PAPER 90 k–GON PARTITIONS

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The object of this note is to present a brief derivation of the main result of Andrews *et al* [1]. We obtain a multivariable generating function associated with $t_k(n)$, the number of k–gon partitions of n, that is, those partitions of n,

$$n = a_1 + a_2 + \cdots + a_k$$

in which

$$1 \le a_1 \le a_2 \cdots \le a_k \text{ and } a_1 + a_2 + \cdots + a_{k-1} > a_k.$$

(The case $k = 3$ gives the number of triangles with integer sides and perimeter n.) Andrews *et al* did this with MacMahon's Partition Analysis (Ω–Calculus), but we do without.

Thus, let

$$a_1 = 1 + \delta_1,\ a_2 = 1 + \delta_1 + \delta_2,\ \cdots, a_k = 1 + \delta_1 + \delta_2 + \cdots + \delta_k,$$

with $\delta_i \ge 0$. Then

$$(k-1) + (k-1)\delta_1 + (k-2)\delta_2 + \cdots + 1\delta_{k-1} > 1 + \delta_1 + \delta_2 + \cdots + \delta_k,$$

or,

$$\delta_k < (k-2) + (k-2)\delta_1 + (k-3)\delta_2 + \cdots + 1\delta_{k-2}.$$

It follows that if S denotes the set of all k–gon partitions,

$$G = \sum_S x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} = \sum x_1^{1+\delta_1} x_2^{1+\delta_1+\delta_2} \cdots x_k^{1+\delta_1+\delta_2+\ \cdots\ +\delta_k}$$

where the sum is taken over all $\{\delta_1,\ \delta_2,\ \cdots\ ,\delta_k\}$ with $\delta_i \geq 0$ and $\delta_k < (k-2) + (k-2)\delta_1 + (k-3)\delta_2 +\ \cdots\ + 1\delta_{k-2}$.

Thus, if we set

$$X_i = x_i x_{i+1} \cdots x_k, \quad i = 1,\ \cdots\ k,$$

we find that

$$\begin{aligned}
G &= \sum X_1^{1+\delta_1} X_2^{\delta_2} X_3^{\delta_3} \cdots X_k^{\delta_k} \\
&= \sum X_1^{1+\delta_1} X_2^{\delta_2} \cdots X_{k-1}^{\delta_{k-1}} \left(\frac{1 - X_k^{(k-2)+(k-2)\delta_1+(k-3)\delta_2+\ \cdots\ +1\delta_{k-2}}}{1 - X_k} \right) \\
&= \frac{X_1}{1-X_k} \sum X_1^{\delta_1} X_2^{\delta_2} \cdots X_{k-1}^{\delta_{k-1}} \\
&\quad - \frac{X_1 X_k^{k-2}}{1-X_k} \sum (X_1 X_k^{k-2})^{\delta_1} (X_2 X_k^{k-3})^{\delta_2} \cdots (X_{k-2}X_k)^{\delta_{k-2}} X_{k-1}^{\delta_{k-1}} \\
&= \frac{X_1}{(1-X_1)(1-X_2)\cdots(1-X_k)} \\
&\quad - \frac{X_1 X_k^{k-2}}{1-X_k} \frac{1}{(1-X_{k-1})(1-X_{k-2}X_k)(1-X_{k-3}X_k^2)\cdots(1-X_1 X_k^{k-2})},
\end{aligned}$$

which is [1, Theorem 1].

Of course, if we put $x_i = q$ for all i, we obtain [1, Corollary 1],

$$\sum_{n\geq 0} t_k(n) q^n = \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)} - \frac{q^{2k-2}}{1-q} \frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2k-2})}.$$

Reference

[1] G. E. Andrews, P. Paule and A. Riese, MacMahon's Partition Analysis IX: k–gon partitions, Bull. Austral. Math. Soc. 64(2001), 321–329.