SUMS INVOLVING SQUARE–FREE INTEGERS

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We are probably all familiar to some degree with the formulae

$$\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6},\quad \sum_{n\geq 1}\frac{1}{n^4}=\frac{\pi^4}{90},\quad \sum_{n\geq 1}\frac{1}{n^6}=\frac{\pi^6}{945},\quad \sum_{n\geq 1}\frac{1}{n^8}=\frac{\pi^8}{9450},$$

and so on.

I have just found that if we sum over the square–free integers S, that is, those not divisible by a square greater than 1, then

$$\sum_{n\in S}\frac{1}{n^2}=\frac{15}{\pi^2},\quad \sum_{n\in S}\frac{1}{n^4}=\frac{105}{\pi^4},\quad \sum_{n\in S}\frac{1}{n^6}=\frac{675675}{691\pi^6},\quad \sum_{n\in S}\frac{1}{n^8}=\frac{34459425}{3617\pi^8}$$

and more generally,

$$(*)\qquad \sum_{n\in S}\frac{1}{n^k}=\frac{\zeta(k)}{\zeta(2k)},\text{ where }\zeta(k)=\sum_{n\geq 1}\frac{1}{n^k}.$$

Thus, in the case where k is a positive integer,

$$\sum_{n\in S}\frac{1}{n^{2k}}=\frac{\zeta(2k)}{\zeta(4k)}$$

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can be written in terms of the Bernoulli numbers B_k since, as is well–known,

$$\zeta(2k) = 2^{2k-1} B_k \pi^{2k}/(2k)!.$$

I offer two proofs of $(*)$.

(1) We have, for each prime p,

$$\left(1+\frac{1}{p^k}\right)\left(1+\frac{1}{p^{2k}}+\frac{1}{p^{4k}}+\ \cdots\right)=\left(1+\frac{1}{p^k}+\frac{1}{p^{2k}}+\ \cdots\right).$$

If we take the product over all primes p we obtain

$$\sum_{n\in S}\frac{1}{n^k}\cdot\sum_{n\geq 1}\frac{1}{n^{2k}}=\sum_{n\geq 1}\frac{1}{n^k},$$

or,

$$\sum_{n\in S}\frac{1}{n^k}\cdot\zeta(2k)=\zeta(k). \quad \square$$

(2) Let

$$A=\sum_{n\in S}\frac{1}{n^k}, \quad B=\sum_{n\notin S}\frac{1}{n^k}.$$

Then

$$A+B=\sum_{n\geq 1}\frac{1}{n^k}=\zeta(k),$$

while

$$B=\sum_{m\geq 2,\ n\in S}\frac{1}{(m^2n)^k}=\sum_{m\geq 2}\frac{1}{m^{2k}}\sum_{n\in S}\frac{1}{n^k}=\left(\zeta(2k)-1\right)A.$$

It follows that

$$A=\frac{\zeta(k)}{\zeta(2k)}, \quad B=\zeta(k)\left(1-\frac{1}{\zeta(2k)}\right). \quad \square$$

We close by noting that if $k=1$, both A and B diverge.