COURSE OUTLINE

MATH1081
Discrete Mathematics

INFORMATION BOOKLET

Semester 1, 2016
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GENERAL INFORMATION
Semester 1, 2016

Assumed Knowledge and Co-requisites
MATH1081, Discrete Mathematics, is a first year 6UOC course available in semester 1 and semester 2. The assumed knowledge for the course is the equivalent of a combined mark of at least 100 in HSC Mathematics and HSC Mathematics Extension 1. There is a formal corequisite of MATH1131 or MATH1141 or MATH1151.

The nature of the course
The subject matter of this course is very different from “high school mathematics” and success at high school is no guarantee of success in Discrete Mathematics. In MATH1081 emphasis is placed on reasoned argument and clarity of exposition as well as algebraic and computational skills.

Lecturers
Dr Dennis Trenerry
Dr Thomas Britz, RC-5111

Lecturer in Charge
Dr T. Britz

Lectures
Lectures are given four times per week, commencing in week 1 and running to week 12. Full details of the timetable are shown in your timetable on myUNSW and the Online Handbook.

The material presented is divided into five sections, and each part will be presented in 2 or 3 week segments as follows:

<table>
<thead>
<tr>
<th>Section</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weeks</td>
<td>1-2</td>
<td>3-4</td>
<td>5-7</td>
<td>8-10</td>
<td>11-12</td>
</tr>
</tbody>
</table>

Tutorials
Each student enrolled in MATH1081 has chosen or been assigned to two tutorial time slots as shown in your timetable. If you wish to change your tutorial times, you should try to do this online using myUNSW before the Monday of week 2. After that time you will not be able to make tutorial changes online. Instead, you will have to go to the Student Services Office, RC-3088.

Each student will have two tutorials per week with the same tutor, with tutorials starting in week 2 and running to week 13.

Attendance at tutorials is compulsory and the roll will be called in tutorials.

There will be a roster of staff available to help students in first-year mathematics courses. This roster is displayed on level 3 of the Red Centre, on the notice boards adjacent to the Student Services Office and opposite the School Office (Room RC-3070).
Assessment

Assessment will be made up of two components - Final Examination 80% and Class Tests 20%. We will count the best 3 of the four class tests. Note that:

- You will **NOT** be allowed to take a calculator into class tests.

- Students can view their marks through the Student Portal, which is accessible via the “Maths & Stats marks” link on the MATH1081 homepage on UNSW Moodle.

- It is **your responsibility** to check that these marks are correct and you should **keep marked tests until the end of semester** in case an error has been made in recording the marks. If there is an error, either speak to your tutor or bring your test paper to the Student Services Office as soon as possible but no later than **no later than Friday week 13**.

UNSW Moodle

The School of Mathematics and Statistics uses the Learning Management System called Moodle. To log in to Moodle use your zID and zPass at the following URL:

http://moodle.telt.unsw.edu.au

Once logged in, you should see a link to MATH1081 that will take you to the MATH1081 homepage in Moodle.

In the general information section there is a link called “Maths & Stats marks”. This takes you to a page where you can log in with your zPass and see the marks recorded for various assessment tasks. After classes have finished and before the start of the exam period, you should log in here and check that your marks have been correctly recorded. This is also where you will find your provisional final mark once it is released by the School.

Textbooks


Reference Books


For interesting applications within Computer Science, try the three part classic - D.E. Knuth, “The Art of Computer Programming”.

Class Tests

There will be one test for each of the first four sections outlined in the syllabus above. The best three will count for assessment. Tests for sections (1), (2), (3) and (4) will be held at the beginning of the first tutorial of the weeks given below.

<table>
<thead>
<tr>
<th>section</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>week</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>
If you are too sick to attend a class test, do not make an application for Special Consideration. Simply give a copy of your medical certificate to your tutor.

Note that

- You MUST be enrolled in a pair of tutorials and you MUST TAKE EVERY TEST IN THE TUTORIAL TO WHICH YOU HAVE BEEN OFFICALLY ALLOCATED.
- To each test you must bring your STUDENT ID card
- You will NOT be allowed to use a calculator in class tests.
- Your best three scores in the four tests will be counted towards your final assessment mark.

Getting help outside tutorials

From week 3 there is a roster which shows for each hour of the week a list of names of members of staff who are available at that time to help students in first year mathematics courses. This roster is displayed on the same noticeboard as timetables, near the School Office (Room 3070, Red Centre) and also outside the Student Services Office (Room 3088, Red Centre). It is also available from the web page

http://www.maths.unsw.edu.au/currentstudents/consultation-mathematics-staff

Calculator Information

For end of semester UNSW exams students must supply their own calculator. Only calculators on the UNSW list of approved calculators may be used in the end of semester exams. This list is similar to the list of calculators approved for HSC examinations. BEFORE the exam period calculators must be given a UNSW “approved calculator” sticker, obtainable from the School of Mathematics and Statistics Office, and other student or Faculty centres.

The UNSW list of calculators approved for use in end of semester exams is available at

https://student.unsw.edu.au/exams

Academic misconduct

It is very important that you understand the University’s Rules for the conduct of Examinations and the penalties for Academic Misconduct. This information can be accessed through myUNSW at:

https://my.unsw.edu.au/student/academiclife/assessment/

{https://student.unsw.edu.au/grades}

Illness and other problems

If your performance in this course is affected by illness or other serious difficulties which are beyond your control, you can apply for Special Consideration and you may be offered the opportunity for Additional Assessment. In order to be offered Additional Assessment it is essential that you follow exactly the procedures set out in the sheet entitled “Application for Special Consideration in First Year Mathematics Courses 2016.” A copy of this document is included in this booklet. Take particular note that

- The School will NOT contact you to tell you that you have been granted Additional Assessment. It is YOUR RESPONSIBILITY to find this out by following the instructions in the document mentioned above.
- If you have a poor record of attendance or performance during the semester you may be failed regardless of illness or compassionate grounds affecting the final exam.
Note also that

- If illness affects your attendance at or performance in a **class test**, do **NOT** make an application for Special Consideration. Simply show a medical certificate to your tutor and this will be taken into account when calculating your final assessment mark.

- If you arrive too late to be admitted to the end of semester exam, go **immediately** to the Student Services Office, Room 3088, Red Centre.

**School of Mathematics and Statistics Policies**

The appropriate pages on the MathsStats web site start at:

http://www.maths.unsw.edu.au/currentstudents/assessment-policies

The School of Mathematics and Statistics will assume that all its students have read and understood the School policies on the above pages and any individual course policies on the Course Handout and Course Home Page. Lack of knowledge about a policy will not be an excuse for failing to follow the procedures in it.

**Contacting the Student Services Office**

The School of Mathematics and Statistics web-site

http://www.maths.unsw.edu.au

contains many pages of useful information on mathematics courses, school policies and how to obtain help, both academic and administrative.

In particular, the URL

http://maths.unsw.edu.au/currentstudents/student-services

provides a range of menus to choose from.

