THERE ARE INFINITELY MANY PRIME NUMBERS

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There are numerous proofs that there are infinitely many primes. Several of the nicest proofs appear in [1]. We give another.

Suppose there are finitely many odd primes, $p_1 < \cdots < p_k$. Then every odd number can be written in the form $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$.

However, suppose $p_1^{\alpha_1} \cdots p_k^{\alpha_k} \le N$. Then $0 \le \alpha_i \le \dfrac{\log N}{\log p_i}$.

The number of odd numbers less than or equal to N of the form $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is therefore not more than

$$\left(\frac{\log N}{\log p_1} + 1\right) \cdots \left(\frac{\log N}{\log p_k} + 1\right) = \frac{\log p_1 N}{\log p_1} \cdots \frac{\log p_k N}{\log p_k}$$

$$\le \frac{(\log p_k N)^k}{(\log p_1)^k} < (\log p_k N)^k < \sqrt{(2k)!}\sqrt{p_k N} < \frac{N}{2} \text{ if } N > 4(2k)!p_k.$$

In particular if $N = 4(2k)!p_k + 1$ there is an odd number in $\{1, 3, \cdots, N\}$ which is not of the form $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, a contradiction.

And, in fact, we have shown that $p_{k+1} \le 4(2k)!p_k + 1$. (This last is a joke. Bertrand's Postulate, proved by Chebychev, is that for $p_k \ge 5$, $p_{k+1} \le 2p_k - 3$.)

Reference

[1] M. Aigner and G. M. Ziegler, Proofs from THE BOOK, 2nd ed., Springer, 2001.