left\{ \sum_{-\infty}^{\infty} q^{24n^2+10n+1} - \sum_{-\infty}^{\infty} q^{24n^2+22n+5} \right\} \\
&= \left(\begin{matrix} q^4, q^{12}, q^{16} \\ -q^2, -q^{14}; q^{16} \end{matrix} \right)_\infty - q \left(\begin{matrix} q^4, q^{12}, q^{16} \\ -q^6, -q^{10}; q^{16} \end{matrix} \right)_\infty \\
&= (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty - q(q^4, q^6, q^{10}, q^{16}, q^{22}, q^{26}, q^{28}, q^{32}; q^{32})_\infty.
\end{aligned}$$

Proof of Theorem 2.

From (5) and (4),

$$\begin{aligned}
&2(q)_\infty / (q^4, q^6, q^8, q^{10}, q^{22}, q^{24}, q^{26}, q^{28}; q^{32})_\infty \\
&= 2 \left(\begin{matrix} q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32} \\ q^4, q^6, q^8, q^{10}, q^{22}, q^{24}, q^{26}, q^{28}; q^{32} \end{matrix} \right)_\infty - 2q \left(\begin{matrix} q^{16}, q^{32} \\ q^8, q^{24}; q^{32} \end{matrix} \right)_\infty \\
&= 2 \left(\begin{matrix} q^{12}, q^{20} \\ q^4, q^{28}; q^{32} \end{matrix} \right)_\infty \left(\begin{matrix} q^2, q^{14}, q^{16} \\ q^6, q^8, q^{10}; q^{16} \end{matrix} \right)_\infty - 2q \left(\begin{matrix} q^{16} \\ q^8; q^{16} \end{matrix} \right)_\infty \\
&= 2 \left(\begin{matrix} q^{12}, q^{20} \\ q^4, q^{28}; q^{32} \end{matrix} \right)_\infty \left\{ \left(\begin{matrix} q^{32} \\ q^{16}; q^{32} \end{matrix} \right)_\infty - q^2 \left(\begin{matrix} q^4, q^{28}, q^{32} \\ q^{12}, q^{16}, q^{20}; q^{32} \end{matrix} \right)_\infty \right\} - 2q \left(\begin{matrix} q^{16} \\ q^8; q^{16} \end{matrix} \right)_\infty \\
&= 2 \left(\begin{matrix} q^{12}, q^{20}, q^{32} \\ q^4, q^{16}, q^{28}; q^{32} \end{matrix} \right)_\infty - 2q^2 \left(\begin{matrix} q^{32} \\ q^{16}; q^{32} \end{matrix} \right)_\infty - 2q \left(\begin{matrix} q^{16} \\ q^8; q^{16} \end{matrix} \right)_\infty \\
&= \left\{ \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{8n^2} \right\} - \sum_{-\infty}^{\infty} q^{2(2n+1)^2} - \sum_{-\infty}^{\infty} q^{(2n+1)^2} \\
&= \left\{ \sum_{-\infty}^{\infty} q^{4n^2} - \sum_{-\infty}^{\infty} q^{(2n+1)^2} \right\} + \left\{ \sum_{-\infty}^{\infty} q^{8n^2} - \sum_{-\infty}^{\infty} q^{2(2n+1)^2} \right\} \\
&= \sum_{-\infty}^{\infty} (-1)^n q^{n^2} + \sum_{-\infty}^{\infty} (-1)^n q^{2n^2}.
\end{aligned}$$

as claimed. The other identity is proved in similar fashion. ■

4. PROOFS OF THEOREMS 3 AND 4

Proof of Theorem 3.

We start by obtaining the 3-dissections

$$(6) \quad \begin{aligned} \left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11} \end{matrix} ; q^{12} \right)_{\infty} &= \left(\begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^9, q^{27}, q^{33} \end{matrix} ; q^{36} \right)_{\infty} + q \left(\begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33} \end{matrix} ; q^{36} \right)_{\infty}, \\ \left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q^3, q^5, q^7, q^9 \end{matrix} ; q^{12} \right)_{\infty} &= \left(\begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33} \end{matrix} ; q^{36} \right)_{\infty} - q^2 \left(\begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^9, q^{15}, q^{21}, q^{27} \end{matrix} ; q^{36} \right)_{\infty}. \end{aligned}$$