The student administration officer for First Year in the Student Services Office of the School of Mathematics and Statistics is Ms M. Lugton (Markie). All administrative enquiries concerning first year Mathematics courses should be sent to Ms Lugton, either:

- by email to fy.MathsStats@unsw.edu.au
- by phone to 9385 7011
- or in person in room RC-3088.

**Aims**

The aim of MATH1081 is that by the time you finish the course you should understand the concepts and techniques covered by the syllabus and have developed skills in applying these concepts and techniques to the solution of appropriate problems. Successful completion of the course will give you a good foundation for understanding many problems that arise in computer science.

**Learning Outcomes**

A student should be able to:

- state definitions as specified in the syllabus,
- state and prove appropriate theorems,
- explain how a theorem applies to specific examples,
• apply the concepts and techniques of the syllabus to solve appropriate problems,

• understand and apply appropriate algorithms,

• use mathematical and other terminology appropriately to communicate information and understanding.

Graduate Attributes

MATH1081 will enhance your research, inquiry and analytical thinking abilities as it will provide you with the mathematical language and mathematical techniques to unravel many seemingly unrelated problems. The course will engage you in independent and reflective learning through your independent mastery of a wide range of tutorial problems. The mathematical problem solving skills that you will develop are generic problem solving skills, based on logical arguments and mathematical language, that can be applied in multidisciplinary work. You will be encouraged to develop your communication skills through active participation in tutorials, and by writing clear, logical arguments when solving problems.

Peter Brown
Director of First Year Studies
School of Mathematics and Statistics
fy.MathsStats@unsw.edu.au
APPLICATIONS FOR SPECIAL CONSIDERATION IN
FIRST YEAR MATHEMATICS COURSES SEMESTER 1 2016

If you feel that your performance in, or attendance at, a final examination has been affected by illness or circumstances beyond your control, or if you missed the examination because of illness or other compelling reasons, you may apply for special consideration. Such an application may lead to the granting of additional assessment.

**It is essential that you take note of the following rules, which apply to applications for special consideration in all first year Mathematics courses.**

1. **Within 3 days** of the affected examination, or at least as soon as possible, you must **submit a request for special consideration to UNSW Student Central ON-LINE.**
   Please refer to link below for How to Apply for Special Consideration,

   https://my.unsw.edu.au/student/atoz/SpecialConsideration.html ApplyingforSpecialConsideration

2. **Please do not expect an immediate response from the School.** All applications will be considered together. See the information below.

3. **You will NOT be granted additional assessment in a course if your performance in the course (judged by attendance, class tests, assignments and examinations) does not meet a minimal standard.** A total mark of at least 40% on all assessment not affected by a request for special consideration will normally be regarded as the minimal standard for award of additional assessment as will at least 80% attendance at tutorial classes.

4. **It is YOUR RESPONSIBILITY** to find out **FROM THE SCHOOL OF MATHEMATICS AND STATISTICS** whether you have been granted additional assessment and when and where the additional assessment examinations will be held. **Do NOT wait to receive official results from the university,** as these results are not normally available until after the Mathematics additional assessment exams have started.
   
   a) A **provisional** list of results in all Mathematics courses and of grants of additional assessment will be available via the “Maths&Stats marks” link in the UNSW Moodle module of your course. The date for this will be announced later.
   
   b) Please read all announcements on Moodle. Failure to read announcements will not be accepted as a reason for missing supplementary exams and for not following the correct procedures.

5. **The timetables** for the additional assessment examinations will be available on the Mathematics website at the same time as the provisional list of results.

   The dates for the mid-year additional assessment examinations will be announced later in the Semester.

6. If you have two additional assessment examinations scheduled for the same time, please consult the School of Mathematics and Statistics Office as soon as possible so that special arrangements can be made.

7. You will need to produce your UNSW Student Card to gain entry to additional assessment examinations.
IMPORTANT NOTES

- The additional assessment examination may be of a different form from the original examination and must be expected to be at least as difficult.

- If you believe that your application for special consideration has not been processed, you should immediately consult the Director of First Year Studies of the School of Mathematics and Statistics (Room 3073 Red Centre).

- If you believe that the above arrangements put you at a substantial disadvantage, you should, at the earliest possible time, send full documentation of the circumstances to the Director of First Year Studies, School of Mathematics and Statistics, University of New South Wales, Sydney, 2052.

In particular, if you suffer from a chronic or ongoing illness that has, or is likely to, put you at a serious disadvantage then you should contact the Student Equity and Disabilities Unit (SEADU) who provide confidential support and advice. Their web site is

http://www.studentequity.unsw.edu.au

SEADU may determine that your condition requires special arrangements for assessment tasks. Once the First Year Office has been notified of these we will make every effort to meet the arrangements specified by SEADU.

Additionally, if you have suffered a serious misadventure during semester then you should provide full documentation to the Director of First Year Studies as soon as possible. In these circumstances it may be possible to arrange discontinuation without failure or to make special examination arrangements.

Professor B. Henry
Head, School of Mathematics and Statistics
UNIVERSITY STATEMENT ON PLAGIARISM

Plagiarism is the presentation of the thoughts or work of another as one’s own. Examples include:

- direct duplication of the thoughts or work of another, including by copying work, or knowingly permitting it to be copied. This includes copying material, ideas or concepts from a book, article, report or other written document (whether published or unpublished), composition, artwork, design, drawing, circuitry, computer program or software, web site, Internet, other electronic resource, or another person’s assignment without appropriate acknowledgement
  - paraphrasing another person’s work with very minor changes keeping the meaning, form and/or progression of ideas of the original;
  - piecing together sections of the work of others into a new whole;
  - presenting an assessment item as independent work when it has been produced in whole or part in collusion with other people, for example, another student or a tutor; and,
  - claiming credit for a proportion a work contributed to a group assessment item that is greater than that actually contributed.

Submitting an assessment item that has already been submitted for academic credit elsewhere may also be considered plagiarism. The inclusion of the thoughts or work of another with attribution appropriate to the academic discipline does not amount to plagiarism.

Students are reminded of their Rights and Responsibilities in respect of plagiarism, as set out in the University Undergraduate and Postgraduate Handbooks, and are encouraged to seek advice from academic staff whenever necessary to ensure they avoid plagiarism in all its forms.

The Learning Centre website is the central University online resource for staff and student information on plagiarism and academic honesty. It can be located at: www.lc.unsw.edu.au/plagiarism

The Learning Centre also provides substantial educational written materials, workshops, and tutorials to aid students, for example, in:

- correct referencing practices;
- paraphrasing, summarising, essay writing, and time management;
- appropriate use of, and attribution for, a range of materials including text, images, formulae and concepts.

Individual assistance is available on request from The Learning Centre. Students are also reminded that careful time management is an important part of study and one of the identified causes of plagiarism is poor time management. Students should allow sufficient time for research, drafting, and the proper referencing of sources in preparing all assessment items.

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1 Based on that proposed to the University of Newcastle by the St James Ethics Centre. Used with kind permission from the University of Newcastle.

