RESULTS OF HURWITZ TYPE FOR FIVE OR MORE SQUARES

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Abstract

Let $r_k(n)$ denote the number of representations of n as a sum of k squares. We give many cases (for $k = 5,\ 6,\ 7$) in which the generating function $\sum_{n\geq 0} r_k(an+b)q^n$ is a simple infinite product. For instance,

$$\sum_{n\geq 0} r_7(24n+23)q^n = 49728 \prod_{n\geq 1} \frac{(1-q^{2n})^{10}(1-q^{3n})^3}{(1-q^n)^6}.$$

keywords

sums of squares, generating function, simple infinite product

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1. Introduction and statement of results

Let $p(n)$ denote the number of partitions of n, and $r_k(n)$ the number of representations of n as a sum of k squares. Inspired by Ramanujan's

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celebrated result

$$\sum_{n\geq 0} p(5n+4)q^n = 5\prod_{n\geq 1}\frac{(1-q^{5n})^5}{(1-q^n)^6},$$

Hirschhorn and McGowan [4] found many instances in which, for $k=2$ or 4, the generating function $\sum_{n\geq 0} r_k(an+b)q^n$ can be written as a product. Cooper and Hirschhorn [1] have since found many similar results when $k=3$, some of which are classical, having been given by Hurwitz [5].

In this paper, we give results of the same type, most discovered by computer search, for $k = 5,\ 6$ and 7. We have found no further such results for $k=5,\ 6,\ 7$, nor indeed for $k>7$.

Before stating our results, we need some definitions. Let

$$\phi(q) = \sum_{-\infty}^{\infty} q^{n^2}, \quad \psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2},$$

$$X(q) = \sum_{-\infty}^{\infty} q^{3n^2+2n}, \quad P(q) = \sum_{-\infty}^{\infty} q^{(3n^2+n)/2},$$

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\right),$$

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} = 3\frac{(q^3)^3_\infty}{(q)_\infty}.$$

Then by Jacobi's triple product identity, each of $\phi(q)$, $\psi(q)$, $X(q)$ and $P(q)$ is a product. For properties of $a(q),\ c(q)$, see [2].

We prove the following.

Theorem 1:

(i) $$\sum_{n\geq 0} r_5(4^\lambda(4n+2))q^n = \frac{40(4\times 8^\lambda+3)}{7}\phi(q)^3\psi(q^2)^2,$$

(ii) $$\sum_{n\geq 0} r_5(4^\lambda(4n+3))q^n = \frac{80(4\times 8^\lambda+3)}{7}\phi(q)^2\psi(q^2)^3,$$

(iii) $$\sum_{n\geq 0} r_5(4^\lambda(8n+3))q^n = \frac{80(4\times 8^\lambda+3)}{7}\phi(q)^2\psi(q)^3,$$

(iv) $$\sum_{n\geq 0} r_5(4^\lambda(8n+5))q^n = \frac{16(5\times 8^{\lambda+1}+9)}{7}\psi(q)^5,$$

(v) $$\sum_{n\geq 0} r_5(4^\lambda(8n+7))q^n = \frac{320(4\times 8^\lambda+3)}{7}\psi(q)^3\psi(q^2)^2,$$

(vi) $$\sum_{n\geq 0} r_5(4^\lambda(24n+13))q^n = \frac{80(5\times 8^{\lambda+1}+9)}{7}\psi(q)^4P(q),$$

(vii) $$\sum_{n\geq 0} r_5(4^\lambda(24n+17))q^n = \frac{480(8^{\lambda+1}-1)}{7}\psi(q)^3\{c(q)/3\},$$

(viii) $$\sum_{n\geq 0} r_5(4^\lambda(72n+69))q^n = \frac{960(5\times 8^{\lambda+1}+9)}{7}\psi(q)^2\psi(q^3)\left\{c(q)/3\right\},$$

(ix) $$\sum_{n\geq 0} r_5(9^\lambda(3n+1))q^n = \frac{10(12\times 27^\lambda+1)}{13}\phi(q)^4 X(q),$$

(x) $$\sum_{n\geq 0} r_5(9^\lambda(9n+6))q^n = \frac{120(27^{\lambda+1}-1)}{13}\phi(q)\phi(q^3)^2 X(q)^3,$$

(xi) $$\sum_{n\geq 0} r_5(9^\lambda(24n+13))q^n = \frac{560(12\times 27^\lambda+1)}{13}\psi(q)^4 P(q),$$

(xii) $$\sum_{n\geq 0} r_5(9^\lambda(24n+17))q^n = \frac{480(15\times 27^\lambda-2)}{13}\psi(q)^3\{c(q)/3\},$$

(xiii) $$\sum_{n\geq 0} r_5(9^\lambda(72n+69))q^n = \frac{3360(27^{\lambda+1}-1)}{13}\psi(q)^2\psi(q^3)\left\{c(q)/3\right\}.$$

Theorem 2:

$$\sum_{n\geq 0} r_6(2^\lambda(4n+3))q^n = 32(4^{\lambda+1}+1)\psi(q)^6.$$

Theorem 3:

(i) $$\sum_{n\geq 0} r_7(4^\lambda(8n+7))q^n = \frac{64(35\times 32^{\lambda+1}+27)}{31}\psi(q)^7,$$

(ii) $$\sum_{n\geq 0} r_7(4^\lambda(24n+23))q^n = \frac{1344(35\times 32^{\lambda+1}+27)}{31}\psi(q)^5\left\{c(q)/3\right\},$$

(iii) $$\sum_{n\geq 0} r_7(9^\lambda(3n+2))q^n = \frac{84(117\times 243^\lambda+4)}{121}\phi(q)^5\left\{c(q^2)/3\right\},$$

(iv) $$\sum_{n\geq 0} r_7(9^\lambda(24n+23))q^n = \frac{49728(117\times 243^\lambda+4)}{121}\psi(q)^5\left\{c(q)/3\right\}.$$

2. Preliminary lemmas

We require the following Lemmas.

(i) $$\phi(q)\psi(q^2)=\psi(q)^2$$

(ii) $$\phi(q)=\phi(q^4)+2q\psi(q^8)$$

(iii) $$\phi(q)^2=\phi(q^2)^2+4q\psi(q^4)^2$$

(iv) $$\phi(q)=\phi(q^9)+2qX(q^3)$$

(v) $$(q)_\infty(\omega q)_\infty(\omega^2 q)_\infty=\frac{(q^3)_\infty^4}{(q^9)_\infty}$$

(vi) $$8q^3X(q^3)^3=\frac{\phi(q^3)^4}{\phi(q^9)}-\phi(q^9)^3$$

(vii) $$\phi(q)^2=\phi(q^9)^2+4q\phi(q^9)X(q^3)+4q^2X(q^3)^3$$

(viii) $$\phi(q)^3 = \frac{\phi(q^3)^4}{\phi(q^9)} + 6q\phi(q^9)^2X(q^3) + 12q^2\phi(q^9)X(q^3)^2$$

(ix)
$$\phi(q)^4 = \left(4\phi(q^3)^4 - 3\phi(q^9)^4\right) + q\left(2\frac{\phi(q^3)^4}{\phi(q^9)} + 6\phi(q^9)^3\right)X(q^3) + q^2\left(24\phi(q^9)^2\right)X(q^3)^2$$