We have

$$\begin{aligned} &\left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11} \end{matrix} ; q^{12} \right)_{\infty} \\ &= \left(\begin{matrix} q^6 \\ q^3, q^9 \end{matrix} ; q^{12} \right)_{\infty} \left(\begin{matrix} q^2, q^{10}, q^{12}, q^{14}, q^{22}, q^{24} \\ q, q^{11}, q^{13}, q^{23} \end{matrix} ; q^{24} \right)_{\infty} \\ &= \frac{(-q^3; q^6)_{\infty}}{(q^{24})_{\infty}} \prod_{n \geq 1} (1 + q^{12n-11})(1 + q^{12n-1})(1 - q^{12n}) \\ &\quad \times \prod_{n \geq 1} (1 - q^{24n-14})(1 - q^{24n-10})(1 - q^{24n}) \\ &= \frac{(-q^3; q^6)_{\infty}}{(q^{24})_{\infty}} \sum_{-\infty}^{\infty} q^{6n^2-5n} \sum_{-\infty}^{\infty} (-1)^n q^{12n^2-2n} \\ &= \frac{(-q^3; q^6)_{\infty}}{(q^{24})_{\infty}} \sum_{m, n = -\infty}^{\infty} (-1)^n q^{6m^2-5m+12n^2-2n}. \end{aligned}$$

We now split the sum into three, according to the residue modulo 3 of $m + n$.

If $m+n \equiv 0 \pmod{3}$, $m-2n \equiv 0$, let $s = \frac{1}{3}(m+n)$, $t = \frac{1}{3}(m-2n)$, $m = 2s+t$, $n = s-t$,

if $m+n \equiv 1$, $s = \frac{1}{3}(m+n-1)$, $t = \frac{1}{3}(m-2n-1)$, $m = 2s+t+1$, $n = s-t$,

if $m+n \equiv -1$, $s = \frac{1}{3}(m+n+1)$, $t = \frac{1}{3}(m-2n+1)$, $m = 2s+t-1$, $n = s-t$

and we obtain

$$\left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11} \end{matrix} ; q^{12} \right)_{\infty}$$

$$\begin{aligned}
&= \frac{(-q^3; q^6)_\infty}{(q^{24})_\infty} \left\{ \sum_{s,t=-\infty}^{\infty} (-1)^{s-t} q^{6(2s+t)^2 - 5(2s+t) + 12(s-t)^2 - 2(s-t)} \right. \\
&\quad + \sum_{s,t=-\infty}^{\infty} (-1)^{s-t} q^{6(2s+t+1)^2 - 5(2s+t+1) + 12(s-t)^2 - 2(s-t)} \\
&\quad \left. + \sum_{s,t=-\infty}^{\infty} (-1)^{s-t} q^{6(2s+t-1)^2 - 5(2s+t-1) + 12(s-t)^2 - 2(s-t)} \right\} \\
&= \frac{(-q^3; q^6)_\infty}{(q^{24})_\infty} \left\{ \sum_{s,t=-\infty}^{\infty} (-1)^{s+t} q^{36s^2 - 12s + 18t^2 - 3t} \right. \\
&\quad + \sum_{s,t=-\infty}^{\infty} (-1)^{s+t} q^{36s^2 + 12s + 18t^2 + 9t + 1} \\
&\quad \left. + \sum_{s,t=-\infty}^{\infty} (-1)^{s+t} q^{36s^2 - 36s + 18t^2 - 15t + 11} \right\} \\
&= (-q^3; q^6)_\infty \{ (q^{15}, q^{21}, q^{36}; q^{36})_\infty + q(q^9, q^{27}, q^{36}; q^{36})_\infty \} \\
&= \left(q^3, q^9, q^{15}, q^{21}, q^{27}, q^{33}; q^{36} \right)_\infty \{ (q^{15}, q^{21}, q^{36}; q^{36})_\infty + q(q^9, q^{27}, q^{36}; q^{36})_\infty \} \\
&= \left(q^6, q^{18}, q^{30}, q^{36}; q^{36} \right)_\infty + q \left(q^6, q^{18}, q^{30}, q^{36}; q^{36} \right)_\infty,
\end{aligned}$$

as claimed. The other identity is proved in similar fashion.

