COMMENT ON "DISSECTING SQUARES"

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On [4, page 427], J. Kingston and D. MacHale make some number theoretic remarks that I would like to comment upon. Throughout, 0 is not considered a square.

They say "It can be shown that every integer greater than 128 can be expressed as the sum of two or more distinct squares, indeed as the sum of at most five distinct squares." The first part of this sentence appears to be correct. Here I am guided by the review of [1] by H. Gupta [2]. But the second part is wrong: 188 is not the sum of five distinct squares; moreover, it seems to me that R. Guy [3] (on whom Kingston and MacHale rely) may be wrong when he asserts that it can be shown that every number greater than 188 is the sum of at most five distinct squares, since the authors of [1] apparently do no more than check up to 100000.

Kingston and MacHale then give a list of those integers that cannot be expressed as the sum of two or more distinct squares. Unfortunately, 3 is missing.

They say "Curiously, this list contains all the small powers of 2." Perhaps this is not so surprising in light of the fact that **no** power of 2, small or large, is the sum of two, three or four distinct squares. I will give a proof of this by descent.

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First we check that none of 1, 2 and 4 is the sum of two, three or four distinct squares. Suppose $n \geq 3$ and $2^n = \sum_{i=1}^{s} x_i^2$ with $2 \leq s \leq 4$ and the x_i distinct. Modulo 8, $\sum_{i=1}^{s} x_i^2 \equiv 0$. Since $s \leq 4$ and squares are congruent to 0, 1 or 4 (mod 8), it follows that each x_i is even, $x_i = 2y_i$, and $2^{n-2} = \sum_{i=1}^{s} y_i^2$ with the y_i distinct. By descent, either 2 or 4 is the sum of two, three or four distinct squares, which is false.

In the same way, we can prove that no number of any of the following forms is the sum of two, three or four distinct squares:

$$3 \times 2^n,\ 9 \times 2^n,\ 11 \times 2^n,\ 7 \times 4^n,\ 15 \times 4^n,\ 19 \times 4^n,\ 23 \times 4^n,\ 27 \times 4^n,$$
$$31 \times 4^n,\ 33 \times 4^n,\ 43 \times 4^n,\ 47 \times 4^n,\ 55 \times 4^n,\ 67 \times 4^n,\ 103 \times 4^n.$$

Moreover, I conjecture that these are **all** the numbers that are not the sum of two, three or four distinct squares. (I have checked up to 2000.)

I suspect the authors' problem, to estimate the number of expressions of n as the sum of two or more distinct squares, is quite difficult.

However, if we let $e_5(n)$ be the number of expressions of n as the sum of no more than five distinct squares, then

$$\sum_{m \leq n} e_5(m) \sim \frac{\pi^2}{7200} n^{5/2},$$

from which we can deduce that

$$e_5(n) \approx \frac{\pi^2}{2880} n^{3/2},$$

though local variations are relatively quite large.

References

[1] J. Bohman, C-E. Fröberg and H. Riesel, Partitions in squares, BIT **19** (1979), 297–301.

[2] H. Gupta, MR **80k**:10043

[3] R. K. Guy, Unsolved Problems in Number Theory, Springer–Verlag, New York, 1994.

[4] J. Kingston and D. MacHale, Dissecting squares, *Math. Gaz.* **85** (November 2001) pp. 403–430.