(x) $$\phi(q)^5 = \left(10\phi(q^9)\phi(q^3)^4 - 9\phi(q^9)^5\right) + q\left(10\phi(q^3)^4\right)X(q^3) + q^2\left(4\frac{\phi(q^3)^4}{\phi(q^9)} + 36\phi(q^9)^3\right)X(q^3)^2$$

(xi) $$\phi(q)^6 = \left(\frac{\phi(q^3)^8}{\phi(q^9)^2} + 18\phi(q^9)^2\phi(q^3)^4 - 18\phi(q^9)^6\right) + q\left(30\phi(q^9)\phi(q^3)^4 - 18\phi(q^9)^5\right)X(q^3) + q^2\left(24\phi(q^3)^4 + 36\phi(q^9)^4\right)X(q^3)^2$$

(xii) $$\phi(q)^7 = \left(7\frac{\phi(q^3)^8}{\phi(q^9)} + 21\phi(q^9)^3\phi(q^3)^4 - 27\phi(q^9)^7\right) + q\left(2\frac{\phi(q^3)^8}{\phi(q^9)^2} + 66\phi(q^9)^2\phi(q^3)^4 - 54\phi(q^9)^6\right)X(q^3) + q^2\left(84\phi(q^9)\phi(q^3)^4\right)X(q^3)^2$$

(xiii)
$$\phi(q)^8 = \left(28\phi(q^3)^8 - 27\phi(q^9)^9\right) + q\left(16\frac{\phi(q^3)^8}{\phi(q^9)} + 108\phi(q^9)^3\phi(q^3)^4 - 108\phi(q^9)^7\right)X(q^3) + q^2\left(4\frac{\phi(q^3)^8}{\phi(q^9)^2} + 216\phi(q^9)^2\phi(q^3)^4 - 108\phi(q^9)^6\right)X(q^3)^2$$

(xiv) $$\frac{1}{\phi(q)} = \frac{\phi(q^9)}{\phi(q^3)^4}\left(\phi(q^9)^2 - 2q\phi(q^9)X(q^3) + 4q^2X(q^3)^2\right)$$

(xv) $$\psi(q) = P(q^3) + q\psi(q^9)$$

(xvi) $$P(q)^3 = \frac{\psi(q)^4}{\psi(q^3)} - q\psi(q^3)^3$$

(xvii) $$X(q)^3 = X(q^3)a(q^6) + 3q\phi(q^9)\{c(q^6)/3\} + 3q^2X(q^3)\{c(q^6)/3\}$$

(xviii) $$P(q)^3 = P(q^3)a(q^3) + 3qP(q^3)\{c(q^3)/3\} + 6q^2\psi(q^9)\{c(q^3)/3\}$$

(xix) $$\begin{aligned}\phi(q)^4 = &\left(\phi(q^3)^4 + 24\phi(q^3)X(q^3)\{c(q^6)/3\}\right)\\ &+ 8q\phi(q^3)X(q^3)a(q^6) + 24q^2\phi(q^3)\phi(q^9)\{c(q^6)/3\}\end{aligned}$$

(xx) $$\begin{aligned}\psi(q)^4 = &\ \psi(q^3)P(q^3)a(q^3)\\ &+ q\left(3\psi(q^3)P(q^3)\{c(q^3)/3\} + \psi(q^3)^4\right)\\ &+ 6q^2\psi(q^3)\psi(q^9)\{c(q^3)/3\}\end{aligned}$$

(xxii) $$\phi(q)X(q) + 2\psi(q^2)P(q^2) = 3\{c(q)/3\}$$

(xxii) $$\phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6) = a(q)$$

(xxiii)
$$\begin{aligned}\phi(q)^4 + 16q\psi(q^2)^4 = &\left(\phi(q^3)^4 + 16q^3\psi(q^6)^4 + 72q^3\{c(q^3)/3\}\{c(q^6)/3\}\right)\\ &+ 24qa(q^6)\{c(q^3)/3\} + 24q^2a(q^3)\{c(q^6)/3\}\end{aligned}$$

(xxiv) $$1+3\sum_{n\geq 1}\left(\frac{q^{6n-5}}{1-q^{6n-5}}-\frac{q^{6n-1}}{1-q^{6n-1}}\right)=\frac{\psi(q)^3}{\psi(q^3)}$$

(xxv) $$a(q)+a(q^2)=2\frac{\psi(q)^3}{\psi(q^3)}$$

(xxvi) $$P(q)\{c(q^2)/3\}=\psi(q^3)\{c(q)/3\}$$

(xxvii) $$\phi(q^3)X(q)^2=\phi(q)\{c(q^2)/3\}$$

(xxviii) $$\psi(q^3)P(q)^2=\psi(q)\{c(q)/3\}$$

Proofs:

(i)

$$\begin{aligned}\phi(q)\psi(q^2)&=\sum_{-\infty}^{\infty}q^{m^2}\sum_{n\geq 0}q^{n^2+n}\\&=\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\\&=(q^2)_\infty^2\prod_{n\geq 1}(1+q^n)^2\\&=(q^2)_\infty^2\frac{(q^2)_\infty^2}{(q)_\infty^2}=\frac{(q^2)_\infty^4}{(q)_\infty^2}=\psi(q)^2.\end{aligned}$$

(ii) 2–dissect $\phi(q)$. Thus

$$\phi(q)=\sum_{n\text{ even}}q^{n^2}+\sum_{n\text{ odd}}q^{n^2}=\sum_{-\infty}^{\infty}q^{4n^2}+\sum_{-\infty}^{\infty}q^{4n^2+4n+1}=\phi(q^4)+2q\psi(q^8).$$

(iii)

$$\begin{aligned}\phi(q^4)^2 + 4q^2\psi(q^8)^2 &= \sum q^{4m^2+4n^2} + \sum q^{(2m+1)^2+(2n+1)^2} \\ &= \sum_{u\equiv v \pmod 2} q^{u^2+v^2} \\ &= \sum q^{(r+s)^2+(r-s)^2} \\ &= \sum q^{2r^2+2s^2} \\ &= \phi(q^2)^2.\end{aligned}$$

Now put q for q^2.

(iv) 3–dissect $\phi(q)$. Thus

$$\begin{aligned}\phi(q) &= \sum_{n\equiv 0 \pmod 3} q^{n^2} + \sum_{n\equiv 1 \pmod 3} q^{n^2} + \sum_{n\equiv -1 \pmod 3} q^{n^2} \\ &= \sum_{-\infty}^{\infty} q^{9n^2} + q\sum_{-\infty}^{\infty} q^{9n^2+6n} + q\sum_{-\infty}^{\infty} q^{9n^2-6n} \\ &= \phi(q^9) + 2qX(q^3).\end{aligned}$$

(v)

$$(q)_\infty(\omega q)_\infty(\omega^2 q)_\infty = \prod_{3|n}(1-q^n)^3 \prod_{3\nmid n}(1-q^{3n}) = (q^3)_\infty^3 \frac{(q^3)_\infty}{(q^9)_\infty} = \frac{(q^3)_\infty^4}{(q^9)_\infty}.$$

