

ON PARTITIONS INTO FOUR DISTINCT SQUARES OF EQUAL PARITY

MICHAEL D. HIRSCHHORN

1. Introduction and statement of results

In recent work [1,2], we studied $p_{4\Box}(n)$, the number of partitions of n into four squares. We found the generating function for $p_{4\Box}(n)$, and showed that the numbers $p_{4\Box}(n)$ possess some (rare) arithmetic properties.

In this paper we make a study of what we will call $p_{4e}(n)$ and $p_{4o}(n)$, the number of partitions of n into four distinct **even** (respectively **odd**) squares. We shall also have reason to consider $p_{4e+}(n)$, where the squares are **strictly positive**.

The main result of this paper is that for $n \equiv 4 \pmod{8}$,

$$p_{4o}(n) = p_{4e}(n) + p_{4e+}(n).$$

As a corollary we obtain the surprising result that for $n \equiv 4 \pmod{8}$,

$$p_{4o}(n) \geq p_{4e}(n) \geq \frac{1}{2}p_{4o}(n).$$

Thus, if $p_{4o}(n) > 0$ (and so $n \equiv 4 \pmod{8}$) then $p_{4e}(n) > 0$, a conjecture announced by R. Wm. Gosper at the Ramanujan Centenary Conference in 1987, the inspiration for this investigation.

We shall give two proofs of our theorem, one arithmetic, the other via generating functions.

2. The arithmetic proof

We shall show that if $n \equiv 4 \pmod{8}$ there are two partitions of n into four distinct odd squares for each partition of n into four distinct even squares greater than zero, and one

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$

partition of n into four distinct odd squares for each partition of n into four distinct even squares one of which is zero.

Suppose $n \equiv 4 \pmod{8}$ and

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

is a partition of n into four distinct even squares. Then w.l.o.g.

$$(x_1, x_2, x_3, x_4) \equiv (0, 2, 2, 2) \text{ or } (2, 0, 0, 0) \pmod{4},$$

$$x_1 + x_2 + x_3 + x_4 \equiv 2 \pmod{8}$$

and

$$x_2 > x_3 > |x_4|.$$

(In the case $(x_1, x_2, x_3, x_4) \equiv (0, 2, 2, 2) \pmod{4}$, the condition $x_1 + x_2 + x_3 + x_4 \equiv 2 \pmod{8}$ determines the sign on x_4 , (the sign of x_1 is undetermined); in the case $(x_1, x_2, x_3, x_4) \equiv (2, 0, 0, 0) \pmod{4}$ the condition determines the sign on x_1 (the sign of x_4 is undetermined).)

In the case $(x_1, x_2, x_3, x_4) \equiv (0, 2, 2, 2) \pmod{4}$, let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} |x_1| \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -|x_1| \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Then

$$n = y_1^2 + y_2^2 + y_3^2 + y_4^2 = y'_1{}^2 + y'_2{}^2 + y'_3{}^2 + y'_4{}^2,$$

$$y_1, y_2, y_3, y_4 \equiv 1 \pmod{4}, \quad y'_1, y'_2, y'_3, y'_4 \equiv 1 \pmod{4},$$

$$y_1 > y_2 > y_3 > y_4, \quad y'_1 > y'_2 > y'_3 > y'_4$$

are partitions of n into four distinct odd squares, different if and only if $x_1 \neq 0$.

Similarly, in the case $(x_1, x_2, x_3, x_4) \equiv (2, 0, 0, 0) \pmod{4}$, let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ |x_4| \end{pmatrix}, \quad \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -|x_4| \end{pmatrix}.$$

Then

$$\begin{aligned} n &= y_1^2 + y_2^2 + y_3^2 + y_4^2 = y_1'^2 + y_2'^2 + y_3'^2 + y_4'^2, \\ y_1, y_2, y_3, y_4 &\equiv 1 \pmod{4}, \quad y_1', y_2', y_3', y_4' \equiv 1 \pmod{4}, \\ y_1 &> y_2 > y_3 > y_4, \quad y_1' > y_2' > y_3' > y_4' \end{aligned}$$

are partitions of n into four distinct odd squares, different if and only if $x_4 \neq 0$.