Now

$$\begin{aligned}
q \left(q^6, q^{18}, q^{30}, q^{36}; q^{36} \right)_\infty &= q \prod_{n \geq 1} (1 + q^{18n-15})(1 + q^{18n-3})(1 - q^{18n}) \\
&= q \sum_{-\infty}^{\infty} q^{9n^2 - 6n} = \sum_{-\infty}^{\infty} q^{(3n-1)^2} = \frac{1}{2} \sum_{3 \nmid n} q^{n^2} \\
&= \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2} \right\}.
\end{aligned}$$

If we write

$$P_1(q) = \left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11} \end{matrix} ; q^{12} \right)_{\infty}, \quad P_2(q) = \left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q^3, q^5, q^7, q^9 \end{matrix} ; q^{12} \right)_{\infty},$$

then (6) becomes

$$P_1(q) = P_1(q^3) + \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2} \right\},$$

$$qP_2(q) = \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2} \right\} - q^3 P_2(q^3).$$

Theorem 3 follows by iteration, since $P_1(q^{3^n}), P_2(q^{3^n}) \rightarrow 1$ as $n \rightarrow \infty$. ■

Proof of Theorem 4.

We start by obtaining the 5-dissections

$$(7) \quad \left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, q^4, q^9, q^{11}, q^{16}, q^{19} \end{matrix} ; q^{20} \right)_{\infty} = \left(\begin{matrix} q^{10}, q^{40}, q^{50}, q^{60}, q^{90}, q^{100} \\ q^5, q^{20}, q^{45}, q^{55}, q^{80}, q^{95} \end{matrix} ; q^{100} \right)_{\infty}$$

$$+ q \left(\begin{matrix} q^{30}, q^{50}, q^{70}, q^{100} \\ q^{15}, q^{35}, q^{65}, q^{85} \end{matrix} ; q^{100} \right)_{\infty}$$

$$+ q^4 \left(\begin{matrix} q^{10}, q^{50}, q^{90}, q^{100} \\ q^5, q^{45}, q^{55}, q^{95} \end{matrix} ; q^{100} \right)_{\infty},$$

$$\left(\begin{matrix} q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20} \\ q^3, q^7, q^8, q^{12}, q^{13}, q^{17} \end{matrix} ; q^{20} \right)_{\infty} = \left(\begin{matrix} q^{30}, q^{50}, q^{70}, q^{100} \\ q^{15}, q^{35}, q^{65}, q^{85} \end{matrix} ; q^{100} \right)_{\infty}$$

$$+ q^3 \left(\begin{matrix} q^{10}, q^{50}, q^{90}, q^{100} \\ q^5, q^{45}, q^{55}, q^{95} \end{matrix} ; q^{100} \right)_{\infty}$$

$$- q^4 \left(\begin{matrix} q^{20}, q^{30}, q^{50}, q^{70}, q^{80}, q^{100} \\ q^{15}, q^{35}, q^{40}, q^{60}, q^{65}, q^{85} \end{matrix} ; q^{100} \right)_{\infty}.$$

We have

$$\left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, q^4, q^9, q^{11}, q^{16}, q^{19} \end{matrix} ; q^{20} \right)_{\infty}$$

$$= \frac{1}{(q^{20})_{\infty}} \left(\begin{matrix} q^8, q^{12}, q^{20} \\ q^4, q^{16} \end{matrix} ; q^{20} \right)_{\infty} \left(\begin{matrix} q^2, q^{10}, q^{18}, q^{20} \\ q, q^9, q^{11}, q^{19} \end{matrix} ; q^{20} \right)_{\infty}$$

$$\begin{aligned}
&= \frac{1}{(q^{20})_\infty} \left(q^8, q^{12}, q^{20} ; q^{20} \right)_\infty \prod_{n \geq 1} (1 + q^{10n-9})(1 + q^{10n-1})(1 - q^{10n}) \\
&= \frac{1}{(q^{20})_\infty} \left\{ \sum_{-\infty}^{\infty} (-1)^n q^{30n^2-2n} + \sum_{-\infty}^{\infty} (-1)^n q^{30n^2-22n+4} \right\} \sum_{-\infty}^{\infty} q^{5n^2-4n} \\
&= \frac{1}{(q^{20})_\infty} \left\{ \sum_{m,n=-\infty}^{\infty} (-1)^m q^{30m^2-2m+5n^2-4n} + \sum_{m,n=-\infty}^{\infty} (-1)^m q^{30m^2-22m+5n^2-4n+4} \right\}.
\end{aligned}$$

We now split each sum into five, according to the residue modulo 5 of $m + 2n$.

