On Some Infinite Product Identities

Shaun Cooper*and Michael Hirschhorn

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Abstract
Uniform proofs of several identities of Blecksmith, Brillhart and Gerst, and a new identity, are given.

1 Introduction

This paper has two main purposes. The first is to present two new identities:

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3, q^5, q^8; q^8)_\infty}{(q, q^4, q^7; q^8)_\infty} \tag{1}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q, q^7, q^8; q^8)_\infty}{(q^3, q^4, q^5; q^8)_\infty} \tag{2}$$

These identities are similar to those in [1, 2] and were apparently missed by these authors. As in [1, 2] identities (1) and (2) were found by computer investigation.

Specifically, we used a computer program written by F. Garvan [4] to check whether certain series were likely to factor into nice products. Here is a sample session using the program to convert the series $\sum_{n=-10}^{10} q^{n^2} + \sum_{n=-10}^{10} q^{2n^2}$ into a product:

```
> with(qseries):
> prodmake(1/2*(sum(q^(n^2),n=-10..10)+sum(q^(2*n^2),n=-10..10)),q,40);
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$$(1-q^3)(1-q^5)(1-q^8)(1-q^{11})(1-q^{13})(1-q^{16})(1-q^{19})(1-q^{21})$$
$$(1-q^{24})(1-q^{27})(1-q^{29})(1-q^{32})(1-q^{35})(1-q^{37})/((1-q)$$
$$(1-q^4)(1-q^7)(1-q^9)(1-q^{12})(1-q^{15})(1-q^{17})(1-q^{20})$$
$$(1-q^{23})(1-q^{25})(1-q^{28})(1-q^{31})(1-q^{33})(1-q^{36})(1-q^{39}))$$

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¿From this output, we conjecture equation (1). Equation (2) was obtained similarly.

The second purpose of this paper is to prove (1) and (2). In fact, we shall show that these, and all of the main theorems in [1, 2] are corollaries of an identity for sigma functions due to Weierstrass [11, p. 451, Ex. 3], which has recently been brought into prominence by Richard Lewis [5, 6, 7, 8].

2 Notation and the main tool

Let q be a complex number satisfying $|q| < 1$, and set

$$\begin{aligned}
(a;q)_\infty &= \prod_{j=0}^{\infty}(1-aq^j) \\
(a_1,a_2,\cdots,a_n;q)_\infty &= (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty \\
\left(\begin{matrix} a_1, & a_2, & \cdots, & a_n \\ b_1, & b_2, & \cdots, & b_n \end{matrix};q\right)_\infty &= (a_1,a_2,\cdots,a_n;q)_\infty/(b_1,b_2,\cdots,b_n;q)_\infty \\
[a;q]_\infty &= (a;q)_\infty(a^{-1}q;q)_\infty \\
[a_1,a_2,\cdots,a_n]_\infty &= [a_1;q]_\infty\,[a_2;q]_\infty\cdots[a_n;q]_\infty\,.
\end{aligned}$$

We will use the standard properties

$$\left[x^{-1};q\right]_\infty = -x^{-1}\,[x;q]_\infty = [qx;q]_\infty \tag{3}$$

$$\left[x,qx;q^2\right]_\infty = [x;q]_\infty \tag{4}$$

$$[x,-x;q]_\infty = \left[x^2;q^2\right]_\infty\,. \tag{5}$$

The main tool that we shall use is

Lemma
Suppose $a_1,a_2,\cdots,a_n;\ b_1,b_2,\cdots,b_n$ are non-zero complex numbers which satisfy
(i) $a_i \neq q^n a_j$ for all $i \neq j$ and all $n \in Z$,
(ii) $a_1a_2\cdots a_n = b_1b_2\cdots b_n$.
Then

$$\sum_{i=1}^{n}\frac{\prod_{j=1}^{n}[a_ib_j^{-1};q]_\infty}{\prod_{j=1,j\neq i}^{n}[a_ia_j^{-1}]_\infty} = 0.$$

This lemma appears in [11, p. 451, Ex. 3] (where the infinite products are expressed in terms of sigma functions), [9], [10, eq. 7.4.3] and [3, p. 138, Ex. 5.23]. A simple proof is given in [5].