2 Adapted with kind permission from the University of Melbourne.
## Syllabus

References are to the textbook by Epp, unless marked otherwise. F indicates the textbook by Franklin and Daoud and R indicates the book *Discrete Mathematics with Applications* by K.H. Rosen (6th edition). The UNSW Library has multiple copies of Rosen numbered P510/482A,B,C, etc.

The references shown in the righthand column are *not* intended to be a definition of what you will be expected to know. They are just intended as a guide to finding relevant material. Some parts of the course are not covered in the textbooks and some parts of the textbooks (even in sections mentioned in the references below) are not included in the course.


Within sections of the course, the topics may not be covered in exactly the order in which they are listed below.

<table>
<thead>
<tr>
<th>Topic</th>
<th>References A</th>
<th>References B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. Sets, functions and sequences</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sets, subsets, power sets. Equality, cardinality.</td>
<td>5.1, 5.3</td>
<td>1.2, 6.1, 6.3</td>
</tr>
<tr>
<td>Set operations: union, intersection, difference, cartesian product. Universal sets, complements.</td>
<td>5.1</td>
<td>6.1</td>
</tr>
<tr>
<td>Russell’s paradox.</td>
<td>5.2</td>
<td>6.2</td>
</tr>
<tr>
<td>Functions. Domain, codomain and range. Arrow diagrams. Ceiling and floor functions. Images and inverse images of sets.</td>
<td>7.1, 3.5</td>
<td>1.3, 7.1, 4.5</td>
</tr>
<tr>
<td>Injective (one-to-one), surjective (onto) and bijective functions.</td>
<td>7.3</td>
<td>7.2</td>
</tr>
<tr>
<td>Composition of functions</td>
<td>7.4</td>
<td>7.3</td>
</tr>
<tr>
<td>Inverse functions.</td>
<td>7.2</td>
<td>7.2</td>
</tr>
<tr>
<td>Sequences, sums and products. Notation. Change of variable in a sum. Telescoping sums.</td>
<td>4.1</td>
<td>5.1</td>
</tr>
</tbody>
</table>

| **2. Integers, Modular Arithmetic and Relations**         |              |              |
| Prime numbers and divisibility                            | 3.1, 3.3     | 4.1, 4.3     |
| Fundamental Theorem of Arithmetic                         | 3.3          | 4.3          |
| Euclidean Algorithm                                        | 3.8          | 4.8          |
| Modular Arithmetic                                         | 3.4          | 4.4, 8.4     |
| Solving Linear Congruences                                 | R2.5         | R3.7         |
| General Relations                                          | 10.1         | 8.1          |
| Reflexivity, symmetry and transitivity                     | 10.2         | 8.2          |
| Equivalence Relations                                      | 10.3         | 8.3          |
| Partially ordered sets and Hasse diagrams                  | 10.5         | 8.5          |
### 3. Logic and Proofs

<table>
<thead>
<tr>
<th>Topic</th>
<th>References A</th>
<th>References B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof versus intuition. Direct proof.</td>
<td>F1</td>
<td>F1</td>
</tr>
<tr>
<td>Propositions, connectives, compound propositions.</td>
<td>1.1</td>
<td>2.1</td>
</tr>
<tr>
<td>Truth tables. Tautology, contingency, logical equivalence.</td>
<td>1.1</td>
<td>2.1</td>
</tr>
<tr>
<td>Implication, converse, inverse, biconditional.</td>
<td>1.2</td>
<td>2.2</td>
</tr>
<tr>
<td>Rules of inference.</td>
<td>1.3</td>
<td>2.3</td>
</tr>
<tr>
<td>Contrapositive, indirect proof, proof by contradiction.</td>
<td>1.2, 3.6, F6,3.7</td>
<td>2.2, 4.6, 4.7, F6</td>
</tr>
<tr>
<td>Quantifiers</td>
<td>2.1</td>
<td>3.1</td>
</tr>
<tr>
<td>Proof of universal statements, exhaustion, proof by cases.</td>
<td>2.1, F2, F3</td>
<td>3.1, F2, F3</td>
</tr>
<tr>
<td>Proof of existential statements. Constructive and non-constructive proofs.</td>
<td>2.1, 3.1, F4, F6</td>
<td>3.1, 4.1, F4, F6</td>
</tr>
<tr>
<td>Counterexamples.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Negation of quantified statements.</td>
<td>2.1</td>
<td>3.2</td>
</tr>
<tr>
<td>Statements with multiple quantifiers.</td>
<td>2.2, 2.3, F5</td>
<td>3.2, 3.3, F5</td>
</tr>
<tr>
<td>Common mistakes in reasoning. Converse and inverse fallacies. Begging</td>
<td>2.3, 3.1</td>
<td>3.3, 3.4, 4.1</td>
</tr>
<tr>
<td>the question, tacit assumption, etc.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mathematical induction</td>
<td>4.2-4.4, F8</td>
<td>5.2-5.4, F8</td>
</tr>
</tbody>
</table>

*Note:* In addition to the sections of Epp mentioned above, sections 4.2-4.5 and 4.7 (3.2-3.5, 3.7 for edition 3) provide many useful worked examples of constructing proofs in elementary number theory.

### 4. Enumeration and Probability

<table>
<thead>
<tr>
<th>Topic</th>
<th>References A</th>
<th>References B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting and Probability</td>
<td>6.1</td>
<td>9.1</td>
</tr>
<tr>
<td>Multiplication Rule</td>
<td>6.2</td>
<td>9.2</td>
</tr>
<tr>
<td>Addition Rule</td>
<td>6.3</td>
<td>9.3</td>
</tr>
<tr>
<td>Principle of Inclusion-Exclusion</td>
<td>6.3</td>
<td>9.3</td>
</tr>
<tr>
<td>Pigeonhole Principle</td>
<td>7.3</td>
<td>9.4</td>
</tr>
<tr>
<td>Permutations and Combinations</td>
<td>6.4, 6.5</td>
<td>9.5, 9.6</td>
</tr>
<tr>
<td>Binomial and Multinomial Theorem</td>
<td>6.7, R4.6</td>
<td>9.7, R5.4</td>
</tr>
<tr>
<td>Discrete Probability</td>
<td>R4.4, 6.1</td>
<td>R6.1, 9.1</td>
</tr>
<tr>
<td>Recurrence Relations</td>
<td>8.2, 8.3</td>
<td>5.6, 5.7, 5.8</td>
</tr>
<tr>
<td>Recursively Defined Sets and Functions</td>
<td>8.1</td>
<td>5.9</td>
</tr>
</tbody>
</table>