If $m + 2n \equiv 0 \pmod{5}$, $2m - n \equiv 0$, $3m + n \equiv 0$, let

$$s = \frac{1}{5}(3m + n), \quad t = \frac{1}{5}(2m - n), \quad m = s + t, \quad n = 2s - 3t,$$

if $m + 2n \equiv 1$, $s = \frac{1}{5}(3m + n + 2)$, $t = \frac{1}{5}(2m - n - 2)$, $m = s + t$, $n = 2s - 3t - 2$,

if $m + 2n \equiv 2$, $s = \frac{1}{5}(3m + n - 1)$, $t = \frac{1}{5}(2m - n + 1)$, $m = s + t$, $n = 2s - 3t + 1$,

if $m + 2n \equiv -1$, $s = \frac{1}{5}(3m + n - 2)$, $t = \frac{1}{5}(2m - n + 2)$, $m = s + t$, $n = 2s - 3t + 2$,

if $m + 2n \equiv -2$, $s = \frac{1}{5}(3m + n + 1)$, $t = \frac{1}{5}(2m - n - 1)$, $m = s + t$, $n = 2s - 3t - 1$.

If we make these substitutions into the two sums, we obtain ten sums, each of which separates into the product of two sums, each summable via the triple product identity. Two of the ten sums are zero and two cancel, leaving six which can be grouped in pairs in an obvious way, and each pair summed by the quintuple product identity, leading to the stated identity. The other identity is proved in similar fashion.

Now,

$$\begin{aligned}
&q \left(\begin{matrix} q^{30}, q^{50}, q^{70}, q^{100} \\ q^{15}, q^{35}, q^{65}, q^{85} \end{matrix} ; q^{100} \right)_\infty + q^4 \left(\begin{matrix} q^{10}, q^{50}, q^{90}, q^{100} \\ q^5, q^{45}, q^{55}, q^{95} \end{matrix} ; q^{100} \right)_\infty \\
&= q \prod_{n \geq 1} (1 + q^{50n-35})(1 + q^{50n-15})(1 - q^{50n}) + q^4 \prod_{n \geq 1} (1 + q^{50n-45})(1 + q^{50n-5})(1 - q^{50n}) \\
&= q \sum_{-\infty}^{\infty} q^{25n^2-10n} + q^4 \sum_{-\infty}^{\infty} q^{25n^2-20n} = \sum_{-\infty}^{\infty} q^{(5n-1)^2} + \sum_{-\infty}^{\infty} q^{(5n-2)^2} = \frac{1}{2} \sum_{5 \nmid n} q^{n^2} \\
&= \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2} \right\}.
\end{aligned}$$

If we write

$$P_1(q) = \left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, q^4, q^9, q^{11}, q^{16}, q^{19} \end{matrix} ; q^{20} \right)_\infty, \quad P_2(q) = \left(\begin{matrix} q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20} \\ q^3, q^7, q^8, q^{12}, q^{13}, q^{17} \end{matrix} ; q^{20} \right)_\infty$$

then (7) becomes

$$P_1(q) = P_1(q^5) + \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2} \right\},$$

$$qP_2(q) = \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2} \right\} - q^5 P_2(q^5).$$

Theorem 4 follows by iteration, since $P_1(q^{5^n}), P_2(q^{5^n}) \rightarrow 1$ as $n \rightarrow \infty$. ■

5. PROOFS OF THEOREMS 5 AND 6

Proof of Theorem 5.