3 The identities

Theorem 1.

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} &= 2\left(\begin{matrix} q^3, & q^5, & q^8 \\ q, & q^4, & q^7 \end{matrix};q^8\right)_\infty, \\
\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} &= 2q\left(\begin{matrix} q, & q^7, & q^8 \\ q^3, & q^4, & q^5 \end{matrix};q^8\right)_\infty.
\end{aligned}$$

Theorem 2. [1, Theorem 1]

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}+\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} &= 2(q;q)_\infty/(q^4,q^6,q^8,q^{10},q^{22},q^{24},q^{26},q^{28};q^{32})_\infty,\\ \sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}-\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} &= 2q(q;q)_\infty/(q^2,q^8,q^{12},q^{14},q^{18},q^{20},q^{24},q^{30};q^{32})_\infty.\end{aligned}$$

Theorem 3. [1, Theorem 3]

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n^2}+\sum_{n=-\infty}^{\infty} q^{3n^2} &= 2\left(\begin{matrix} q^2, & q^6, & q^{10}, & q^{12} \\ q, & q^3, & q^9, & q^{11}\end{matrix};q^{12}\right)_\infty,\\ \sum_{n=-\infty}^{\infty} q^{n^2}-\sum_{n=-\infty}^{\infty} q^{3n^2} &= 2q\left(\begin{matrix} q^2, & q^6, & q^{10}, & q^{12} \\ q^3, & q^5, & q^7, & q^{9}\end{matrix};q^{12}\right)_\infty.\end{aligned}$$

Theorem 4. [2, Theorem 3]

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n^2}+\sum_{n=-\infty}^{\infty} q^{5n^2} &= 2\left(\begin{matrix} q^2, & q^8, & q^{10}, & q^{12}, & q^{18}, & q^{20} \\ q, & q^4, & q^9, & q^{11}, & q^{16}, & q^{19}\end{matrix};q^{20}\right)_\infty,\\ \sum_{n=-\infty}^{\infty} q^{n^2}-\sum_{n=-\infty}^{\infty} q^{5n^2} &= 2q\left(\begin{matrix} q^4, & q^6, & q^{10}, & q^{14}, & q^{16}, & q^{20} \\ q^3, & q^7, & q^8, & q^{12}, & q^{13}, & q^{17}\end{matrix};q^{20}\right)_\infty.\end{aligned}$$

Theorem 5. [2, Theorem 1]

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{2n(n+1)}+\sum_{n=-\infty}^{\infty} q^{6n(n+1)+1} &= 2\left(\begin{matrix} q^2, & q^5, & q^7, & q^{12}, & q^{17}, & q^{19}, & q^{22}, & q^{24} \\ q, & q^4, & q^6, & q^{11}, & q^{13}, & q^{18}, & q^{20}, & q^{23}\end{matrix};q^{24}\right)_\infty,\\ \sum_{n=-\infty}^{\infty} q^{2n(n+1)}-\sum_{n=-\infty}^{\infty} q^{6n(n+1)+1} &= 2\left(\begin{matrix} q, & q^{10}, & q^{11}, & q^{12}, & q^{13}, & q^{14}, & q^{23}, & q^{24} \\ q^4, & q^5, & q^6, & q^{7}, & q^{17}, & q^{18}, & q^{19}, & q^{20}\end{matrix};q^{24}\right)_\infty.\end{aligned}$$

Theorem 6. [2, Theorem 4]

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n(n+1)}+\sum_{n=-\infty}^{\infty} q^{5n(n+1)+1} &= 2\left(\begin{matrix} q^3, & q^7, & q^{10}, & q^{13}, & q^{17}, & q^{20} \\ q, & q^6, & q^9, & q^{11}, & q^{14}, & q^{19}\end{matrix};q^{20}\right)_\infty,\\ \sum_{n=-\infty}^{\infty} q^{n(n+1)}-\sum_{n=-\infty}^{\infty} q^{5n(n+1)+1} &= 2\left(\begin{matrix} q, & q^9, & q^{10}, & q^{11}, & q^{19}, & q^{20} \\ q^2, & q^3, & q^7, & q^{13}, & q^{17}, & q^{18}\end{matrix};q^{20}\right)_\infty.\end{aligned}$$

Proof of Theorem 1
We manipulate the identities in Theorem 1 into a pair of equivalent identities, which are then shown to be true.
Adding the identities in Theorem 1 and dividing by 2 gives

$$\sum_{n=-\infty}^{\infty} q^{n^2}=\frac{(q^3,q^5,q^8;q^8)_\infty}{(q,q^4,q^7;q^8)_\infty}+q\frac{(q,q^7,q^8;q^8)_\infty}{(q^3,q^4,q^5;q^8)_\infty} \tag{6}$$