### 5. Graphs

<table>
<thead>
<tr>
<th>Topic</th>
<th>References A</th>
<th>References B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic terminology. simple graphs, $K_n$. Directed graphs. Subgraphs, complementary graphs.</td>
<td>11.1</td>
<td>10.1</td>
</tr>
<tr>
<td>Degree, the Handshaking Theorem (Epp Theorem 10.1.1 (11.1.1 in ed. 3))</td>
<td>11.1</td>
<td>10.1</td>
</tr>
<tr>
<td>Bipartite graphs, $K_{m,n}$.</td>
<td>11.1</td>
<td>10.1</td>
</tr>
<tr>
<td>Adjacency and incidence matrices.</td>
<td>11.3</td>
<td>10.3</td>
</tr>
<tr>
<td>Isomorphism, isomorphism invariants.</td>
<td>11.4</td>
<td>10.4</td>
</tr>
<tr>
<td>Walks, paths and circuits. Euler and Hamilton paths. Connected graphs, connected components.</td>
<td>11.2</td>
<td>10.2</td>
</tr>
<tr>
<td>Trees, spanning trees.</td>
<td>11.5, 11.6</td>
<td>10.5, 10.7</td>
</tr>
<tr>
<td>Weighted graphs. Minimal spanning trees. Kruskal and Dijkstra algorithms.</td>
<td>11.6</td>
<td>10.6, 10.7</td>
</tr>
</tbody>
</table>
PROBLEM SETS

**Recommended Problems:** It is strongly recommended that you attempt all questions marked by †. You should regard these questions as the minimum that you should attempt if you are to pass this course. However, the more practice in solving problems you get the better you are likely to do in class tests and exams, and so you should aim to solve as many of the problems on this sheet as possible. Ask your tutor about any problems you cannot solve. Problems marked by a star (∗) are more difficult, and should only be attempted after you are sure you can do the unstarred problems.

**PROBLEM SET 1**

**Basic Set Theory**

1. Are any of the sets \( A = \{1, 1, 2, 3\} \), \( B = \{3, 1, 2, 2\} \), \( C = \{1, 2, 1, 2, 4, 2, 3\} \) equal?

†2. Show that

\[ A = \{ x \in \mathbb{R} \mid \cos x = 1 \} \]

is a subset of

\[ B = \{ x \in \mathbb{R} \mid \sin x = 0 \} \].

Is the first a proper subset of the second? Give reasons.

3. a) List all the subsets of the set \( A = \{a, b, c\} \).

b) List all the elements of \( A \times B \) where \( A = \{a, b, c\} \) and \( B = \{1, 2\} \).

4. Given the sets \( X = \{24k + 7 \mid k \in \mathbb{Z}\} \), \( Y = \{4n + 3 \mid n \in \mathbb{Z}\} \), \( Z = \{6m + 1 \mid m \in \mathbb{Z}\} \), prove that \( X \subseteq Y \) and \( X \subseteq Z \) but \( Y \nsubseteq Z \).

†5. If \( S = \{0, 1\} \), find

a) \( |P(S)| \),

b) \( |P(P(S))| \),

c) \( |P(P(P(S)))| \).

†6. Determine whether the following are true or false

a) \( a \in \{a\} \),

b) \( \{a\} \in \{a\} \),

c) \( \{a\} \subseteq \{a\} \),

d) \( a \subseteq \{a\} \),

e) \( \{a\} = \{a, \{a\}\} \),

f) \( \{a\} \in \{a, \{a\}\} \),

g) \( \{a\} \subseteq \{a, \{a\}\} \),

7. If \( A, B, C \) are sets such that \( A \subseteq B \) and \( B \subseteq C \), prove that \( A \subseteq C \).

8. Is it true that if \( P(A) = P(B) \) for two sets \( A, B \) then \( A = B \)?
Set Operations and Algebra

9. If \( A = \{ \text{letters in the word } \text{mathematics} \} \) and 
\[ B = \{ \text{letters in the words } \text{set theory} \} \], list the elements of the sets
   
   a) \( A \cup B \),
   
   b) \( A \cap B \),
   
   c) \( A - B \),
   
   d) \( B - A \).

10. Define the sets \( R, S \) and \( T \) by
   
   \[ R = \{ x \in \mathbb{Z} \mid x \text{ is divisible by } 2 \} \]
   
   \[ S = \{ x \in \mathbb{Z} \mid x \text{ is divisible by } 3 \} \]
   
   \[ T = \{ x \in \mathbb{Z} \mid x \text{ is divisible by } 6 \} \].

   a) Is \( S = T \)?
   
   b) Is \( R \subseteq T \)?
   
   c) Is \( T \subseteq R \)?
   
   d) Is \( T \subseteq S \)?
   
   e) Find \( R \cap S \).

†11. In a class of 40 people studying music: 2 play violin, piano and recorder, 7 play at least violin and piano, 6 play at least piano and recorder, 5 play at least recorder and violin, 17 play at least violin, 19 play at least piano, and 14 play at least recorder. How many play none of these instruments?

†12. Prove the following statements if they are true and give a counter-example if they are false.
   
   a) For all sets \( A, B \) and \( C \), if \( A \cap C \subseteq B \cap C \) and \( A \cup C \subseteq B \cup C \) then \( A \subseteq B \).
   
   b) For all sets \( A, B \) and \( C \), \( (A \cup B) \cap C = A \cup (B \cap C) \).

†13. Let \( A \) and \( B \) be general sets. Determine the containment relation (\( \subseteq, \supseteq, =, \) none) that holds between
   
   a) \( P(A \cup B) \) and \( P(A) \cup P(B) \),
   
   b) \( P(A \cap B) \) and \( P(A) \cap P(B) \).

†14. Let \( A \) and \( B \) be general sets. Determine the containment relation (\( \subseteq, \supseteq, =, \) none) that holds between
   
   \[ P(A \times B) \text{ and } P(A) \times P(B) \]

15. Show that \( A - B = A \cap B^c \) and hence simplify the following using the laws of set algebra.
   
   a) \( A \cap (A - B) \).
   
   b) \( (A - B) \cup (A \cap B) \).
   
   c) \( (A \cup B) \cup (C \cap A) \cup (A \cap B)^c \).

†16. Use the laws of set algebra to simplify
   
   \( (A - B^c) \cup (B \cap (A \cap B)^c) \).
17. Simplify
\[ [A \cap (A \cap B^c)] \cup [(A \cap B) \cup (B \cap A^c)] , \]
and hence simplify
\[ [A \cup (A \cup B^c)] \cap [(A \cup B) \cap (B \cup A^c)] . \]

18. Draw a Venn diagram for the general situation for three sets \( A, B \) and \( C \). Use it to answer the following:
   a) If \( B - A = C - A \), what subregions in your diagram must be empty?
   b) Prove or produce a counter-example to the following statement:
      \[
      \text{If } B - A = C - A \quad \text{then} \quad B = C .
      \]

19. Use the laws of set algebra to prove that
\[
(R - P) - Q = R - (P \cup Q)
\]
for any sets \( R, P \) and \( Q \).

*20. Define the symmetric difference \( A \oplus B \) of two sets \( A \) and \( B \) to be
\[
A \oplus B = (A - B) \cup (B - A).
\]
   a) Draw a Venn diagram illustrating \( A \oplus B \).
   b) If \( A = \{ \text{even numbers strictly between } 0 \text{ and } 20 \} \),
      \( B = \{ \text{multiples of } 3 \text{ strictly between } 0 \text{ and } 20 \} \), write down the set \( A \oplus B \).
   c) Explain why \( A \oplus B \) can also be written as \( (A \cup B) - (A \cap B) \).
   d) Suppose \( A, B \) and \( C \) are sets such that \( A \oplus C = B \oplus C \). Prove that \( A = B \).
      (Hint: You may use a Venn diagram to assist your argument.)