(vi) Put $q,\ \omega q,\ \omega^2 q$ for q in (iv), multiply the three results and use (v). Thus

$$\begin{aligned}\phi(q^9)^3 + 8q^3X(q^3)^3 &= \phi(q)\phi(\omega q)\phi(\omega^2 q) \\ &= \frac{(q^2)_\infty^5}{(q)_\infty^2(q^4)_\infty^2}\frac{(\omega^2 q^2)_\infty^5}{(\omega q)_\infty^2(\omega q^4)_\infty^2}\frac{(\omega q^2)_\infty^5}{(\omega^2 q)_\infty^2(\omega^2 q^4)_\infty^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(q^2)_\infty^5(\omega q^2)_\infty^5(\omega^2 q^2)_\infty^5}{(q)_\infty^2(\omega q)_\infty^2(\omega^2 q)_\infty^2(q^4)_\infty^2(\omega q^4)_\infty^2(\omega^2 q^4)_\infty^2} \\
&= \frac{(q^6)_\infty^{20}}{(q^{18})_\infty^5}\cdot\frac{(q^9)_\infty^2}{(q^3)_\infty^8}\cdot\frac{(q^{36})_\infty^2}{(q^{12})_\infty^8} \\
&= \left(\frac{(q^6)_\infty^5}{(q^3)_\infty^2(q^{12})_\infty^2}\right)^4 \Big/ \left(\frac{(q^{18})_\infty^5}{(q^9)_\infty^2(q^{36})_\infty^2}\right) \\
&= \frac{\phi(q^3)^4}{\phi(q^9)}.
\end{aligned}
$$

(vii) Square (iv).

(viii) Multiply (vii) by (iv), then use (vi).

(ix) Multiply (viii) by (iv), then use (vi).

(x) Multiply (ix) by (iv), then use (vi).

(xi) Multiply (x) by (iv), then use (vi).

(xii) Multiply (xi) by (iv), then use (vi).

(xiii) Multiply (xii) by (iv), then use (vi).

(xiv)

$$
\begin{aligned}
\frac{1}{\phi(q)} &= \frac{\phi(\omega q)\phi(\omega^2 q)}{\phi(q)\phi(\omega q)\phi(\omega^2 q)} \\
&= \frac{\phi(q^9)}{\phi(q^3)^4}\left(\phi(q^9)+2\omega q X(q^3)\right)\left(\phi(q^9)+2\omega^2 q X(q^3)\right) \\
&= \frac{\phi(q^9)}{\phi(q^3)^4}\left(\phi(q^9)^2-2q\phi(q^9)X(q^3)+4q^2X(q^3)^2\right).
\end{aligned}
$$

(xv) 2–dissect $X(q)$.

(xvi) Put q, ωq and $\omega^2 q$ for q in (xv), multiply the three results and use (v).

(xvii)

$$
\begin{aligned}
q^3X(q^3)^3 &= \sum_{k,l,m=-\infty}^{\infty} q^{(3k+1)^2+(3l+1)^2+(3m+1)^2} \\
&= \sum_{a\equiv b\equiv c\equiv 1 \pmod 3} q^{a^2+b^2+c^2}.
\end{aligned}
$$

Now, $a+b+c=3r$ for some r, so let

$$a=r+s, \quad b=r+t, \quad c=r+u.$$

Then

$$s+t+u=0, \quad s\equiv t\equiv u\equiv -r+1 \pmod 3$$

and

$$
\begin{aligned}
q^3X(q^3)^3 &= \sum q^{(r+s)^2+(r+t)^2+(r+u)^2} \\
&= \sum q^{3r^2+s^2+t^2+u^2} \\
&= \sum_{r=-\infty}^{\infty} q^{3r^2} \sum_{s\equiv t\equiv -r+1 \pmod 3} q^{2s^2+2st+2t^2} \\
&= \sum_{-\infty}^{\infty} q^{3(3r)^2} \sum q^{2(3a+1)^2+2(3a+1)(3b+1)+2(3b+1)^2} \\
&+ \sum_{-\infty}^{\infty} q^{3(3r+1)^2} \sum q^{2(3a)^2+2(3a)(3b)+2(3b)^2} \\
&+ \sum_{-\infty}^{\infty} q^{3(3r-1)^2} \sum q^{2(3a-1)^2+2(3a-1)(3b-1)+2(3b-1)^2} \\
&= q^6\phi(q^{27})c(q^{18}) + q^3X(q^9)a(q^{18}) + q^9X(q^9)c(q^{18}) \\
&= q^3X(q^9)a(q^{18}) + 3q^6\phi(q^{27})\{c(q^{18}/3\} + 3q^9X(q^9)\{c(q^{18})/3\}.
\end{aligned}
$$

(xviii)

$$q^3P(q^{24})^3 = \sum_{k,l,m=-\infty}^{\infty} q^{(6k+1)^2+(6l+1)^2+(6m+1)^2}$$
$$= \sum_{a\equiv b\equiv c\equiv 1 \pmod 6} q^{a^2+b^2+c^2}.$$

Now, $a+b+c=6r+3$ for some r, so let

$$a=2r+1+s, \quad b=2r+1+t, \quad c=2r+1+u.$$

Then

$$s+t+u=0, \quad s\equiv t\equiv u\equiv -2r \pmod 6$$

and

$$\begin{aligned}
q^3P(q^{24})^3 &= \sum q^{(2r+1+s)^2+(2r+1+t)^2+(2r+1+u)^2}\\
&= \sum q^{12r^2+12r+3+s^2+t^2+u^2}\\
&= \sum_{r=-\infty}^{\infty} q^{12r^2+12r+3} \sum_{s\equiv t\equiv -2r \pmod 6} q^{2s^2+2st+2t^2}\\
&= \sum_{-\infty}^{\infty} q^{12(3r)^2+12(3r)+3} \sum q^{2(6a)^2+2(6a)(6b)+2(6b)^2}\\
&+ \sum_{-\infty}^{\infty} q^{12(3r-1)^2+12(3r-1)+3} \sum q^{2(6a+2)^2+2(6a+2)(6b+2)+2(6b+2)^2}\\
&+ \sum_{-\infty}^{\infty} q^{12(3r+1)^2+12(3r+1)+3} \sum q^{2(6a-2)^2+2(6a-2)(6b-2)+2(6b-2)^2}\\
&= q^3P(q^{72})a(q^{72}) + q^{27}P(q^{72})c(q^{72}) + 2q^{51}\psi(q^{216})c(q^{72})\\
&= q^3P(q^{72})a(q^{72}) + 3q^{27}P(q^{72})\{c(q^{72})/3\} + 6q^{51}\psi(q^{216})\{c(q^{72})/3\}.
\end{aligned}$$

(xix) From lemmas (vi) and (xvii),

$$\begin{aligned}\phi(q)^4 &= \phi(q^3)\left(\phi(q^3)^3 + 8qX(q)^3\right)\\ &= \left(\phi(q^3)^4 + 24q^3\phi(q^3)X(q^3)\{c(q^6)/3\}\right)\\ &+ 8q\phi(q^3)X(q^3)a(q^6) + 24q^2\phi(q^3)\phi(q^9)\{c(q^6)/3\}.\end{aligned}$$

(xx) From lemmas (xvi) and (xviii),

$$\begin{aligned}\psi(q)^4 &= \psi(q^3)\left(P(q)^3 + q\psi(q^3)^3\right)\\ &= \psi(q^3)P(q^3)a(q^3)\\ &+ q\left(3\psi(q^3)P(q^3)\{c(q^3)/3\} + \psi(q^3)^4\right)\\ &+ 6q^2\psi(q^3)\psi(q^9)\{c(q^3)/3\}.\end{aligned}$$

