SOME COUPLED SECOND–ORDER RECURRENCES AND THEIR SOLUTIONS

Michael D. Hirschhorn

There is a lot of interest in the Fibonacci numbers and related mathematics. (There is a journal devoted to this area.) Recently I received a copy of a somewhat unusual book on the subject [1]. In the first section the authors consider various systems of coupled second–order recurrence relations, but do not state the solutions as succinctly as is possible.

My aim in this short note is to simply state the solutions of these systems, all of which can be proved by induction (though I must admit to having used generating functions to find them).

The Fibonacci numbers are given by

$$\{F_n\}_{n\geq 0} = \{0, 1, 1, 2, 3, 5, \cdots\}.$$

In order to state our results, we define two periodic sequences with period 6 by

$$\{u_n\}_{n\geq 0} = \{0, 1, -1, 0, 1, -1, \cdots\},$$
$$\{v_n\}_{n\geq 0} = \{0, 1, 1, 0, -1, -1, \cdots\}.$$

Incidentally,

$$u_n = \frac{2}{\sqrt{3}} \sin \frac{2n\pi}{3}, \quad v_n = \frac{2}{\sqrt{3}} \sin \frac{n\pi}{3}.$$

The systems and their solutions are

$$a_{n+2} = b_{n+1} + b_n, \qquad b_{n+2} = a_{n+1} + a_n, \tag{1}$$

$$a_n = \frac{1}{2}(F_{n-1} - u_{n-1})a_0 + \frac{1}{2}(F_{n-1} + u_{n-1})b_0 + \frac{1}{2}(F_n + u_n)a_1 + \frac{1}{2}(F_n - u_n)b_1,$$
$$b_n = \frac{1}{2}(F_{n-1} + u_{n-1})a_0 + \frac{1}{2}(F_{n-1} - u_{n-1})b_0 + \frac{1}{2}(F_n - u_n)a_1 + \frac{1}{2}(F_n + u_n)b_1.$$

(2)
$$a_{n+2} = a_{n+1} + b_n,$$
$$b_{n+2} = b_{n+1} + a_n,$$

$$a_n = \frac{1}{2}(F_{n-1} - v_{n-1})a_0 + \frac{1}{2}(F_{n-1} + v_{n-1})b_0 + \frac{1}{2}(F_n + v_n)a_1 + \frac{1}{2}(F_n - v_n)b_1,$$
$$b_n = \frac{1}{2}(F_{n-1} + v_{n-1})a_0 + \frac{1}{2}(F_{n-1} - v_{n-1})b_0 + \frac{1}{2}(F_n - v_n)a_1 + \frac{1}{2}(F_n + v_n)b_1.$$

(3)
$$a_{n+2} = b_{n+1} + a_n,$$
$$b_{n+2} = a_{n+1} + b_n,$$

$$a_{2n} = F_{2n-1}a_0 + F_{2n}b_1,\ a_{2n+1} = F_{2n}b_0 + F_{2n+1}a_1,$$
$$b_{2n} = F_{2n-1}b_0 + F_{2n}a_1,\ b_{2n+1} = F_{2n}a_0 + F_{2n+1}b_1.$$

(4)
$$a_{n+2} = b_{n+1} - a_n,$$
$$b_{n+2} = a_{n+1} - b_n,$$

$$a_n = -\frac{1}{2}(u_{n-1} + v_{n-1})a_0 + \frac{1}{2}(u_{n-1} - v_{n-1})b_0 + \frac{1}{2}(u_n + v_n)a_1 - \frac{1}{2}(u_n - v_n)b_1,$$
$$b_n = \frac{1}{2}(u_{n-1} - v_{n-1})a_0 - \frac{1}{2}(u_{n-1} + v_{n-1})b_0 - \frac{1}{2}(u_n - v_n)a_1 + \frac{1}{2}(u_n + v_n)b_1.$$

(5)
$$a_{n+2} = a_{n+1} - b_n,$$
$$b_{n+2} = b_{n+1} - a_n,$$

$$a_n = \frac{1}{2}(F_{n-1} - v_{n-1})a_0 - \frac{1}{2}(F_{n-1} + v_{n-1})b_0 + \frac{1}{2}(F_n + v_n)a_1 - \frac{1}{2}(F_n - v_n)b_1,$$
$$b_n = -\frac{1}{2}(F_{n-1} + v_{n-1})a_0 + \frac{1}{2}(F_{n-1} - v_{n-1})b_0 - \frac{1}{2}(F_n - v_n)a_1 + \frac{1}{2}(F_n + v_n)b_1.$$

(6)
$$a_{n+2} = b_{n+1} - b_n,$$
$$b_{n+2} = a_{n+1} - a_n,$$

$$a_n = (-1)^n\{\frac{1}{2}(F_{n-1} - u_{n-1})a_0 - \frac{1}{2}(F_{n-1} + u_{n-1})b_0 - \frac{1}{2}(F_n + u_n)a_1 + \frac{1}{2}(F_n - u_n)b_1\},$$
$$b_n = (-1)^n\{-\frac{1}{2}(F_{n-1} + u_{n-1})a_0 + \frac{1}{2}(F_{n-1} - u_{n-1})b_0 + \frac{1}{2}(F_n - u_n)a_1 - \frac{1}{2}(F_n + u_n)b_1\}.$$

Reference

[1] K. Atanassov, V. Atanassova, A. Shannon, J. Turner, New Visual Perspectives on Fibonacci Numbers, World Scientific, 2002.