†21. Prove if true or give a counter example if false:
   For all sets \( A, B \) and \( C \), \( A \times (B \cup C) = (A \times B) \cup (A \times C) \).

†22. Let
\[
A_k = \{ n \in \mathbb{N} \mid k \leq n \leq k^2 + 5 \}
\]
for \( k = 1, 2, 3, \ldots \). Find
   a) \( \bigcup_{k=1}^{4} A_k \);
   b) \( \bigcap_{k=10}^{90} A_k \);
   c) \( \bigcap_{k=1}^{\infty} A_k \).

23. Repeat the previous question if
\[
A_k = \{ x \in \mathbb{R} \mid 1 - \frac{1}{k} < x \leq k \} .
\]
Functions

24. Which of the following are functions?
   a) \( f : \mathbb{R} \to \mathbb{R}, \, f(x) = \sqrt{x^2 - 1} \).
   b) \( f : \mathbb{Z} \to \mathbb{Z}, \, f(x) = 2x + 1 \).
   c) \( f : \mathbb{R} \to \mathbb{R}, \, f(x) = \frac{1}{x} \).
   d) \( f : \mathbb{Q} \to \mathbb{Q}, \, f(x) = q \) where \( x = \frac{p}{q}, \, p, q \) integers.
   e) \( f : \mathbb{Q} \to \mathbb{Q}, \, f(x) = q \) where \( x = \frac{p}{q}, \, p, q \) integers with \( q > 0 \) and no common factor except 1.

25. Recall from lectures that \( \lfloor x \rfloor \) is the largest integer less than or equal to \( x \), and that \( \lceil x \rceil \) is the smallest integer greater than or equal to \( x \). Evaluate
   a) \( \lfloor \pi \rfloor \),
   b) \( \lceil \pi \rceil \),
   c) \( \lfloor -\pi \rfloor \),
   d) \( \lceil -\pi \rceil \).

*26. Prove that if \( n \) is an integer, then
   \[ n - \left\lfloor \frac{1}{3^n} \right\rfloor - \left\lceil \frac{2}{3^n} \right\rceil \]
   equals either 0 or 1.
   (Hint: Write \( n \) as \( 3k, 3k + 1 \) or \( 3k + 2 \), where \( k \) is an integer.)

†27. Determine which of the following functions are one-to-one, which are onto, and which are bijections.
   a) \( f : \mathbb{Z} \to \mathbb{Z}, \, f(x) = 2x \).
   b) \( f : \mathbb{Q} \to \mathbb{Q}, \, f(x) = 2x + 3 \).
   c) \( f : \mathbb{R} \to \mathbb{Z}, \, f(x) = \lfloor x \rfloor \).
   *d) \( f : \mathbb{R} \to \mathbb{R}, \, f(x) = x - \lfloor x \rfloor \).

†28. a) Let \( S \) be the set \( \{ n \in \mathbb{N} \mid 0 \leq n \leq 11 \} \) and define \( f : S \to S \) by letting \( f(n) \) be the remainder when \( 5n + 2 \) is divided by 12. Is \( f \) one-to-one? Is \( f \) onto?
   b) Repeat part (a) with \( 5n + 2 \) replaced by \( 4n + 2 \).

†29. Define \( f : \mathbb{R} \to \mathbb{R} \) by
   \[ f(x) = x^2 - 4x + 6 \].
   a) What is the range of \( f \)?
   b) Is \( f \) onto? Explain.
   c) Is \( f \) one-to-one? Explain.

30. Repeat the above question with \( g : \mathbb{R} \to \mathbb{R} \) defined by
   \[ g(x) = x^3 - x + 1 \]
   and \( h : \mathbb{R} \to \mathbb{R} \) defined by
   \[ h(x) = x^3 + x + 1 \]
   (Hint: You may use differentiation.)
Let $\mathbb{Z}^+$ be the set of all positive integers and $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be the function defined by $f(m, n) = mn$ for all $(m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Determine whether $f$ is one-to-one or onto.

Find $g \circ f$ for each of the following pairs of functions

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 3$  \quad  g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sqrt{x^2 + 2}$,

b) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$  \quad  g : \{\text{odd integers}\} \to \mathbb{Z}$, $g(x) = \frac{2x + 1}{2}$.

Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions.

a) Show that if $f$ and $g$ are both onto then $g \circ f$ is also onto.

b) Is it true that if $f$ and $g$ are both one-to-one then $g \circ f$ is also one-to-one?

If $f, g : \mathbb{N} \to \mathbb{N}$ are functions defined by $f(n) = 2n$ and

$$g(n) = \begin{cases} \frac{8}{2n} & \text{if } n \text{ is even} \\ \frac{n - 1}{2} & \text{if } n \text{ is odd} \end{cases}$$

show that $g \circ f = \iota$ but $f \circ g \neq \iota$ where $\iota$ is the identity function.

For each of the following bijections find the inverse and the domain and range of the inverse.

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 5x + 3$.

b) $g : \mathbb{Z} \to \mathbb{N}$

$$g(x) = \begin{cases} 2|x| - 1 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x^2 - 1$. Find

a) $f(A)$ if $A = \{x \mid -2 \leq x \leq 3\}$,

b) $f^{-1}(B)$ if $B = \{y \mid 1 \leq y \leq 17\}$.

Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the function given by

$$f(m, n) = m^2 - n^2.$$  

a) Show that $f$ is not onto.

b) Find $f^{-1}(\{8\})$.

Suppose $f$ is a function from $X$ to $Y$ and $A, B$ are subsets of $X$, and suppose that $S, T$ are subsets of $Y$.

a) What containment relation (if any) is there between

i) $f(A) \cup f(B)$ and $f(A \cup B)$,

ii) $f(A) \cap f(B)$ and $f(A \cap B)$?

b) Show that $f^{-1}(S) \cup f^{-1}(T) = f^{-1}(S \cup T)$.

c) Show that $f^{-1}(S) \cap f^{-1}(T) = f^{-1}(S \cap T)$. 
Sequences and Summation

39. Use the formulae
\[ \sum_{k=1}^{N} k = \frac{N}{2} (N + 1) \quad \text{and} \quad \sum_{k=1}^{N} k^2 = \frac{N}{6} (N + 1)(2N + 1) \]
to evaluate
\[ \sum_{k=3}^{22} (3k + 4)^2. \]

40. a) Make use of changes of summation index to show that
\[ \sum_{k=1}^{N} (k + 1)^3 - \sum_{k=1}^{N} (k - 1)^3 = (N + 1)^3 + N^3 - 1. \]

b) Hence show that
\[ \sum_{k=1}^{N} k^2 = \frac{N}{6} (N + 1)(2N + 1). \]