(xxi)

$$\begin{aligned}q^4\Big(\phi(q^{12})X(q^{12}) + 2\psi(q^{24})P(q^{24})\Big)&\\ &= \sum q^{3(2m)^2+(6n+2)^2} + \sum q^{3(2m+1)^2+(6n-1)^2}\\ &= \sum_{u\not\equiv v \pmod 2} q^{3u^2+(3v-1)^2}\\ &= \sum q^{3(r-s)^2+(3r+3s+2)^2}\\ &= q^4\sum q^{12r^2+12rs+12s^2+12r+12s}\\ &= q^4 c(q^{12})\\ &= 3q^4\{c(q^{12})/3\}.\end{aligned}$$

(xxii)

$$\phi(q^4)\phi(q^{12}) + 4q^4\psi(q^8)\psi(q^{24}) = \sum q^{4m^2+12n^2} + \sum q^{(2m+1)^2+3(2n+1)^2}$$

$$
\begin{aligned}
&= \sum_{u \equiv v \pmod 2} q^{u^2+3v^2} \\
&= \sum q^{(r-s)^2+3(r+s)^2} \\
&= \sum q^{4r^2+4rs+4s^2} \\
&= a(q^4).
\end{aligned}
$$

(xxiii) From lemmas (xix) and (xx), (xxi) and (xxii),

$$
\begin{aligned}
&\phi(q)^4 + 16q\psi(q^2)^4 \\
&= \Big(\phi(q^3)^4 + 24q^3\phi(q^3)X(q^3)\{c(q^6)/3\} \\
&\qquad + 16q^3\left(3\psi(q^6)P(q^6)\{c(q^6)/3\} + \psi(q^6)^4\right)\Big) \\
&+ q\Big(8\phi(q^3)X(q^3)a(q^6) + 16\psi(q^6)P(q^6)a(q^6)\Big) \\
&+ q^2\Big(24\phi(q^3)\phi(q^9)\{c(q^6)/3\} + 96q^3\psi(q^6)\psi(q^{18})\{c(q^6)/3\}\Big) \\
&= \Big(\phi(q^3)^4 + 16q^3\psi(q^6)^4 \\
&\qquad + 24q^3\{c(q^6)/3\}\left(\phi(q^3)X(q^3) + 2\psi(q^6)P(q^6)\right)\Big) \\
&+ 8qa(q^6)\Big(\phi(q^3)X(q^3) + 2\psi(q^6)P(q^6)\Big) \\
&+ 24q^2\{c(q^6)/3\}\Big(\phi(q^3)\phi(q^9) + 4q^3\psi(q^6)\psi(q^{18})\Big) \\
&= \Big(\phi(q^3)^4 + 16q^3\psi(q^6)^4 + 72q^3\{c(q^3)/3\}\{c(q^6)/3\}\Big) \\
&+ 24qa(q^6)\{c(q^3)/3\} + 24q^2a(q^3)\{c(q^6)/3\}.
\end{aligned}
$$

(xxiv) See [3, (8)].

(xxv) We have

$$\begin{aligned} a(q)+a(q^2) &= 1+6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\\ &\qquad +1+6\sum_{n\geq 1}\left(\frac{q^{6n-4}}{1-q^{6n-4}}-\frac{q^{6n-2}}{1-q^{6n-2}}\right)\\ &= 2+6\sum_{n\geq 1}\left(\frac{q^{6n-5}}{1-q^{6n-5}}+\frac{q^{6n-2}}{1-q^{6n-2}}-\frac{q^{6n-4}}{1-q^{6n-4}}-\frac{q^{6n-1}}{1-q^{6n-1}}\right)\\ &\qquad +6\sum_{n\geq 1}\left(\frac{q^{6n-4}}{1-q^{6n-4}}-\frac{q^{6n-2}}{1-q^{6n-2}}\right)\\ &= 2\left\{1+3\sum_{n\geq 1}\left(\frac{q^{6n-5}}{1-q^{6n-5}}-\frac{q^{6n-1}}{1-q^{6n-1}}\right)\right\}\\ &= 2\frac{\psi(q)^3}{\psi(q^3)}. \end{aligned}$$

(xxvi),(xxvii),(xxviii)

$$\phi(q)=\frac{(q^2)_\infty^5}{(q)_\infty^2(q^4)_\infty^2},\quad \psi(q)=\frac{(q^2)_\infty^2}{(q)_\infty},\quad X(q)=\frac{(q^2)_\infty^2(q^3)_\infty(q^{12})_\infty}{(q)_\infty(q^4)_\infty(q^6)_\infty},$$

$$P(q)=\frac{(q^2)_\infty(q^3)_\infty^2}{(q)_\infty(q^6)_\infty},\quad \{c(q)/3\}=\frac{(q^3)_\infty^3}{(q)_\infty}.$$

The results follow.

3. Proof of Theorem 1

(i)—(v) We start by proving that for $\lambda \geq 0$

$$(\alpha)\qquad \sum_{n\geq 0} r_5(4^\lambda n)q^n = \phi(q)^5+\frac{80(8^\lambda-1)}{7}q\phi(q)\psi(q^2)^4.$$

(α) is true for $\lambda = 0$. Suppose (α) true for some $\lambda \geq 0$. Then

$$\sum_{n\geq 0} r_5(4^\lambda n)q^n = \left(\phi(q^4) + 2q\psi(q^8)\right)^5 + \frac{80(8^\lambda - 1)}{7} q\left(\phi(q^4) + 2q\psi(q^8)\right)\phi(q^2)^2\psi(q^4)^2$$

$$= \left(\phi(q^4) + 2q\psi(q^8)\right)^5 + \frac{80(8^\lambda - 1)}{7} q\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^4)^2 + 4q^2\psi(q^8)^2\right)\phi(q^4)\psi(q^8)$$

$$= \phi(q^4)^5 + 10q\phi(q^4)^4\psi(q^8) + 40q^2\phi(q^4)^3\psi(q^8)^2 + 80q^3\phi(q^4)^2\psi(q^8)^3 + 80q^4\phi(q^4)\psi(q^8)^4 + 32q^5\psi(q^8)^5$$
$$+ \frac{80(8^\lambda - 1)}{7}\left(q\phi(q^4)^4\psi(q^8) + 2q^2\phi(q^4)^3\psi(q^8)^2 + 4q^3\phi(q^4)^2\psi(q^8)^3 + 8q^4\phi(q^4)\psi(q^8)^4\right)$$

$$= \phi(q^4)^5 + \frac{10(8^{\lambda+1} - 1)}{7} q\phi(q^4)^4\psi(q^8) + \frac{40(4\times 8^\lambda + 3)}{7} q^2\phi(q^4)^3\psi(q^8)^2$$
$$+ \frac{80(4\times 8^\lambda + 3)}{7} q^3\phi(q^4)^2\psi(q^8)^3 + \frac{80(8^{\lambda+1} - 1)}{7} q^4\phi(q^4)\psi(q^8)^4 + 32q^5\psi(q^8)^5.$$

It follows that

$$\sum_{n\geq 0} r_5(4^{\lambda+1} n)q^n = \phi(q)^5 + \frac{80(8^{\lambda+1} - 1)}{7} q\phi(q)\psi(q^2)^4.$$

