COMMENT ON "A CURIOUS IDENTITY"

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Dear Editor,

I would like to point out that S. Simon's "curious identity" is a special case of an identity known to Johann Friedrich Pfaff (in the late 18th Century). (For information regarding Pfaff, visit
http//:www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Pfaff).
In order to realise this, we must first write each side of the identity in standard hypergeometric series notation:

The left hand sum is

$$\sum_{r=0}^{\infty} \frac{(-1)^{q+r}(q+r)!(1+x)^r}{(q-r)!(r!)^2}$$

$$= \sum_{r\geq 0} (-1)^{q+r} \frac{q!(q+1)(q+2)\ \cdots\ (q+r)}{q!/(q(q-1)\ \cdots\ (q-r+1))r!r!}(1+x)^r$$

$$= (-1)^q \sum_{r\geq 0} \frac{(q+1)(q+2)\ \cdots\ (q+r)(-q)(-q+1)\ \cdots\ (-q+r-1)}{r!(1)(2)\ \cdots\ (r)}(1+x)^r$$

$$= (-1)^q \sum_{r\geq 0} \frac{(q+1)_r(-q)_r}{r!(1)_r}(1+x)^r$$

$$= (-1)^q\ {}_2F_1\left(\begin{matrix} q+1, & -q \\ & 1 \end{matrix}; 1+x\right),$$

while the right hand sum is

$$\begin{aligned}
&\sum_{r=0}^{\infty} \frac{(q+r)!x^r}{(q-r)!(r!)^2} \\
&= \sum_{r\geq 0} \frac{q!(q+1)(q+2)\ \cdots\ (q+r)}{q!/(q(q-1)\ \cdots\ (q-r+1))r!r!} x^r \\
&= \sum_{r\geq 0} \frac{(q+1)(q+2)\ \cdots\ (q+r)(-q)(-q+1)\ \cdots\ (-q+r-1)}{r!(1)(2)\ \cdots\ (r)} (-x)^r \\
&= \sum_{r\geq 0} \frac{(q+1)_r(-q)_r}{r!(1)_r} (-x)^r \\
&= {}_2F_1\left(\begin{matrix} q+1, & -q \\ & 1 \end{matrix}; -x\right).
\end{aligned}$$

Thus his identity is

$$(1) \qquad (-1)^q\ {}_2F_1\left(\begin{matrix} q+1, & -q \\ & 1 \end{matrix}; 1+x\right) = {}_2F_1\left(\begin{matrix} q+1, & -q \\ & 1 \end{matrix}; -x\right).$$

This is the special case $n = q,\ b = q+1,\ c = 1$ with $-x$ for x of [1, (2.3.14)].

In proving his identity, S. Simons needs to show that

$$(2) \qquad \sum_{t=0}^{q-s} \frac{(-1)^{q+s+t}(q+s+t)!}{(q-s-t)!(s+t)!t!} = \frac{(q+s)!}{s!(q-s)!}.$$

The sum is

$$\begin{aligned}
&(-1)^{q+s} \sum_{t\geq 0} (-1)^t \frac{(q+s)!(q+s+1)\ \cdots\ (q+s+t)}{(q-s)!/((q-s)\ \cdots\ (q-s-t+1))s!(s+1)\ \cdots\ (s+t)t!} \\
&= (-1)^{q+s} \frac{(q+s)!}{(q-s)!s!} \sum_{t\geq 0} \frac{(q+s+1)_t(-(q-s))_t}{(s+1)_t t!}
\end{aligned}$$

$$= (-1)^{q+s} \frac{(q+s)!}{(q-s)!s!} \; {}_2F_1 \left(\begin{matrix} q+s+1, & -(q-s) \\ & s+1 \end{matrix} ; 1 \right).$$

Thus (2) is

$$(-1)^{q+s} \frac{(q+s)!}{(q-s)!s!} \; {}_2F_1 \left(\begin{matrix} q+s+1, & -(q-s) \\ & s+1 \end{matrix} ; 1 \right) = \frac{(q+s)!}{(q-s)!s!}.$$

The ${}_2F_1$ is a terminating ${}_2F_1$ with base 1, so we recognise this as the special case $a = q+s+1$, $c = s+1$, $n = q-s$ of the Chu–Vandermonde identity [1,Corollary 2.2.3].

Reference

[1] George E. Andrews, Richard Askey and Ranjan Roy, Special Functions, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge University Press, 1999.