41. Show that
\[ \frac{2}{k(k+2)} = \frac{1}{k} - \frac{1}{k+2} \]
and hence, for \( N \geq 1 \)
\[ \sum_{k=1}^{N} \frac{2}{k(k+2)} = 3 - \frac{1}{N+1} - \frac{1}{N+2}. \]

42. Show that
\[ \frac{5k - 2}{k(k-1)(k-2)} = 4 \frac{3}{k-2} - \frac{1}{k-1} - \frac{1}{k} \]
and hence evaluate
\[ \sum_{k=3}^{n} \frac{5k - 2}{k(k-1)(k-2)}. \]

43. By writing out the terms, show that for \( N > 0 \)
\[ \prod_{k=1}^{N} \frac{k}{k + 2} = \frac{2}{(N + 1)(N + 2)}. \]

44. Using the fact that
\[ 1 - \frac{1}{k^2} = \frac{(k - 1)(k + 1)}{k^2} \]
find an expression for \( \prod_{k=2}^{N} \left( 1 - \frac{1}{k^2} \right) \) in terms of \( N \).
PROBLEM SET 2

Integers and Modular Arithmetic

†1. Find the quotient and (non-negative) remainder when
   a) 19 is divided by 7,
   b) −111 is divided by 11,
   c) 1001 is divided by 13.

2. Are the following true or false?
   7 | 161, 7 | 162, 17 | 68, 17 | 1001.

3. Which of the following are prime?
   17, 27, 37, 111, 1111, 11111.

†4. Find the prime factorization of the following
   117, 143, 3468, 75600.

†5. Find the gcd and lcm of the following pairs
   a) $2^2 \cdot 3^5 \cdot 5^3$ and $2^5 \cdot 3^3 \cdot 5^2$,
   b) $2^2 \cdot 3 \cdot 5^3$ and $3^2 \cdot 7$,
   c) 0 and 3.

†6. Evaluate $13 \mod 3$, $155 \mod 19$, $(-97) \mod 11$.

†7. Prove for $a, b, c, d \in \mathbb{Z}$, $k, m \in \mathbb{Z}^+$ that
   a) if $a | c$ and $b | d$ then $ab | cd$,
   b) if $ab | bd$ and $b \neq 0$, then $a | d$,
   c) if $a | b$ and $b | c$ then $a | c$,
   d) if $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$ then $a - b \equiv c - d \pmod{m}$,
   e) if $k | m$ and $a \equiv b \pmod{m}$ then $a \equiv b \pmod{k}$,
   f) if $d = \gcd(a, b)$ then $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$,
   *g) if $a \equiv b \pmod{m}$ then $\gcd(a, m) = \gcd(b, m)$.

†8. a) Find the least positive integer $n$ for which

   $3^n \equiv 1 \pmod{7}$.

   Hence evaluate $3^{100} \pmod{7}$.

   b) Find the least positive integer $n$ for which

   $5^n \equiv 1 \pmod{17}$ or $5^n \equiv -1 \pmod{17}$.

   Hence evaluate $5^{243} \pmod{17}$.

   c) Evaluate $2^4 \pmod{18}$ and hence evaluate $2^{300} \pmod{18}$.
9. For each of the following, use the Euclidean Algorithm to find \( d = \gcd(a, b) \) and \( x, y \in \mathbb{Z} \) with \( d = ax + by \)

a) \( \gcd(12, 18) \),

b) \( \gcd(111, 201) \),

c) \( \gcd(13, 21) \),

d) \( \gcd(83, 36) \),

e) \( \gcd(22, 54) \),

f) \( \gcd(112, 623) \).

10. Solve, or prove there are no solutions. Give your answer in terms of the original modulus and also, where possible, in terms of a smaller modulus.

a) \( 151x - 294 \equiv 44 \pmod{7} \),

b) \( 45x + 113 \equiv 1 \pmod{20} \),

c) \( 25x \equiv 7 \pmod{11} \),

d) \( 2x \equiv 3 \pmod{1001} \),

 e) \( 111x \equiv 75 \pmod{321} \),

f) \( 1215x \equiv 560 \pmod{2755} \),

 g) \( 182x \equiv 481 \pmod{533} \).

11. Let \( a, b \) be integers, not both zero, let \( S \) be the set of integers defined by

\[ S = \{ax + by \mid x, y \in \mathbb{Z}\}, \]

and let \( d_0 \) be the smallest positive integer in the set \( S \).

The aim of this question is to use the Division Algorithm and the definition of greatest common divisor (gcd) to show that \( d_0 = \gcd(a, b) \).

Prove the following

a) If \( s \in S \), then \( d_0 \) is a divisor of \( s \). (Hint: Write \( s = d_0q + r \) and show \( r \in S \).)

b) \( d_0 \) is a divisor of both \( a \) and \( b \).

c) If \( d \) is a divisor of both \( a \) and \( b \), then \( d \) is a divisor of \( d_0 \).

d) \( d_0 = \gcd(a, b) \), and hence there exist integers \( x, y \) such that \( ax + by = \gcd(a, b) \).

12. a) Prove that if \( a, b \) and \( m \) are integers with the properties \( \gcd(a, b) = 1 \) and \( a \mid m \) and \( b \mid m \),

then \( ab \mid m \). (Hint: use 11(d).)

b) Prove that if \( a \) and \( b \) are coprime integers and \( a \mid bc \), then \( a \mid c \). (Hint: use 11(d).)
Relations

13. List the ordered pairs in the relations \( R_i \), for \( i = 1, 2, 3 \), from \( A = \{2, 3, 4, 5\} \) to \( B = \{2, 4, 6\} \) where

a) \( (m, n) \in R_1 \) iff \( m - n = 1 \),
b) \( (m, n) \in R_2 \) iff \( m \mid n \),
c) \( (m, n) \in R_3 \) iff \( \gcd(m, n) = 1 \).

14. Represent each relation \( R_i \) of Question 13 by:

a) an arrow diagram,
b) a matrix \( M_{R_i} \).

15. Construct arrow diagrams representing relations on \( \{a, b, c\} \) that have the following properties.

a) Reflexive, but neither transitive nor symmetric.
b) Symmetric, but neither transitive nor reflexive.
c) Transitive, but neither symmetric nor reflexive.
d) Symmetric and transitive, but not reflexive.
e) Transitive and reflexive, but not symmetric.
f) Symmetric and antisymmetric and reflexive.

16. A relation \( R \) on \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \) is defined by \( a R b \) iff \( a - b \) has either 2 or 3 as a divisor. Show that \( R \) is reflexive and symmetric, but not transitive.

17. Define an equivalence relation \( \sim \) on the set \( S = \{0, 1, 2, 3, 4, 5, 6\} \) by \( x \sim y \) if and only if \( x \equiv y \pmod{3} \). Partition \( S \) into equivalence classes.

18. Define a relation \( \sim \) on the set \( S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \) by \( x \sim y \) if and only if \( x^2 \equiv y^2 \pmod{5} \).

a) Prove that \( \sim \) is an equivalence relation on \( S \).
b) Partition \( S \) into equivalence classes.