Thus (α) follows by induction. Also

$$\sum_{n\geq 0} r_5(4^\lambda(4n+1))q^n = \frac{10(8^{\lambda+1} - 1)}{7}\phi(q)^4\psi(q^2) + 32q\psi(q^2)^5,$$

$$\sum_{n\geq 0} r_5(4^\lambda(4n+2))q^n = \frac{40(4\times 8^\lambda+3)}{7}\phi(q)^3\psi(q^2)^2,$$

$$\sum_{n\geq 0} r_5(4^\lambda(4n+3))q^n = \frac{80(4\times 8^\lambda+3)}{7}\phi(q)^2\psi(q^2)^3.$$

Further,

$$\begin{aligned}\sum_{n\geq 0} r_5(4^\lambda(4n+1))q^n &= \frac{10(8^{\lambda+1}-1)}{7}\phi(q)^4\psi(q^2)+32q\psi(q^2)^5\\ &= \frac{10(8^{\lambda+1}-1)}{7}\left(\phi(q^2)^2+4q\psi(q^4)^2\right)^2\psi(q^2)+32q\psi(q^2)^5\end{aligned}$$

so

$$\begin{aligned}\sum_{n\geq 0} r_5(4^\lambda(8n+5))q^n &= \frac{10(8^{\lambda+1}-1)}{7}\cdot 8\phi(q)^2\psi(q^2)^2\cdot\psi(q)+32\psi(q)^5\\ &= \frac{16(5\times 8^{\lambda+1}+9)}{7}\psi(q)^5.\end{aligned}$$

Also

$$\sum_{n\geq 0} r_5(4^\lambda(4n+3))q^n = \frac{80(4\times 8^\lambda+3)}{7}\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\psi(q^2)^3$$

so

$$\sum_{n\geq 0} r_5(4^\lambda(8n+3))q^n = \frac{80(4\times 8^\lambda+3)}{7}\phi(q)^2\psi(q)^3,$$

$$\sum_{n\geq 0} r_5(4^\lambda(8n+7))q^n = \frac{320(4\times 8^\lambda+3)}{7}\psi(q)^3\psi(q^2)^2.$$

(vi) We have

$$\sum_{n\geq 0} r_5(4^\lambda(8n+5))q^n = \frac{16(5\times 8^{\lambda+1}+9)}{7}\left(P(q^3)+q\psi(q^9)\right)^5$$

from which it follows that

$$\begin{aligned}\sum_{n\geq 0} r_5(4^\lambda(24n+13))q^n &= \frac{16(5\times 8^{\lambda+1}+9)}{7}\cdot 5P(q)\psi(q^3)\left(P(q)^3+q\psi(q^3)^3\right)\\ &= \frac{80(5\times 8^{\lambda+1}+9)}{7}P(q)\psi(q)^4.\end{aligned}$$

(vii) We have

$$\begin{aligned}&\sum_{n\geq 0} r_5(4^\lambda(8n+1))q^n\\ &= \frac{10(8^{\lambda+1}-1)}{7}\left(\phi(q)^4+16q\psi(q^2)^4\right)\psi(q)\\ &= \frac{10(8^{\lambda+1}-1)}{7}\Big(\left(\phi(q^3)^4+16q^3\psi(q^6)^4+72q^3\{c(q^3)/3\}\{c(q^6)/3\}\right)\\ &\qquad + 24qa(q^6)\{c(q^3)/3\}+24q^2a(q^3)\{c(q^6)/3\}\Big)\\ &\qquad \times\Big(P(q^3)+q\psi(q^9)\Big).\end{aligned}$$

It follows that

$$\sum_{n\geq 0} r_5(4^\lambda(24n+17))q^n$$

$$= \frac{10(8^{\lambda+1}-1)}{7}\Big(24P(q)a(q)\{c(q^2)/3\} + 24\psi(q^3)a(q^2)\{c(q)/3\}\Big)$$
$$= \frac{240(8^{\lambda+1}-1)}{7}\psi(q^3)\{c(q)/3\}\left(a(q)+a(q^2)\right)$$
$$= \frac{240(8^{\lambda+1}-1)}{7}\psi(q^3)\{c(q)/3\}\cdot 2\frac{\psi(q)^3}{\psi(q^3)}$$
$$= \frac{480(8^{\lambda+1}-1)}{7}\psi(q)^3\{c(q)/3\}.$$

(viii) We have

$$\sum_{n\geq 0} r_5(4^\lambda(8n+5))q^n = \frac{16(5\times 8^{\lambda+1}+9)}{7}\psi(q)^5$$
$$= \frac{16(5\times 8^{\lambda+1}+9)}{7}\Big(P(q^3)+q\psi(q^9)\Big)^5.$$

It follows that

$$\sum_{n\geq 0} r_5(4^\lambda(24n+21))q^n = \frac{16(5\times 8^{\lambda+1}+9)}{7}\Big(10P(q)^3\psi(q^3)^2 + q\psi(q^3)^5\Big)$$
$$= \frac{16(5\times 8^{\lambda+1}+9)}{7}$$
$$\times\Big(10\left(P(q^3)a(q^3)+3qP(q^3)\{c(q^3)/3\}+6q^2\psi(q^9)\{c(q^3)/3\}\right)\psi(q^3)^2 + q\psi(q^3)^5\Big).$$

It follows that

$$\sum_{n\geq 0} r_5(4^\lambda(72n+69))q^n = \frac{960(5\times 8^{\lambda+1}+9)}{7}\psi(q)^2\psi(q^3)\{c(q)/3\}.$$

(ix), (x) We start by proving that for $\lambda \geq 0$

$$(\alpha)\qquad \sum_{n\geq 0} r_5(3^{2\lambda}n)q^n = \frac{15\times 27^\lambda - 2}{13}\phi(q)^5 - \frac{15(27^\lambda-1)}{13}\phi(q^3)^4\phi(q),$$

$$(\beta)\quad \sum_{n\ge 0} r_5(3^{2\lambda+1}n) = \frac{5(27^{\lambda+1}-1)}{13}\phi(q^3)\phi(q)^4 - \frac{5\times 27^{\lambda+1}-18}{13}\phi(q^3)^5.$$

(α) is true for $\lambda = 0$. Suppose (α) true for some $\lambda \ge 0$. Then

$$\begin{aligned}
\sum_{n\ge 0} r_5(3^{2\lambda}n)q^n = \frac{15\times 27^{\lambda}-2}{13}\Big\{ &\left(10\phi(q^9)\phi(q^3)^4 - 9\phi(q^9)^5\right) \\
&+ q\left(10\phi(q^3)^4 X(q^3)\right) \\
&+ q^2\left(4\frac{\phi(q^3)^4}{\phi(q^9)} + 36\phi(q^9)^3\right) X(q^3)^2 \Big\} \\
- \frac{15(27^{\lambda}-1)}{13}&\phi(q^3)^4\left(\phi(q^9) + 2qX(q^3)\right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{5(27^{\lambda+1}-1)}{13}\phi(q^9)\phi(q^3)^4 - \frac{5\times 27^{\lambda+1}-18}{13}\phi(q^9)^5\right) \\
&\quad + \frac{120\times 27^{\lambda}+10}{13} q\phi(q^3)^4 X(q^3) \\
&\quad + \frac{15\times 27^{\lambda}-2}{13} q^2\left(4\frac{\phi(q^3)^4}{\phi(q^9)} + 36\phi(q^9)^3\right) X(q^3)^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n\ge 0} r_5(3^{2\lambda+1}n)q^n &= \frac{5(27^{\lambda+1}-1)}{13}\phi(q^3)\phi(q)^4 - \frac{5\times 27^{\lambda+1}-18}{13}\phi(q^3)^5 \\
&= \frac{5(27^{\lambda+1}-1)}{13}\Big\{ \left(4\phi(q^3)^5 - 3\phi(q^9)^4\phi(q^3)\right) \\
&\qquad + q\left(2\frac{\phi(q^3)^5}{\phi(q^9)} + 6\phi(q^9)^3\phi(q^3)\right) X(q^3) \\
&\qquad + q^2\left(24\phi(q^9)^2\phi(q^3)X(q^3)^2\right) \Big\}
\end{aligned}$$

