

ON SUMS OF SQUARES

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In the November 1999 issue of the *Mathematical Gazette*, Canon D.B. Eperson asked an interesting question: What proportion of numbers are sums of two squares? Unfortunately, I was beaten to the punch, but I think I still have something worth saying.

In what I am going to say, 0 is a square, so a square is also a sum of two squares, a sum of two squares is a sum of three squares, and so on.

The proportion of numbers which are squares is 0, for there are roughly \sqrt{n} squares not exceeding n , and $\frac{\sqrt{n}}{n} \rightarrow 0$ as $n \rightarrow \infty$.

The proportion of numbers which are sums of four squares is 1. Every number is the sum of four squares.

The proportion of numbers which are sums of three squares is $\frac{5}{6}$. A number is not a sum of three squares if and only if it is of the form $4^a(8k+7)$. The proportion of numbers which are of this form is

$$\frac{1}{8} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots \right) = \frac{1}{6}.$$

Now, turning to Canon Eperson's question, the proportion of numbers which are sums of two squares is 0. A number n is the sum of two squares

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if and only if in the prime factorisation of n the primes congruent to -1 modulo 4 occur to even powers. The proportion of numbers which are divisible by the prime p to an even power is

$$1 - \frac{1}{p} + \frac{1}{p^2} - + \cdots = 1 - \frac{1}{p+1}.$$

Thus, the proportion of numbers which are sums of two squares is

$$\prod \left(1 - \frac{1}{p+1}\right) = \prod \frac{p}{p+1} = 1 / \prod \left(1 + \frac{1}{p}\right)$$

where the product is taken over all primes $p \equiv -1 \pmod{4}$.

Now,

$$\prod_{\text{all primes } p} \left(1 - \frac{1}{p}\right)^{-1} = \sum_{n \geq 1} \frac{1}{n}$$

diverges, so

$$\sum_{\text{all primes } p} \frac{1}{p}$$

diverges.

Since about half the primes are congruent to $-1 \pmod{4}$, it is also true that

$$\sum_{\text{prime } p \equiv -1 \pmod{4}} \frac{1}{p}$$

diverges, so

$$\prod_{\text{prime } p \equiv -1 \pmod{4}} \left(1 + \frac{1}{p}\right)$$

diverges, and the proportion of numbers which are sums of two squares is 0.

I read in Bruce Berndt's "Ramanujan's Notebooks", Part IV, Springer 1994, pp.60–66 that Ramanujan gave an estimate for $\rho(n)$, the proportion of numbers less than or equal to n which are sums of two squares, and that Landau proved that

$$\rho(n) = \frac{K}{\sqrt{\log n}}(1 + o(1)) \text{ as } n \rightarrow \infty$$

where

$$K = \left\{ \frac{1}{2} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 - p^{-2}} \right\}^{1/2} \approx 0.764.$$