19. a) Let \( A_1 = \{0\} \), \( A_2 = \{1, 2\} \), \( A_3 = \{3, 4\} \) be subsets of \( S = \{0, 1, 2, 3, 4\} \). Define a relation \( \sim \) on the set \( S \) by \( x \sim y \) if and only if \( x, y \in A_i \) for some \( i \in \{1, 2, 3\} \). Show that \( \sim \) is an equivalence relation on \( S \).
b) Let \( B_1 = \{0\} \), \( B_2 = \{1, 2\} \), \( B_3 = \{3\} \) be subsets of \( S = \{0, 1, 2, 3, 4\} \). Explain why the relation \( \sim \) on \( S \) defined by \( x \sim y \) if and only if \( x, y \in B_i \) for some \( i \in \{1, 2, 3\} \) is not an equivalence relation on \( S \).
c) Let \( C_1 = \{0, 1\} \), \( C_2 = \{1, 2\} \), \( C_3 = \{3, 4\} \) be subsets of \( S = \{0, 1, 2, 3, 4\} \). Explain why the relation \( \sim \) on \( S \) defined by \( x \sim y \) if and only if \( x, y \in C_i \) for some \( i \in \{1, 2, 3\} \) is not an equivalence relation on \( S \).
20. The following diagram represents a set \( V = \{ u, v, x, y, z \} \) of six cities and direct flights between them.

\[
\begin{align*}
v & \quad w & \quad x \\
\quad u & \quad z & \\
\end{align*}
\]

a) Define a relation \( \sim \) on \( V \) by \( a \sim b \) if and only if it is possible to fly from \( a \) to \( b \) using an even number of flights (including 0 flights).

i) Prove that \( \sim \) is an equivalence relation on \( V \).

ii) Partition the set \( V \) into equivalence classes.

b) Define a relation \( R \) on \( V \) by \( aRb \) if and only if it is possible to fly from \( a \) to \( b \) using an odd number of flights. Prove that \( R \) is not an equivalence relation.

\[ \text{†21.} \]

Consider the set \( S = \{0, 1, 2, \ldots, 11\} \) of integers modulo 12. Define the relation \( \sim \) on \( S \) by \( x \sim y \) iff \( x^2 \equiv y^2 \mod 12 \). Given that \( \sim \) is an equivalence relation, partition \( S \) into equivalence classes.

\[ \text{†22.} \]

Let \( a \) and \( b \) be two fixed real numbers. Define a relation \( \sim \) on \( \mathbb{R}^2 \) by \((x_1, y_1) \sim (x_2, y_2)\) iff \( ax_1 + by_1 = ax_2 + by_2 \). Prove that \( \sim \) is an equivalence relation and give a geometric description of the equivalence class of \((1, 1)\).

23. Answer the following questions for the Poset \( (\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq) \):

a) Draw a Hasse diagram for the Poset.

b) Find the maximal elements.

c) Find the minimal elements.

d) Is there a greatest element?

e) Is there a least element?

f) Find all upper bounds of \( \{\{2\}, \{4\}\} \).

g) Find the least upper bound of \( \{\{2\}, \{4\}\} \) if it exists.

h) Find all lower bounds of \( \{\{1, 3, 4\}, \{2, 3, 4\}\} \).

i) Find the greatest lower bound of \( \{\{1, 3, 4\}, \{2, 3, 4\}\} \) if it exists.

24. Answer the following questions for the Poset \( (\{2, 4, 6, 9, 12, 27, 36, 54, 60, 72\}, \mid) \):

a) Draw a Hasse diagram for the Poset.

b) Find the maximal elements.

c) Find the minimal elements.

d) Is there a greatest element?

e) Is there a least element?

f) Find the set of upper bounds of \( \{2, 9\} \).

g) Find the least upper bound of \( \{2, 9\} \) if it exists.

h) Find the set of lower bounds of \( \{60, 72\} \).

i) Find the greatest lower bound of \( \{60, 72\} \) if it exists.
25. Give an example of a poset that
   a) Has a minimal element but no least element.
   b) Has a maximal element but no greatest element.

†26. A relation $|$ is defined on $A = \{1, 2, 4, 6, 8, 9, 12, 18, 36, 72, 108\}$ by $a \mid b$ iff $a$ divides $b$.
   a) Show that $(A, \mid)$ is a poset.
   b) Construct its Hasse diagram.
   c) Which members of $A$ are minimal elements? Which are maximal elements? Which are greatest elements? Which are least elements?

†27. Let $S = \{0, 1, 2, 3\}$ and $P(S)$ denote the power set of $S$.
Define the relation $A \preceq B$ for $A, B \in P(S)$ by $A \preceq B$ iff $A \subseteq B$.
   a) Show that $\preceq$ is a partial order.
   b) Construct the Hasse diagram of $(P(S), \preceq)$.
   c) Which members of $P(S)$ are minimal elements? Which are maximal elements? Which are greatest elements? Which are least elements?
PROBLEM SET 3

Note that the subject matter of this section of the course is mathematical proof itself, and not the particular results proved in classes or posed in the following problem set. You should be prepared to prove results from any area of school or first-year university mathematics.

In this problem set, F&D refers to the book by Franklin and Daoud, and the problems have been printed here with the permission of the author. Note the solutions to some of the F&D questions are available in the book “Introduction to Proofs in Mathematics” by J. Franklin and A. Daoud.

Basic Proofs

1. (F&D Chapter 1 Q1) Show that: $\frac{1}{1,000} - \frac{1}{1,002} < \frac{2}{1,000,000}$

2. (F&D Chapter 1 Q3) Show that: $\sqrt{1,001} - \sqrt{1,000} < \frac{1}{2\sqrt{1,000}}$
   
   Hint: multiply $\sqrt{1,001} - \sqrt{1,000}$ by $\frac{\sqrt{1,001} + \sqrt{1,000}}{\sqrt{1,001} + \sqrt{1,000}}$

3. (F&D Chapter 1 Q9) Prove that: $7\sqrt{7!} > 6\sqrt{6!}$

4. (F&D Chapter 1 Q12) Show that: $\sqrt{2} + \sqrt{2} + \sqrt{2} - \sqrt{2} < 2\sqrt{2}$

5. (F&D Chapter 1 Q14)
   
   a) Prove that: $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$
   
   b) Hence show that $\cos \frac{2\pi}{3}$ is a root of the equation $4x^3 - 3x - 1 = 0$

6. a) Suppose that $n$ is a positive integer. Use the Binomial theorem and appropriate inequalities to prove that

   $0 < \left(1 + \frac{1}{n}\right)^n < 3.$

   b) (F&D Chapter 1 Q20) Prove that: $99^{100} > 100^{99}$

7. (F&D Chapter 1 Q21) Show that: $\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$

Generalisation; “all” statements

8. Prove by exhaustion of cases that for any real number $x$ we have

   $|x(x-2)| \leq 2x^2 + 1.$

9. (F&D Chapter 2 Q3) Prove that the product of any two odd numbers is an odd number.