We start by proving

$$(8) \quad \prod_{n \geq 1} (1 + q^{12n-11})(1 + q^{12n-9})(1 + q^{12n-3})(1 + q^{12n-1})(1 - q^{12n})^2$$

$$= \prod_{n \geq 1} (1 + q^{24n-20})(1 + q^{24n-14})(1 + q^{24n-10})(1 + q^{24n-4})(1 - q^{24n})^2$$

$$+ q \prod_{n \geq 1} (1 + q^{24n-22})(1 + q^{24n-16})(1 + q^{24n-8})(1 + q^{24n-2})(1 - q^{24n})^2.$$

We have

$$\prod_{n \geq 1} (1 + q^{12n-11})(1 + q^{12n-9})(1 + q^{12n-3})(1 + q^{12n-1})(1 - q^{12n})^2$$

$$= \sum_{m, n = -\infty}^{\infty} q^{6m^2 - 5m + 6n^2 - 3n}$$

$$= \sum_{m \equiv n \pmod{2}} q^{6m^2 - 5m + 6n^2 - 3n} + \sum_{m \not\equiv n \pmod{2}} q^{6m^2 - 5m + 6n^2 - 3n}$$

$$= \sum_{u, v = -\infty}^{\infty} q^{6(u+v)^2 - 5(u+v) + 6(u-v)^2 - 3(u-v)}$$

$$+ \sum_{u, v = -\infty}^{\infty} q^{6(u+v+1)^2 - 5(u+v+1) + 6(u-v)^2 - 3(u-v)}$$

$$= \sum_{u, v = -\infty}^{\infty} q^{12u^2 - 8u + 12v^2 - 2v} + q \sum_{u, v = -\infty}^{\infty} q^{12u^2 + 4u + 12v^2 + 10v}$$

$$\begin{aligned}
&= \prod_{n \geq 1} (1 + q^{24n-20})(1 + q^{24n-14})(1 + q^{24n-10})(1 + q^{24n-4})(1 - q^{24n})^2 \\
&+ q \prod_{n \geq 1} (1 + q^{24n-22})(1 + q^{24n-16})(1 + q^{24n-8})(1 + q^{24n-2})(1 - q^{24n})^2.
\end{aligned}$$

We can write (8) as follows.

$$\begin{aligned}
&\left(q^2, q^6, q^{12}, q^{12}, q^{18}, q^{22}, q^{24}, q^{24} \right. \\
&\left. q, q^3, q^9, q^{11}, q^{13}, q^{15}, q^{21}, q^{23}; q^{24} \right)_{\infty} \\
&= \left(q^8, q^{24}, q^{24}, q^{40}, q^{48}, q^{48} \right. \\
&\left. q^4, q^{10}, q^{14}, q^{34}, q^{38}, q^{44}; q^{48} \right)_{\infty} + q \left(q^4, q^{24}, q^{24}, q^{44}, q^{48}, q^{48} \right. \\
&\left. q^2, q^8, q^{22}, q^{26}, q^{40}, q^{46}; q^{48} \right)_{\infty}.
\end{aligned}$$

If we multiply by $\left(\frac{q^{10}, q^{14}}{q^{12}, q^{24}}; q^{24} \right)_{\infty}$ we obtain

$$\begin{aligned}
(9) \quad &\left(q^2, q^6, q^{10}, q^{12}; q^{12} \right)_{\infty} = \left(q^8, q^{24}, q^{40}, q^{48} \right. \\
&\left. q^4, q^{12}, q^{36}, q^{44}; q^{48} \right)_{\infty} \\
&+ q \left(q^4, q^{10}, q^{14}, q^{24}, q^{34}, q^{38}, q^{44}, q^{48} \right. \\
&\left. q^2, q^8, q^{12}, q^{22}, q^{26}, q^{36}, q^{40}, q^{46}; q^{48} \right)_{\infty}.
\end{aligned}$$

Combining Theorem 3 and (9), we find

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{3n^2} = \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{12n^2} + 2q \left(\frac{q^4, q^{10}, q^{14}, q^{24}, q^{34}, q^{38}, q^{44}, q^{48}}{q^2, q^8, q^{12}, q^{22}, q^{26}, q^{36}, q^{40}, q^{46}}; q^{48} \right)_{\infty}.$$

If we extract odd powers, divide by q and replace q^2 by q we obtain

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} + \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2 \left(\frac{q^2, q^5, q^7, q^{12}, q^{17}, q^{19}, q^{22}, q^{24}}{q, q^4, q^6, q^{11}, q^{13}, q^{18}, q^{20}, q^{23}}; q^{24} \right)_{\infty}.$$

If we put $-q$ for q we find

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} - \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2 \left(\frac{q, q^{10}, q^{11}, q^{12}, q^{13}, q^{14}, q^{23}, q^{24}}{q^4, q^5, q^6, q^7, q^{17}, q^{18}, q^{19}, q^{20}}; q^{24} \right)_{\infty}. \blacksquare$$

Proof of Theorem 6.

