BINOMIAL COEFFICIENT IDENTITIES AND HYPERGEOMETRIC SERIES

Michael D. Hirschhorn

In recent months I have come across many instances in which someone has found what they believe is a new result, in which they evaluate in closed form a sum involving binomial coefficients or factorials. In each case they have managed to do that either by using the recent powerful method of Wilf and Zeilberger (the W–Z method) [6], or by comparing coefficients in some *ad hoc* algebraic identity. The aim of this note is to describe, using a few examples, a **purely algorithmic** method for re–casting the sum as a (multiple of a) hypergeometric series in standard notation, so that one can then simply **look up** standard tables of hypergeometric series to see if the series under investigation is "summable" via known results.

I do not claim any originality in this idea. I got it from Richard Askey, who claims [2] that "at least 90% if not 95% of the formulas in Table 3 of Henry Gould's Tables [3] yield to this approach". Indeed, to quote Askey further, "For years [before working with George E. Andrews in 1973] I had been trying to point out that the rather confused world of binomial coefficient summations is best understood in the language of hypergeometric series identities. Time and again I would find first–rate mathematicians who had never heard of this insight and who would waste considerable time proving some apparently new binomial coefficient summation which almost always turned out to be a special case of one of a handful of classical hypergeometric identities."

The identities I will use to illustrate the method are the following. The first came to me in a paper I was asked to referee, but is to be found in Wang and Guo [8] (1989). The second was found in 2001 by an Honours

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student at UNSW, T. T. To [7], in his fourth–year essay, and the third appeared in a very recent article by Victor Moll [5] in the Notices of the American Mathematical Society. They are

$$\sum_k \binom{m}{2k}\binom{k}{n} = 2^{m-2n-1}\left\{\binom{m-n}{n}+\binom{m-n-1}{n-1}\right\}, \tag{1}$$

$$\sum_j \frac{1}{(m-k-2j)!(k+j)!(j-l)!4^j} = 2^{k-m}\frac{(2m-2l)!}{(m-k-2l)!(m-l)!(k+m)!} \tag{2}$$

and

$$\sum_j (-1)^j\binom{m-1}{j}\binom{2m-2j-1}{k+m-1} = 2^{m-k-1}\frac{k+m}{m}\binom{m}{k}. \tag{3}$$

Before showing how to transform the various sums into hypergeometric series, I must introduce the standard notation for hypergeometric series. My reference is Andrews, Askey and Roy [1]. We start with what they call the shifted factorial (note that it is a rising factorial),

$$(a)_n = a(a+1)\ \cdots\ (a+n-1).$$

Now define

$${}_2F_1\left(\begin{matrix}a, & b\\ & c\end{matrix}; x\right) = \sum_{k\ge 0}\frac{(a)_k(b)_k}{k!(c)_k}x^k.$$

We note at the outset that if either of the parameters a, b is a negative integer $-n$ then the series terminates, since $(-n)_k = 0$ for $k > n$. Also, x is called the base of the ${}_2F_1$. We will see that all our illustrative examples involve a terminating ${}_2F_1$ with base 1. (Incidentally, we can define a ${}_pF_q$ in analogous fashion.)

We start by considering the sum on the left of (1),

$$\sum_k \binom{m}{2k}\binom{k}{n}. \tag{4}$$

First of all we notice that for any term to be non–zero, we require

$$m \geq 2k, \quad k \geq n,$$

so in particular

$$m \geq 2n$$

and then the sum becomes

$$S = \sum_{k \geq n} \binom{m}{2k} \binom{k}{n} = \sum_{k \geq 0} \binom{m}{2n+2k} \binom{n+k}{n}. \tag{5}$$

We now observe that when $k = 0$, the term $\binom{m}{2n+2k}$ assumes the value $\binom{m}{2n}$. We have to start by extracting this term. Thus

$$\binom{m}{2n+2k} = \binom{m}{2n} \frac{(2n)!(m-2n)!}{(2n+2k)!(m-2n-2k)!}. \tag{6}$$

We then take each of the quotients on the right of (7), and write

$$\frac{(2n)!}{(2n+2k)!} = \frac{1}{(2n+1)(2n+2) \ \cdots \ (2n+2k)} = \frac{1}{(2n+1)_{2k}}, \tag{7}$$

and

(8)

$$\begin{aligned} \frac{(m-2n)!}{(m-2n-2k)!} &= (m-2n-2k+1)(m-2n-2k+2) \ \cdots \ (m-2n) \\ &= (m-2n-2k+1)_{2k}. \end{aligned}$$