Conversely, suppose $n \equiv 4 \pmod{8}$ and

$$n = y_1^2 + y_2^2 + y_3^2 + y_4^2$$

is a partition of n into four odd squares. Then w.l.o.g.

$$y_1, y_2, y_3, y_4 \equiv 1 \pmod{4} \text{ and } y_1 > y_2 > y_3 > y_4.$$

If we now let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

then

$$\begin{aligned} n &= x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ (x_1, x_2, x_3, x_4) &\equiv (0, 2, 2, 2) \text{ or } (2, 0, 0, 0) \pmod{4}, \\ x_1 + x_2 + x_3 + x_4 &\equiv 2 \pmod{8} \end{aligned}$$

and

$$x_2 > x_3 > |x_4|.$$

This establishes the (one-to-one-or-two) correspondence described earlier between the partitions of n into four distinct even squares and the partitions of n into four distinct odd squares. ■

An example

Consider $n = 420$. The partitions of 420 into four distinct even squares are

$$420 = 20^2 + 4^2 + 2^2 = 16^2 + 12^2 + 4^2 + 2^2 = 16^2 + 10^2 + 8^2 = 14^2 + 12^2 + 8^2 + 4^2.$$

Corresponding to these we have, respectively,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 20 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \\ -7 \\ -11 \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 16 \\ 12 \\ \pm 4 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 17 \\ 1 \\ -3 \\ -11 \end{pmatrix}, \quad \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \\ 1 \\ -15 \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 16 \\ 8 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 17 \\ 9 \\ 1 \\ -7 \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -14 \\ 12 \\ 8 \\ \pm 4 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ -11 \\ -15 \end{pmatrix}, \quad \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -7 \\ -19 \end{pmatrix}.$$

Thus

$$\begin{aligned} 420 &= 20^2 + 4^2 + 2^2 = 16^2 + 12^2 + 4^2 + 2^2 = 16^2 + 10^2 + 8^2 = 14^2 + 12^2 + 8^2 + 4^2 \\ &= 19^2 + 7^2 + 3^2 + 1^2 = 17^2 + 11^2 + 3^2 + 1^2 = 17^2 + 9^2 + 7^2 + 1^2 \\ &= 15^2 + 13^2 + 5^2 + 1^2 = 15^2 + 11^2 + 7^2 + 5^2 = 13^2 + 11^2 + 9^2 + 7^2, \end{aligned}$$

$$p_{4o}(420) = 6, \quad p_{4e}(420) = 4, \quad p_{4e+}(420) = 2, \quad \text{and} \quad p_{4o}(420) = p_{4e}(420) + p_{4e+}(420).$$

3. The generating function proof

Let x_i , $i = 0, \dots, N$ be arbitrary. It is easy to verify by expanding the expression on the right that

$$\begin{aligned} (1) \quad & 24 \sum_{i>j>k>l} x_i x_j x_k x_l \\ &= \left(\sum_{i=0}^N x_i \right)^4 - 6 \left(\sum_{i=0}^N x_i \right)^2 \sum_{i=0}^N x_i^2 + 3 \left(\sum_{i=0}^N x_i^2 \right)^2 + 8 \sum_{i=0}^N x_i \sum_{i=0}^N x_i^3 - 6 \sum_{i=0}^N x_i^4. \end{aligned}$$

[The author discovered this formula recently, but feels that it is unlikely to be new.]