$$- \frac{5 \times 27^{\lambda+1} - 18}{13} \phi(q^3)^5$$

$$= \left(\frac{15 \times 27^{\lambda+1} - 2}{13} \phi(q^3)^5 - \frac{15(27^{\lambda+1} - 1)}{13} \phi(q^9)^4 \phi(q^3) \right)$$
$$+ \frac{5(27^{\lambda+1} - 1)}{13} q \left(2 \frac{\phi(q^3)^5}{\phi(q^9)} + 6\phi(q^9)^3 \phi(q^3) \right) X(q^3)$$
$$+ \frac{120(27^{\lambda+1} - 1)}{13} q^2 \phi(q^9)^2 \phi(q^3) X(q^3)^2.$$

It follows that

$$\sum_{n \geq 0} r_5(3^{2\lambda+2} n) q^n = \frac{15 \times 27^{\lambda+1} - 2}{13} \phi(q)^5 - \frac{15(27^{\lambda+1} - 1)}{13} \phi(q^3)^4 \phi(q).$$

Thus (α) and (β) follow by induction. Additionally,

$$\sum_{n \geq 0} r_5(9^\lambda (3n+1)) q^n = \frac{10(12 \times 27^\lambda + 1)}{13} \phi(q)^4 X(q)$$

and

$$\sum_{n \geq 0} r_5(9^\lambda (9n+6)) q^n = \frac{120(27^{\lambda+1} - 1)}{13} \phi(q^3)^2 \phi(q) X(q)^2.$$

(xi) We have

$$\sum_{n \geq 0} r_5(9^\lambda (3n+1)) q^n = \frac{10(12 \times 27^\lambda + 1)}{13} \phi(q)^4 X(q).$$

If we write

$$\phi(q)^4 X(q) = F_0(q^8) + qF_1(q^8) + \cdots + q^7 F_7(q^8)$$

then

$$\sum_{n\geq 0} r_5(9^\lambda(24n+13))q^n = \frac{10(12\times 27^\lambda+1)}{13}F_4(q).$$

If we compare this result for $\lambda=0$ with Theorem 1(vi) for $\lambda=0$ we find

$$\sum_{n\geq 0} r_5(24n+13)q^n = 10F_4(q) = 560\psi(q)^4P(q),$$

so

$$F_4(q) = 56\psi(q)^4P(q)$$

and

$$\sum_{n\geq 0} r_5(9^\lambda(24n+13))q^n = \frac{560(12\times 27^\lambda+1)}{13}\psi(q)^4P(q).$$

(xii) We have

$$\sum_{n\geq 0} r_5(9^\lambda(3n+2))q^n = \frac{15\times 27^\lambda-2}{13}\left(4\frac{\phi(q)^4}{\phi(q^3)}+36\phi(q^3)^3\right)X(q)^2.$$

If we write

$$\left(4\frac{\phi(q)^4}{\phi(q^3)}+36\phi(q^3)^3\right)X(q)^2 = G_0(q^8)+qG_1(q^8)+\ \cdots\ +q^7G_7(q^8)$$

we find

$$\sum_{n\geq 0} r_5(9^\lambda(24n+17))q^n = \frac{15\times 27^\lambda-2}{13}G_5(q).$$

If we compare this for $\lambda=0$ with Theorem 1(vii) for $\lambda=0$ we find

$$\sum_{n\geq 0} r_5(24n+17)q^n = G_5(q) = 480\psi(q)^3\{c(q)/3\}$$

and

$$\sum_{n\geq 0} r_5(9^\lambda(24n+17))q^n = \frac{480(15\times 27^\lambda - 2)}{13}\psi(q)^3\{c(q)/3\}.$$

(xiii) We have

$$\sum_{n\geq 0} r_5(3^{2\lambda+1}n)q^n = \frac{5(27^{\lambda+1}-1)}{13}\phi(q^3)\phi(q)^4 - \frac{5\times 27^{\lambda+1}-18}{13}\phi(q^3)^5$$

$$= \frac{5(27^{\lambda+1}-1)}{13}\Big(\big(\phi(q^3)^5 + 24q^3\phi(q^3)^2X(q^3)\{c(q^6)/3\}\big)$$

$$+ 8q\phi(q^3)^2X(q^3)a(q^6) + 24q^2\phi(q^3)^2\phi(q^9)\{c(q^6)/3\}\Big)$$

$$- \frac{5\times 26^{\lambda+1}-18}{13}\phi(q^3)^5$$

so

$$\sum_{n\geq 0} r_5(3^{2\lambda+1}(3n+2))q^n = \frac{120(27^{\lambda+1}-1)}{13}\phi(q)^2\phi(q^3)\{c(q^2)/3\}.$$

If we write

$$\phi(q)^2\phi(q^3)\{c(q^2)/3\} = H_0(q^8) + qH_1(q^8) + \cdots + q^7H_7(q^8)$$

we find

$$\sum_{n\geq 0} r_5(3^{2\lambda+1}(24n+23))q^n = \frac{120(27^{\lambda+1}-1)}{13}H_7(q).$$

If we compare this for $\lambda = 0$ with Theorem 1(viii) for $\lambda = 0$ we find

$$\sum_{n\geq 0} r_5(72n+69)q^n = 240H_7(q) = 6720\psi(q)^2\psi(q^3)\{c(q)/3\}$$

so

$$H_7(q) = 28\psi(q)^2\psi(q^3)\{c(q)/3\}$$

and

$$\sum_{n\geq 0} r_5(9^\lambda(72n+69))q^n = \frac{3360(27^{\lambda+1}-1)}{13}\psi(q)^2\psi(q^3)\{c(q)/3\}.$$

4. Proof of Theorem 2

We start by proving that for $\lambda \geq 0$

$$(\alpha) \qquad \sum_{n\geq 0} r_6(2^\lambda n)q^n = \phi(q)^6 + (4^{\lambda+2}-16)q\phi(q)^2\psi(q^2)^4.$$

(α) is true for $\lambda = 0$. Suppose (α) is true for some $\lambda \geq 0$. Then

$$\begin{aligned}\sum_{n\geq 0} r_6(2^\lambda n)q^n &= \left(\phi(q^2)^2 + 4q\psi(q^4)^2\right)^3 \\ &\quad + (4^{\lambda+2}-16)q\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\phi(q^2)^2\psi(q^4)^2 \\ &= \phi(q^2)^6 + 12q\phi(q^2)^4\psi(q^4)^2 + 48q^2\phi(q^2)^2\psi(q^4)^4 + 64q^3\psi(q^4)^6 \\ &\quad + (4^{\lambda+2}-16)\left(q\phi(q^2)^4\psi(q^4)^2 + 4q^2\phi(q^2)^2\psi(q^4)^4\right) \\ &= \phi(q^2)^6 + (4^{\lambda+2}-4)q\phi(q^2)^4\psi(q^4)^2 \\ &\quad + (4^{\lambda+3}-16)q^2\phi(q^2)^2\psi(q^4)^4 + 64q^3\psi(q^4)^6.\end{aligned}$$

It follows that

$$\sum_{n\geq 0} r_6(2^{\lambda+1}n)q^n = \phi(q)^6 + (4^{\lambda+3}-16)q\phi(q)^2\psi(q^2)^4,$$

and (α) follows by induction.