By the Jacobi Triple Product Identity, the left hand side is

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n^2} &= (-q,-q,q^2;q^2)_\infty\\ &= \frac{(q^2;q^2)_\infty(q^2,q^2;q^4)_\infty}{(q,q;q^2)_\infty}\\ &= \frac{(q^2,q^2,q^2,q^4,q^6,q^6,q^6,q^8;q^8)_\infty}{(q,q,q^3,q^3,q^5,q^5,q^7,q^7;q^8)_\infty}.\end{aligned}$$

Substituting this into (6) and multiplying both sides by

$$(q,q,q^3,q^3,q^4,q^5,q^5,q^7,q^7;q^8)_\infty/(q^8;q^8)_\infty$$

gives

$$\begin{aligned} &(q^2,q^2,q^2,q^4,q^4,q^6,q^6,q^6;q^8)_\infty \\ = \quad &(q,q^3,q^3,q^3,q^5,q^5,q^5,q^7;q^8)_\infty + q(q,q,q,q^3,q^5,q^7,q^7,q^7;q^8)_\infty, \end{aligned}$$

or

$$[q^2,q^2,q^2,q^4;q^8]_\infty = [q,q^3,q^3,q^3;q^8]_\infty + q[q,q,q,q^3;q^8]_\infty. \tag{7}$$

Similarly, subtracting the identities in Theorem 1 and performing similar manipulations to the above, gives

$$[q,q^3,q^4,q^4;q^8]_\infty = [q^2,q^2,q^3,q^3;q^8]_\infty - q[q,q,q^2,q^2;q^8]_\infty. \tag{8}$$

In the Lemma, replace q with q^8 and take $n=3$. Then (7) and (8) follow by taking $(a_1,a_2,a_3;b_1,b_2,b_3) = (1,q,q^3;q^2,q^4,q^{-2})$ and $(a_1,a_2,a_3;b_1,b_2,b_3) = (1,q^2,q^6;q,q^3,q^4)$, respectively, and simplifying using (3).
Theorem 1 follows by working backwards through the above, starting with(7) and (8).

Proof of Theorem 2
Adding the two identities in Theorem 2 and applying the Jacobi Triple Product Identity on the left gives

$$\begin{aligned} (q^2,q^2,q^4;q^4)_\infty \quad = \quad &(q;q^2)_\infty(q^{16};q^{16})_\infty \left[(q^2,q^{12},q^{14},q^{18},q^{20},q^{30};q^{32})_\infty \right. \\ &\left. +q(q^4,q^6,q^{10},q^{22},q^{26},q^{28};q^{32})_\infty\right]. \end{aligned}$$

Use

$$\begin{aligned} (q^2,q^{14},q^{18},q^{30};q^{32})_\infty &= [q^2;q^{16}]_\infty \\ (q^6,q^{10},q^{22},q^{26};q^{32})_\infty &= [q^6;q^{16}]_\infty \\ (q^{12},q^{20};q^{32})_\infty &= [q^6,-q^6;q^{16}]_\infty \\ (q^4,q^{28};q^{32})_\infty &= [q^2,-q^2;q^{16}]_\infty \end{aligned}$$

and divide by $(q;q^2)_\infty(q^{16};q^{16})_\infty$ to get

$$\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q^2)_\infty(q^{16};q^{16})_\infty} = [q^2,q^6,-q^6;q^{16}]_\infty + q[q^2,-q^2,q^6;q^{16}]_\infty.$$

Multiply this by $[q^3,-q^4,q^5;q^{16}]_\infty/[q^2,q^6;q^{16}]_\infty$ and simplify to obtain

$$[-q,q^6,-q^7,q^8;q^{16}]_\infty = [q^3,-q^4,q^5,-q^6;q^{16}]_\infty + q[-q^2,q^3,-q^4,q^5;q^{16}]_\infty. \tag{9}$$

If the identities in Theorem 2 are subtracted instead, then manipulations like those above lead to

$$[q,q^6,q^7,q^8;q^{16}]_\infty = [-q^3,-q^4,-q^5,-q^6;q^{16}]_\infty - q[-q^2,-q^3,-q^4,-q^5;q^{16}]_\infty. \tag{10}$$

In the Lemma, replace q with q^{16} and take $n=3$. Then (9) and (10) follow by taking $(a_1,a_2,a_3;b_1,b_2,b_3) = (1,-q^2,q^3;q^6,q^8,-q^{-9})$ and $(a_1,a_2,a_3;b_1,b_2,b_3) = (1,-q^2,-q^3;q^6,q^8,q^{-9})$, respectively (or notice that (10) is just (9) with q replaced with $-q$). Theorem 2 then follows from (9) and (10).

The proofs of Theorems 3 – 6 are similar to the above, so we give outlines only.

Proof of Theorem 3

By addition and subtraction, Theorem 3 is equivalent to

$$[q^2, q^2, q^4, q^6; q^{12}]_\infty = [q, q^3, q^5, q^5; q^{12}]_\infty + q[q, q, q^3, q^5; q^{12}]_\infty \tag{11}$$

and