But each of these rising factorials is of length $2k$, and what we need is rising factorials of length k. To deal with the first of these we require one trick, for the other, three tricks. In the first, we pull out every second term, thus:

$$\begin{aligned}(2n+1)_{2k} &= (2n+1)(2n+2)\ \cdots\ (2n+2k-1)(2n+2k) \\ &= (2n+1)(2n+3)\ \cdots\ (2n+2k-1) \\ &\quad \times (2n+2)(2n+4)\ \cdots\ (2n+2k) \\ &= 2^k(n+\frac{1}{2})(n+\frac{3}{2})\ \cdots\ (n+k-\frac{1}{2}) \\ &\quad \times 2^k(n+1)(n+2)\ \cdots\ (n+k) \\ &= 2^{2k}(n+\frac{1}{2})_k(n+1)_k.\end{aligned} \tag{9}$$

In the second, we must first reverse the product so as to have the first term **not** depend on k, then insert $-$ signs in every term to re–obtain a rising factorial, then pull out every second term, thus:

(10)

$$\begin{aligned}&(m-2n-2k+1)_{2k} = (m-2n-2k+1)(m-2n-2k+2)\ \cdots\ (m-2n) \\ &= (m-2n)(m-2n-1)\ \cdots\ (m-2n-2k+1) \\ &= (-m+2n)(-m+2n+1)\ \cdots\ (-m+2n+2k-1) \\ &= (-m+2n)(-m+2n+2)\ \cdots\ (-m+2n+2k-2) \\ &\quad \times (-m+2n+1)(-m+2n+3)\ \cdots\ (-m+2n+2k-1) \\ &= 2^k(\frac{-m+2n}{2})(\frac{-m+2n}{2}+1)\ \cdots\ (\frac{-m+2n}{2}+k-1) \\ &\quad \times 2^k(\frac{-m+2n+1}{2})(\frac{-m+2n+1}{2}+1)\ \cdots\ (\frac{-m+2n+1}{2}+k-1) \\ &= 2^{2k}(\frac{-m+2n}{2})_k(\frac{-m+2n+1}{2})_k.\end{aligned}$$

Also, returning to (5), we need

$$\binom{n+k}{n} = \frac{(n+1)(n+2)\ \cdots\ (n+k)}{k!} = \frac{(n+1)_k}{k!}. \tag{11}$$

If we make use of (6), (7), (8), (9), (10) and (11), we find

$$S = \binom{m}{2n}\sum_{k\geq 0}\frac{(\frac{-m+2n}{2})_k(\frac{-m+2n+1}{2})_k}{k!(n+\frac{1}{2})_k}$$

or,

$$S = \binom{m}{2n} {}_2F_1\left(\begin{matrix} \dfrac{-m+2n}{2}, & \dfrac{-m+2n+1}{2} \\ & n+\dfrac{1}{2} \end{matrix} ;1\right); \tag{12}$$

here the ${}_2F_1$ terminates and has base 1.

Now consider our second example, (2). The sum on the left is

$$\sum_j \frac{1}{(m-k-2j)!(k+j)!(j-l)!4^j}. \tag{13}$$

Here we need

$$j \geq l \text{ and } m-k-2j \geq 0, \text{ so } m-k-2l \geq 0,$$

and the sum becomes

$$\begin{aligned} S &= \sum_{j\geq l} \frac{1}{(m-k-2j)!(k+j)!(j-l)!4^j} \\ &= \sum_{j\geq 0} \frac{1}{(m-k-2l-2j)!(k+l+j)!j!4^{l+j}}. \end{aligned}$$

Now,

$$\begin{aligned} (m-k-2l-2j)! &= \frac{(m-k-2l)!}{(m-k-2l)(m-k-2l-1)\ \cdots\ (m-k-2l-2j+1)}, \\ (k+l+j)! &= (k+l)!(k+l+1)(k+l+2)\ \cdots\ (k+l+j), \end{aligned}$$

so

$$\begin{aligned} S = {} & \frac{1}{(k+l)!(m-k-2l)!4^l} \\ & \times \sum_{j\geq 0} \frac{(m-k-2l)(m-k-2l-1)\ \cdots\ (m-k-2l-2j+1)}{(k+l+1)(k+l+2)\ \cdots\ (k+l+j)j!2^{2j}}. \end{aligned}$$