Also

$$\sum_{n\geq 0} r_6(2^\lambda(2n+1))q^n = (4^{\lambda+2}-4)\phi(q)^4\psi(q^2)^2 + 64q\psi(q^2)^6$$

$$= (4^{\lambda+2}-4)\left(\phi(q^2)^2 + 4q\psi(q^4)^2\right)^2\psi(q^2)^2 + 64q\psi(q^2)^6$$

and therefore

$$\sum_{n\geq 0} r_6(2^\lambda(4n+3))q^n = 8(4^{\lambda+2}-4)\phi(q)^2\psi(q^2)^2\psi(q)^2 + 64\psi(q)^6$$

$$= 32(4^{\lambda+1}+1)\psi(q)^6.$$

5. Proof of Theorem 3

(i) We start by proving that for $\lambda \geq 0$

$$(\alpha) \qquad \sum_{n\geq 0} r_7(4^\lambda n)q^n = \phi(q)^7 + \frac{560(32^\lambda - 1)}{31} q\phi(q)^3\psi(q^2)^4.$$

(α) is true for $\lambda = 0$. Suppose (α) true for some $\lambda \geq 0$. Then

$$\sum_{n\geq 0} r_7(4^\lambda n)q^n$$

$$= \left(\phi(q^4) + 2q\psi(q^8)\right)^7$$

$$+ \frac{560(32^\lambda - 1)}{31} q\left(\phi(q^4) + 2q\psi(q^8)\right)^3\left(\phi(q^4)^2 + 4q^2\psi(q^8)^2\right)\phi(q^4)\psi(q^8)$$

$$= \phi(q^4)^7 + 14q\phi(q^4)^6\psi(q^8) + 84q^2\phi(q^4)^5\psi(q^8)^2 + 280q^3\phi(q^4)^4\psi(q^8)^3$$

$$+ 560q^4\phi(q^4)^3\psi(q^8)^4 + 672q^5\phi(q^4)^2\psi(q^8)^5 + 448q^6\phi(q^4)\psi(q^8)^6 + 128q^7\psi(q^8)^7$$

$$+\frac{560(32^{\lambda}-1)}{31}\left(q\phi(q^4)^6\psi(q^8)\ +6q^2\phi(q^4)^5\psi(q^8)^2+16q^3\phi(q^4)^4\psi(q^8)^3\right.$$
$$\left.+32q^4\phi(q^4)^3\psi(q^8)^4\ +48q^5\phi(q^4)^2\psi(q^8)^5\ +32q^6\phi(q^4)\psi(q^8)^6\right)$$

$$=\phi(q^4)^7+\frac{560\times 32^{\lambda}-126}{31}q\phi(q^4)^6\psi(q^8)+\frac{3360\times 32^{\lambda}-756}{31}q^2\phi(q^4)^5\psi(q^8)^2$$
$$+\frac{280(\times 32^{\lambda+1}-1)}{31}q^3\phi(q^4)^4\psi(q^8)^3+\frac{560(32^{\lambda+1}-1)}{31}q^4\phi(q^4)^3\psi(q^8)^4$$
$$+\frac{840\times 32^{\lambda+1}-6048}{31}q^5\phi(q^4)^2\psi(q^8)^5+\frac{560\times 32^{\lambda+1}-4032}{31}q^6\phi(q^4)\psi(q^8)^6$$
$$+128q^7\psi(q^8)^7.$$

It follows that

$$\sum_{n\geq 0} r_7(4^{\lambda+1}n)q^n=\phi(q)^7+\frac{560(32^{\lambda+1}-1)}{31}q\phi(q)^3\psi(q^2)^4.$$

Thus (α) follows by induction. We also have

$$\sum_{n\geq 0} r_7(4^{\lambda}(4n+3))q^n$$
$$=\frac{280(32^{\lambda+1}-1)}{31}\phi(q)^4\psi(q^2)^3+128q\psi(q^2)^7$$

$$=\frac{280(32^{\lambda+1}-1)}{31}\left(\phi(q^2)^2+4q\psi(q^4)^2\right)^2\psi(q^2)^3+128q\psi(q^2)^7,$$

so

$$\sum_{n\geq 0} r_7(4^{\lambda}(8n+7))q^n=\frac{280(32^{\lambda+1}-1)}{31}\cdot 8\phi(q)^2\psi(q^2)^2\psi(q)^3+128\psi(q)^7$$
$$=\frac{64(35\times 32^{\lambda+1}+27)}{31}\psi(q)^7.$$

(ii) We have

$$\sum_{n\geq 0} r_7(4^\lambda(8n+7))q^n = \frac{64(35\times 32^{\lambda+1}+27)}{31}\left(P(q^3)+q\psi(q^9)\right)^7.$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} r_7(4^\lambda(24n+23))q^n &= \frac{64(35\times 32^{\lambda+1}+27)}{31}\cdot 21P(q)^2\psi(q^3)^2\left(P(q)^3+q^3\psi(q^3)^3\right)\\ &= \frac{1344(35\times 32^{\lambda+1}+27)}{31}P(q)^2\psi(q^3)^2\frac{\psi(q)^4}{\psi(q^3)}\\ &= \frac{1344(35\times 32^{\lambda+1}+27)}{31}\psi(q)^5\left\{c(q)/3\right\}.\end{aligned}$$

(iii) We start by proving that for $\lambda \geq 0$

(α)

$$\begin{aligned}\sum_{n\geq 0} r_7(3^{2\lambda}n)q^n = \frac{252\times 243^\lambda-10}{242}\phi(q)^7 &- \frac{63(243^\lambda-1)}{242}\phi(q^3)^4\phi(q)^3\\ &- \frac{189(243^\lambda-1)}{242}\frac{\phi(q^3)^8}{\phi(q)},\end{aligned}$$

(β)

$$\begin{aligned}\sum_{n\geq 0} r_7(3^{2\lambda+1}n)q^n = \frac{7(243^{\lambda+1}-1)}{242}\frac{\phi(q)^8}{\phi(q^3)} &+ \frac{21(243^{\lambda+1}-1)}{242}\phi(q^3)^3\phi(q)^4\\ &- \frac{28\times 243^{\lambda+1}-270}{242}\phi(q^3)^7.\end{aligned}$$