If in (1) we set $x_i = q^{(2i+1)^2}$, let $N \rightarrow \infty$ and use the fact that

$$\sum_{i \geq 0} q^{(2i+1)^2} = q\psi(q^8)$$

where

$$\psi(q) = \sum_{i \geq 0} q^{(i^2+i)/2},$$

we obtain

(2)

$$\sum_{n \geq 0} p_{4o}(n)q^n = \frac{1}{24} (q^4\psi(q^8)^4 - 6q^4\psi(q^8)^2\psi(q^{16}) + 3q^4\psi(q^{16})^2 + 8q^4\psi(q^8)\psi(q^{24}) - 6q^4\psi(q^{32}))$$

or,

$$\sum_{n \equiv 4 \pmod{8}} p_{4o}(n)q^n = \frac{1}{24} (q^4\psi(q^8)^4 - 6q^4\psi(q^8)^2\psi(q^{16}) + 3q^4\psi(q^{16})^2 + 8q^4\psi(q^8)\psi(q^{24}) - 6q^4\psi(q^{32}))$$

since the series in (2) contains only powers of q which are 4 modulo 8.

If in (1) we set $x_i = q^{(2i)^2}$, let $N \rightarrow \infty$ and use the fact that

$$\sum_{i \geq 0} q^{(2i)^2} = \frac{1}{2}(\phi(q^4) + 1)$$

where

$$\phi(q) = \sum_{i=-\infty}^{\infty} q^{i^2},$$

we obtain, after simplification,

(3)

$$\begin{aligned} \sum_{n \geq 0} p_{4e}(n)q^n &= \frac{1}{384} (\phi(q^4)^4 + 4\phi(q^4)^3 - 6\phi(q^4)^2 + 12\phi(q^4) - 15 + 12\phi(q^8)^2 + 12\phi(q^8) \\ &\quad + 32\phi(q^{12}) - 48\phi(q^{16}) - 12\phi(q^4)^2\phi(q^8) - 24\phi(q^4)\phi(q^8) + 32\phi(q^4)\phi(q^{12})). \end{aligned}$$

If in (1) we set $x_i = q^{(2i+2)^2}$, let $N \rightarrow \infty$ and use the fact that

$$\sum_{i \geq 0} q^{(2i+2)^2} = \frac{1}{2}(\phi(q^4) - 1),$$

we obtain

$$(4) \quad \sum_{n \geq 0} p_{4e+}(n)q^n = \frac{1}{384} (\phi(q^4)^4 - 4\phi(q^4)^3 + 18\phi(q^4)^2 - 60\phi(q^4) + 105 + 12\phi(q^8)^2 - 36\phi(q^8) - 32\phi(q^{12}) - 48\phi(q^{16}) - 12\phi(q^4)^2\phi(q^8) + 24\phi(q^4)\phi(q^8) + 32\phi(q^4)\phi(q^{12})).$$

It follows from (3) and (4) that

$$\sum_{n \geq 0} (p_{4e}(n) + p_{4e+}(n))q^n = \frac{1}{192} (\phi(q^4)^4 + 6\phi(q^4)^2 - 24\phi(q^4) + 45 + 12\phi(q^8)^2 - 12\phi(q^8) - 48\phi(q^{16}) - 12\phi(q^4)^2\phi(q^8) + 32\phi(q^4)\phi(q^{12})).$$