We shall start by proving

$$(10) \quad \prod_{n \geq 1} (1 + q^{10n-9})(1 + q^{10n-1})(1 - q^{10n}) \\ = \prod_{n \geq 1} (1 + q^{40n-28})(1 + q^{40n-12})(1 - q^{40n}) + q \prod_{n \geq 1} (1 + q^{40n-32})(1 + q^{40n-8})(1 - q^{40n}).$$

We have

$$\prod_{n \geq 1} (1 + q^{10n-9})(1 + q^{10n-1})(1 - q^{10n}) \\ = \sum_{-\infty}^{\infty} q^{5n^2-4n} = \sum_{-\infty}^{\infty} q^{20n^2-8n} + q \sum_{-\infty}^{\infty} q^{20n^2+12n} \\ = \prod_{n \geq 1} (1 + q^{40n-28})(1 + q^{40n-12})(1 - q^{40n}) + q \prod_{n \geq 1} (1 + q^{40n-32})(1 + q^{40n-8})(1 - q^{40n}).$$

We can write (10) as follows.

$$\left(\begin{matrix} q^2, q^{10}, q^{18}, q^{20} \\ q, q^9, q^{11}, q^{19}; q^{20} \end{matrix} \right)_{\infty} = \left(\begin{matrix} q^{24}, q^{40}, q^{56}, q^{80} \\ q^{12}, q^{28}, q^{52}, q^{68}; q^{80} \end{matrix} \right)_{\infty} + q \left(\begin{matrix} q^{16}, q^{40}, q^{64}, q^{80} \\ q^8, q^{32}, q^{48}, q^{72}; q^{80} \end{matrix} \right)_{\infty}.$$

If we multiply by $\left(\begin{matrix} q^8, q^{12} \\ q^4, q^{16}; q^{20} \end{matrix} \right)_{\infty}$ we obtain

$$(11) \quad \left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, q^4, q^9, q^{11}, q^{16}, q^{19}; q^{20} \end{matrix} \right)_{\infty} \\ = \left(\begin{matrix} q^8, q^{32}, q^{40}, q^{48}, q^{72}, q^{80} \\ q^4, q^{16}, q^{36}, q^{44}, q^{64}, q^{76}; q^{80} \end{matrix} \right)_{\infty} + q \left(\begin{matrix} q^{12}, q^{28}, q^{40}, q^{52}, q^{68}, q^{80} \\ q^4, q^{24}, q^{36}, q^{44}, q^{56}, q^{76}; q^{80} \end{matrix} \right)_{\infty}.$$

Combining Theorem 4 and (11) we obtain

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{5n^2} = \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{20n^2} + 2q \left(\begin{matrix} q^{12}, q^{28}, q^{40}, q^{52}, q^{68}, q^{80} \\ q^4, q^{24}, q^{36}, q^{44}, q^{56}, q^{76}; q^{80} \end{matrix} \right)_{\infty}.$$

If we extract odd powers, divide by q and replace q^4 by q we obtain

$$\sum_{-\infty}^{\infty} q^{n(n+1)} + \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2 \left(\begin{matrix} q^3, q^7, q^{10}, q^{13}, q^{17}, q^{20} \\ q, q^6, q^9, q^{11}, q^{14}, q^{19}; q^{20} \end{matrix} \right)_{\infty}.$$

If we put $-q$ for q we find

$$\sum_{-\infty}^{\infty} q^{n(n+1)} - \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2 \left(\frac{q, q^9, q^{10}, q^{11}, q^{19}, q^{20}}{q^2, q^3, q^7, q^{13}, q^{17}, q^{18}; q^{20}} \right)_{\infty}. \blacksquare$$

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Department of Mathematics,
 Massey University, Albany Campus,
 Private Bag 102 904, North Shore Mail Centre, Auckland, New Zealand
 e-mail s.cooper@massey.ac.nz

School of Mathematics
 UNSW SYDNEY NSW 2052
 Australia
 e-mail m.hirschhorn@unsw.edu.au