(α) is true for $\lambda = 0$. Suppose (α) is true for some $\lambda \geq 0$. Then

$$\sum_{n\geq 0} r_7(3^{2\lambda}n)q^n$$

$$= \frac{252\times 243^{\lambda}-10}{242}\Big\{ \left(7\frac{\phi(q^3)^8}{\phi(q^9)} + 21\phi(q^9)^3\phi(q^3)^4 - 27\phi(q^9)^7\right)$$
$$+ q\left(2\frac{\phi(q^3)^8}{\phi(q^9)^2} + 66\phi(q^9)^2\phi(q^3)^4 - 54\phi(q^9)^6\right)X(q^3)$$
$$+ q^2\left(84\phi(q^9)\phi(q^3)^4X(q^3)^2\right)\Big\}$$
$$- \frac{63(243^{\lambda}-1)}{242}\Big\{\frac{\phi(q^3)^8}{\phi(q^9)} + 6q\phi(q^9)^2\phi(q^3)^4X(q^3)$$
$$+ 12q^2\phi(q^9)\phi(q^3)^4X(q^3)^2\Big\}$$
$$- \frac{189(243^{\lambda}-1)}{242}\Big\{\phi(q^9)^3\phi(q^3)^4 - 2q\phi(q^9)^2\phi(q^3)^4X(q^3)$$
$$+ 4q^2\phi(q^9)\phi(q^3)^4X(q^3)^2\Big\}$$

$$= \Bigg(\frac{7(243^{\lambda+1}-1)}{242}\frac{\phi(q^3)^8}{\phi(q^9)} + \frac{21(243^{\lambda+1}-1)}{242}\phi(q^9)^3\phi(q^3)^4$$
$$-\frac{28\times 243^{\lambda+1}-270}{242}\phi(q^9)^7\Bigg)$$
$$+ q\Bigg(\frac{504\times 243^{\lambda}-20}{242}\frac{\phi(q^3)^8}{\phi(q^9)^2} + \frac{16632\times 243^{\lambda}-660}{242}\phi(q^9)^2\phi(q^3)^4$$
$$-\frac{56\times 243^{\lambda+1}-540}{242}\phi(q^9)^6\Bigg)X(q^3)$$
$$+ q^2\frac{19656\times 243^{\lambda}+672}{242}\phi(q^9)\phi(q^3)^4X(q^3)^2.$$

It follows that

$$\sum_{n\ge 0} r_7(3^{2\lambda+1}n)q^n = \frac{7(243^{\lambda+1}-1)}{242}\frac{\phi(q)^8}{\phi(q^3)} + \frac{21(243^{\lambda+1}-1)}{242}\phi(q^3)^3\phi(q)^4$$

$$- \frac{28 \times 243^{\lambda+1} - 270}{242} \phi(q^3)^7$$

$$= \frac{7(243^{\lambda+1}-1)}{242} \Big\{ \left(28\phi(q^3)^7 - 27\frac{\phi(q^9)^8}{\phi(q^3)} \right)$$

$$+ q \left(16\frac{\phi(q^3)^7}{\phi(q^9)} + 108\phi(q^9)^3\phi(q^3)^3 - 108\frac{\phi(q^9)^7}{\phi(q^3)} \right) X(q^3)$$

$$+ q^2 \left(4\frac{\phi(q^3)^7}{\phi(q^9)^2} + 216\phi(q^9)^2\phi(q^3)^3 - 108\frac{\phi(q^9)^6}{\phi(q^3)} \right) X(q^3)^2 \Big\}$$

$$+ \frac{21(243^{\lambda+1}-1)}{242} \Big\{ \left(4\phi(q^3)^7 - 3\phi(q^9)^4\phi(q^3)^3 \right)$$

$$+ q \left(2\frac{\phi(q^3)^7}{\phi(q^9)} + 6\phi(q^9)^3\phi(q^3)^3 \right) X(q^3)$$

$$+ q^2 \left(24\phi(q^9)^2\phi(q^3)^3 X(q^3)^2 \right) \Big\}$$

$$- \frac{28 \times 243^{\lambda+1} - 270}{242} \Big\{ \phi(q^3)^7 \Big\}$$

$$= \Bigg(\frac{252 \times 243^{\lambda+1} - 10}{242} \phi(q^3)^7 - \frac{63(243^{\lambda+1}-1)}{242} \phi(q^9)^4\phi(q^3)^3$$

$$- \frac{189(243^{\lambda+1}-1)}{242} \frac{\phi(q^9)^8}{\phi(q^3)} \Bigg)$$

$$+ q \Bigg(\frac{154(243^{\lambda+1}-1)}{242} \frac{\phi(q^3)^7}{\phi(q^9)} + \frac{882(243^{\lambda+1}-1)}{242} \phi(q^9)^3\phi(q^3)^3$$

$$- \frac{756(243^{\lambda+1}-1)}{242} \frac{\phi(q^9)^7}{\phi(q^3)} \Bigg) X(q^3)$$

$$+ q^2 \Bigg(\frac{28(243^{\lambda+1}-1)}{242} \frac{\phi(q^3)^7}{\phi(q^9)^2} + \frac{2016(243^{\lambda+1}-1)}{242} \phi(q^9)^2\phi(q^3)^3$$

$$- \frac{756(243^{\lambda+1}-1)}{242} \frac{\phi(q^9)^6}{\phi(q^3)} \Bigg) X(q^3)^2.$$

It follows that

$$\sum_{n\ge 0} r_7(3^{2\lambda+2}n)q^n = \frac{252\times 243^{\lambda+1}-10}{242}\phi(q)^7 - \frac{63(243^{\lambda+1}-1)}{242}\phi(q^3)^4\phi(q)^3$$
$$- \frac{189(243^{\lambda+1}-1)}{242}\frac{\phi(q^3)^8}{\phi(q)}.$$

Thus (α) and (β) follow by induction. Also

$$\sum_{n\ge 0} r_7(9^{\lambda}(3n+2))q^n = \frac{19656\times 243^{\lambda}+672}{242}\phi(q^3)\phi(q)^4X(q)^2$$
$$= \frac{84(117\times 243^{\lambda}+4)}{121}\phi(q)^5\left\{c(q^2)/3\right\}.$$

(iv) We have

$$\sum_{n\ge 0} r_7(9^{\lambda}(3n+2))q^n = \frac{84(117\times 243^{\lambda}+4)}{121}\phi(q)^5\{c(q^2)/3\}.$$

If we write

$$\phi(q)^5\{c(q^2)/3\} = J_0(q^8) + qJ_1(q^8) + \cdots + q^7J_7(q^8)$$

we find

$$\sum_{n\ge 0} r_7(9^{\lambda}(24n+23))q^n = \frac{84(117\times 243^{\lambda}+4)}{121}J_7(q).$$

If we compare this for $\lambda = 0$ with Theorem 3(ii) for $\lambda = 0$ we find

$$\sum_{n\ge 0} r_7(24n+23)q^n = 84J_7(q) = 49728\psi(q)^5\{c(q)/3\}$$

so

$$J_7(q) = 592\psi(q)^5\{c(q)/3\}$$

and

$$\sum_{n\geq 0} r_7(9^\lambda(24n+23))q^n = \frac{49728(117\times 243^\lambda + 4)}{121}\psi(q)^5\{c(q)/3\}.$$

References

[1] S. Cooper and M. D. Hirschhorn, Results of Hurwitz type for three squares, Discrete Math., 274(2004), 9–24.

[2] M. Hirschhorn, F. Garvan and J. Borwein, Cubic analogues of the Jacobian theta function $\theta(z,q)$, Canad. J. Math. 45(1993), 673–694.

[3] Michael D. Hirschhorn, Jacobi's two–square theorem and related results, The Ramanujan Journal, 3(1999), 153–158.

[4] Michael D. Hirschhorn and James A. McGowan, Algebraic consequences of Jacobi's two– and four–square theorems, Frank G. Garvan, Mourad E. H. Ismail (eds), Developments in Mathematics, Vol. 4, Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics. Kluwer, 2001, 107–132.

[5] Adolf Hurwitz, Ueber die Anzahl der Classen quadratischer Formen von negativer Determinante, Mathematische Werke, Band II, Birkhauser, Basel, 1933, 68–71.