We now extract the terms in which $n \equiv 4 \pmod{8}$. We use the fact that

$$\phi(q) = \phi(q^4) + 2q\psi(q^8)$$

to write

$$\begin{aligned} \sum_{n \geq 0} (p_{4e}(n) + p_{4e+}(n))q^n &= \frac{1}{192} ((\phi(q^{16}) + 2q^4\psi(q^{32}))^4 + 6(\phi(q^{16}) + 2q^4\psi(q^{32}))^2 \\ &\quad - 24(\phi(q^{16}) + 2q^4\psi(q^{32})) + 45 + 12\phi(q^8)^2 - 12\phi(q^8) - 48\phi(q^{16}) \\ &\quad - 12(\phi(q^{16}) + 2q^4\psi(q^{32}))^2\phi(q^8) \\ &\quad + 32(\phi(q^{16}) + 2q^4\psi(q^{32}))(\phi(q^{48}) + 2q^{12}\psi(q^{96}))). \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{n \equiv 4 \pmod{8}} (p_{4e}(n) + p_{4e+}(n))q^n \\ &= \frac{1}{24} (q^4\phi(q^{16})^3\psi(q^{32}) + 4q^{12}\phi(q^{16})\psi(q^{32})^3 + 3q^4\phi(q^{16})\psi(q^{32}) - 6q^4\psi(q^{32}) \\ &\quad - 6q^4\phi(q^8)\phi(q^{16})\psi(q^{32}) + 8q^4\phi(q^{48})\psi(q^{32}) + 8q^{12}\phi(q^{16})\psi(q^{96})) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{24} (q^4 \phi(q^8)^2 \phi(q^{16}) \psi(q^{32}) - 6q^4 \phi(q^8) \phi(q^{16}) \psi(q^{32}) + 3q^4 \phi(q^{16}) \psi(q^{32}) \\
&\quad + 8q^4 \psi(q^8) \psi(q^{24}) - 6q^4 \psi(q^{32})).
\end{aligned}$$

Here we have used the facts that

$$\phi(q^2)^2 + 4q\psi(q^4)^2 = \phi(q)^2$$

and

$$\phi(q^6)\psi(q^4) + q\phi(q^2)\psi(q^{12}) = \psi(q)\psi(q^3).$$

So our theorem is equivalent to the identity

$$\psi(q)^4 - 6\psi(q)^2\psi(q^2) + 3\psi(q^2) = \phi(q)^2\phi(q^2)\psi(q^4) - 6\phi(q)\phi(q^2)\psi(q^4) + 3\phi(q^2)\psi(q^4).$$

This is straightforward, since, with $(q)_\infty = \prod_{n \geq 1} (1 - q^n)$,

$$\begin{aligned}
\psi(q) &= \frac{(q^2)_\infty^2}{(q)_\infty}, \quad \phi(q) = \frac{(q^2)_\infty^5}{(q)_\infty^2 (q^4)_\infty}, \\
\phi(q)^2 \phi(q^2) \psi(q^4) &= \frac{(q^2)_\infty^{10}}{(q)_\infty^4 (q^4)_\infty^4} \cdot \frac{(q^4)_\infty^5}{(q^2)_\infty^2 (q^8)_\infty^2} \cdot \frac{(q^8)_\infty^2}{(q^4)_\infty} = \frac{(q^2)_\infty^8}{(q)_\infty^4} = \psi(q)^4, \\
\phi(q) \phi(q^2) \psi(q^4) &= \frac{(q^2)_\infty^5}{(q)_\infty^2 (q^4)_\infty^2} \cdot \frac{(q^4)_\infty^5}{(q^2)_\infty^2 (q^8)_\infty^2} \cdot \frac{(q^8)_\infty^2}{(q^4)_\infty} = \frac{(q^2)_\infty^3 (q^4)_\infty^2}{(q)_\infty^2} = \frac{(q^2)_\infty^4}{(q)_\infty^2} \cdot \frac{(q^4)_\infty^2}{(q^2)_\infty} \\
&= \psi(q)^2 \psi(q^2)
\end{aligned}$$

and

$$\phi(q^2)\psi(q^4) = \frac{(q^4)_\infty^5}{(q^2)_\infty^2 (q^8)_\infty^2} \cdot \frac{(q^8)_\infty^2}{(q^4)_\infty} = \frac{(q^4)_\infty^4}{(q^2)_\infty^2} = \psi(q^2)^2. \blacksquare$$

References

- [1] M. D. Hirschhorn, Some formulae for partitions into squares, *Discrete Math*, 211(2000),225–228.
- [2] M. D. Hirschhorn and J. A. Sellers, Some relations for partitions into four squares, in *Special Functions 1999, Proceedings of the International Workshop on Special Functions*, Hong Kong, World Scientific.