Next,

$$\begin{aligned}
&\frac{(m-k-2l)(m-k-2l-1)\ \cdots\ (m-k-2l-2j+1)}{2^{2j}}\\
&=\frac{(m-k-2l)(m-k-2l-2)\ \cdots\ (m-k-2l-2j+2)}{2^{j}}\\
&\times\frac{(m-k-2l-1)(m-k-2l-3)\ \cdots\ (m-k-2l-2j+1)}{2^{j}}\\
&=(\frac{-m+k+2l}{2})(\frac{-m+k+2l}{2}+1)\ \cdots\ (\frac{-m+k+2l}{2}+(j-1))\\
&\times(\frac{-m+k+2l+1}{2})(\frac{-m+k+2l+1}{2}+1)\ \cdots\ (\frac{-m+k+2l+1}{2}+(j-1))
\end{aligned}$$

so

$$S=\frac{1}{(k+l)!(m-k-2l)!4^l}\sum_{j\geq 0}\frac{(\frac{-m+k+2l}{2})_j(\frac{-m+k+2l+1}{2})_j}{j!(k+l+1)_j}$$

or,

(14)

$$S=\frac{1}{(k+l)!(m-k-2l)!4^l}\ {}_2F_1\left(\begin{matrix}\frac{-m+k+2l}{2}, & \frac{-m+k+2l+1}{2} \\ & k+l+1\end{matrix};1\right).$$

Once again the ${}_2F_1$ terminates and has base 1.

The sum in our third example, (3), is easily shown to be

$$\binom{2m-1}{k+m-1}\ {}_2F_1\left(\begin{matrix}\frac{-m+k}{2}, & \frac{-m+k+1}{2} \\ & -m+\frac{1}{2}\end{matrix};1\right). \tag{15}$$

Now it will probably come as no surprise that there is a theorem which says that a terminating ${}_2F_1$ with base 1 is summable. The consequence of this is that we now know that each of the sums in (1), (2) and (3) can be written in closed form. The theorem, known as the Chu–Vandermonde theorem [1, Corollary 2.2.3] states

$$ {}_2F_1\left(\begin{matrix} -n, & a \\ & c \end{matrix}; 1\right) = \frac{(c-a)_n}{(c)_n}. \tag{16} $$

As corollaries of (16), we can show that if $n,\ N$ are non–negative integers,

$$ {}_2F_1\left(\begin{matrix} -\frac{N}{2}-\frac{1}{2}, & -\frac{N}{2} \\ & n+\frac{1}{2} \end{matrix}; 1\right) = 2^N \frac{(2n)!(n+N)!}{n!(2n+N)!}, \tag{17} $$

$$ {}_2F_1\left(\begin{matrix} -\frac{N}{2}, & -\frac{N}{2}+\frac{1}{2} \\ & n+1 \end{matrix}; 1\right) = 2^{-N} \frac{(2n+2N)!n!}{(2n+N)!(n+N)!}, \tag{18} $$

and

$$ {}_2F_1\left(\begin{matrix} -\frac{N}{2}, & -\frac{N}{2}+\frac{1}{2} \\ & -n-N+\frac{1}{2} \end{matrix}; 1\right) = 2^N \frac{(n+N)!(2n+N)!}{(2n+2N)!n!}. \tag{19} $$

It is now an easy matter to deduce (1), (2) and (3) from (12), (14) and (15).

The referee of an earlier version of this paper wanted me to relate the Chu–Vandermonde theorem (16) to two other identities which have the name "Vandermonde" attached to them, namely

$$ (x+a)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (a)_{n-k} $$

and, for positive integers $x,\ a$,

$$ \binom{x+a}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{a}{n-k}. $$

Since the right hand sides of both of these are (essentially) binomial coefficient sums, each can be written, **following the above algorithm**, as hypergeometric series. Each is a (multiple of a) terminating ${}_2F_1$ with base 1, so is summable via (16).

Note that I have not given a proof of (16) here. I refer the reader to Hirschhorn [4].

Bibliography

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