$$[q, q^5, q^6; q^{12}]_\infty = [q^2, q^3, q^5; q^{12}]_\infty - q[q, q^2, q^3; q^{12}]_\infty. \tag{12}$$

In the Lemma, replace q with q^{12} and take $n = 3$. Then (11) and (12) follow by taking $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, q, q^2; q^3, q^5, q^{-5})$ and $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, q, q^2; q^4, q^5, q^{-6})$, respectively, and simplifying.

Proof of Theorem 4

By addition and subtraction, Theorem 4 is equivalent to

$$\begin{aligned} & [q^2, q^2, q^2, q^4, q^4, q^6, q^6, q^6, q^8, q^8, q^{10}; q^{20}]_\infty \\ = \; & [q, q^2, q^3, q^3, q^5, q^5, q^7, q^7, q^8, q^8, q^9; q^{20}]_\infty + q[q, q, q^3, q^4, q^4, q^5, q^5, q^6, q^7, q^9, q^9; q^{20}]_\infty \end{aligned} \tag{13}$$

and

$$\begin{aligned} & [q, q^3, q^4, q^7, q^8, q^9, q^{10}; q^{20}]_\infty \\ = \; & [q^2, q^3, q^5, q^5, q^7, q^8, q^8; q^{20}]_\infty - q[q, q^4, q^4, q^5, q^5, q^6, q^9; q^{20}]_\infty. \end{aligned} \tag{14}$$

These can be manipulated to

$$[q^2, q^2, -q^3, -q^5; q^{10}]_\infty = [q, q^3, -q^4, -q^4; q^{10}]_\infty + q[q, q, -q^2, -q^4; q^{10}]_\infty \tag{15}$$

and

$$[q^2, -q^2, q^3, -q^5; q^{10}]_\infty = [-q, q^3, q^4, -q^4; q^{10}]_\infty - q[q, -q, -q^2, q^4; q^{10}]_\infty, \tag{16}$$

respectively. In the Lemma, replace q with q^{10} and take $n = 3$. Then (15) and (16) follow by taking $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, -q^5, q; q^{-1}, -q^{-2}, q^9)$ and $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, -q^5, q^6; q^{-1}, -q^{-2}, q^{14})$, respectively, and simplifying.

Proof of Theorem 5

By addition and subtraction, Theorem 5 is equivalent to

$$[-1, -q^4, q^4, q^6; q^{12}]_\infty = [-q, -q^3, q^3, q^5; q^{12}]_\infty + [q, q^3, -q^3, -q^5; q^{12}]_\infty \tag{17}$$

and

$$q[-1, -q^2, q^2, q^6; q^{12}]_\infty = [-q, -q^3, q^3, q^5; q^{12}]_\infty - [q, q^3, -q^3, -q^5; q^{12}]_\infty. \tag{18}$$

In the Lemma, replace q with q^{12} and take $n = 3$. Then (17) and (18) follow by taking $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, -1, q^9; q^5, -q, q^3)$ and $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, -1, q^3; q^5, -q, q^{-3})$, respectively.

Proof of Theorem 6

By addition and subtraction, Theorem 6 is equivalent to

$$[-1, -q^4, q^4, q^4; q^{10}]_\infty = [-q, -q^3, q^3, q^3; q^{10}]_\infty + [q, q^3, -q^3, -q^3; q^{10}]_\infty \tag{19}$$

and

$$q[-1, -q, q, q^4; q^{10}]_\infty = [-q, -q^2, q^2, q^3; q^{10}]_\infty - [q, q^2, -q^2, -q^3; q^{10}]_\infty. \tag{20}$$

In the Lemma, replace q with q^{10} and take $n = 3$. Then (19) and (20) follow by taking $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, -1, q^7; q^3, -q, q^3)$ and $(a_1, a_2, a_3; b_1, b_2, b_3) = (1, -1, q^2; q^3, -q, q^{-2})$, respectively.

Remark Equations (19) and (20) are due to Lewis [6, eq 1.13] who used them to compute the 2-dissection of Ramanujan's continued fraction.

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Department of Mathematics, Massey University, Albany Campus, Private Bag 102 904, North Shore Mail Centre, Auckland, New Zealand
e-mail: s.cooper@massey.ac.nz

School of Mathematics, UNSW, Sydney, Australia 2052
e-mail: mikeh@maths.unsw.edu.au

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