10. (F&D Chapter 2 Q11) Prove that $\sqrt{n!} < \sqrt[n]{(n+1)!}$ for any whole number $n$ (imitate the proof in Chapter 1).

11. (F&D Chapter 2 Q12) Prove that the product of three consecutive whole numbers, of which the middle one is odd, is divisible by 24.
12. (F&D Chapter 2 Q19) Find a generalisation of:
\[
\frac{1}{1000} - \frac{1}{1002} < \frac{2}{(1000)^2}
\]
and prove it.

13. (F&D Chapter 2 Q21) Prove that for all whole numbers \(n, (n + 1)(n + 2)\ldots(2n - 1)(2n) = 2^n\cdot1.3.5\ldots(2n - 1)\)

\[\dagger\] 14. (F&D Chapter 5 Q9) Consider the following statement concerning a positive integer \(x \geq 2\).

“If \(x\) is not divisible by any positive integer \(n\) satisfying \(2 \leq n \leq \sqrt{x}\) then \(x\) is a prime number.”

a) Show that the above statement is true.
b) Is the statement still true if the condition on \(n\) is replaced by \(2 \leq n < \sqrt{x}\)?

*15. a) Let \(a\) and \(n\) be integers greater than 1. Prove that \(a^n - 1\) is prime only if \(a = 2\) and \(n\) is prime. Is the converse of this statement true?
b) [V] Show that \(2^n + 1\) is prime only if \(n\) is a power of 2.

*16. Consider the following statement concerning a positive integer \(n\):

“For all \(a, b \in \mathbb{Z}\), if \(n \mid ab\) then either \(n \mid a\) or \(n \mid b\).”

a) Prove that if \(n\) is prime then the statement is true.
b) Prove that the statement is false when \(n = 30\).
c) Prove that if \(n\) is composite then the statement is false.

17. a) Use the result of part (a) of the previous question to prove that if \(p\) is prime, \(a\) is an integer and \(p \mid a^2\), then \(p \mid a\).

* b) Are there any integers \(n\) other than primes for which it is true that for all integers \(a\), if \(n \mid a^2\) then \(n \mid a\)? If so, describe all such \(n\).

* c) Prove that if \(p\) is prime then \(\sqrt{p}\) is irrational.

18. a) Show that 2 is a multiplicative inverse for 4 (mod 7) and 3 is a multiplicative inverse for 5 (mod 7). Hence determine the value of 5! (mod 7)

* b) [V] Prove that if \(p\) is prime then

\[(p - 2)! \equiv 1 \pmod{p}\]

Hint: What is the multiplicative inverse of \(p - 1\) (mod \(p\)) when \(p\) is prime?
Writing proofs

†19. In the following questions you are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow; and any necessary algebraic details.

a) Theorem. If \( x^3 + x^2 + x + 2 \equiv 0 \pmod{5} \) then \( x \equiv 1 \pmod{5} \).
   Basic ideas: if \( x \equiv 0, 2, 3, 4 \pmod{5} \) then \( x^3 + x^2 + x + 2 = 2, 1, 1, 1 \pmod{5} \).

b) Theorem. Let \( x, y \) and \( m \) be integers. If \( m \mid (4x + y) \) and \( m \mid (7x + 2y) \) then \( m \mid x \) and \( m \mid y \).
   Basic ideas: \( 2(4x + y) - (7x + 2y) = x \) and \( m \mid \text{LHS} \).

c) Theorem. \( \log_2 7 \) is irrational.
   Basic ideas: if \( \log_2 7 = \frac{p}{q} \) then \( 2^p = 7^q \), but then LHS is even and RHS is odd.

d) Theorem. If \( n \) is a non-negative integer then 11 is a factor of \( 2^{4n+3} + 3 \times 5^n \).
   Basic ideas: if \( 2^{4n+3} = 11k - 3 \times 5^n \) then \( 2^{4n+7} = 11(16k - 3 \times 5^n) - 3 \times 5^{n+1} \).

Converse; if and only if

†20. Prove the following statement, then write down its converse. Is the converse true or false? Prove your answer.

   “For all \( x \in \mathbb{Z} \), if \( x \equiv -1 \pmod{7} \) then \( x^3 \equiv -1 \pmod{7} \).”

21. a) Prove that the following proposition is true.
   If \( x \equiv 3 \pmod{4} \) then \( x^3 + 2x - 1 \equiv 0 \pmod{4} \).

b) Write down the contrapositive of the proposition in part (a). Is it true? Explain.

c) Write down the converse of the proposition in part (a). Is it true? Explain.

†22. (F&D Chapter 3 Q6) Prove that a whole number is odd if and only if its square is odd.

†23. (F&D Chapter 3 Q10) Prove that a triangle is isosceles if and only if two of its angles are equal.
   (An isosceles triangle is by definition a triangle with two equal sides.)

24. (F&D Chapter 3 Q13) Show that a number is divisible by 3 if and only if the sum of its digits is divisible by 3.

25. (F&D Chapter 3 Q18) Prove that a real number is rational if and only if its decimal expansion is terminating or (eventually) repeating.

*26. For integers \( x \) and \( y \), show that \( 7 \mid x^2 + y^2 \) if and only if \( 7 \mid x \) and \( 7 \mid y \).

27. A parallelogram is defined to be a quadrilateral with both pairs of opposite sides parallel. Use properties of congruent and similar triangles to prove that a quadrilateral is a parallelogram if and only if two opposite sides are equal and parallel.
“Some” statements

†28. (F&D Chapter 4 Q11) Show that there is a solution of, \( x^{100} + 5x - 2 = 0 \) between \( x = 0 \) and \( x = 1 \).

29. [V] (F&D Chapter 4 Q4) A perfect number is one which equals the sum of its factors (counting 1 as a factor, but not the number itself). Show that there exists a perfect number.

†30. (F&D Chapter 4 Q13) Consider the infinite geometric progression,

\[
\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n + \cdots
\]

Prove that there exists an integer \( N \) such that the sum of the first \( N \) terms of the above series differs from 1 by less than 10\(^{-6}\).

31. (F&D Chapter 4 Q16) A formula for \( e^x \) is,

\[
1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots
\]

Show that \( e^3 = 20.1 \), correct to 1 decimal place, by showing:

a) For any \( n \geq 4 \),

\[
\frac{3^n}{n!} < \frac{9}{2} \left(\frac{3}{4}\right)^{n-3}
\]

b) Hence, that there exists \( N \) such that,

\[
\frac{3^{N+1}}{(N+1)!} + \frac{3^{N+2}}{(N+2)!} + \cdots < \frac{1}{20}
\]

c) Hence, that \( e^3 = 20.1 \) correct to 1 decimal place.

*32. (F&D Chapter 5 Q11) Every one of six points is joined to every other one by either a red or a blue line. Show that there exist three of the points joined by lines of the same colour.

Multiple quantifiers

*33. [V] (F&D Chapter 5 Q8) Prove that between any two irrational numbers there is a rational number.

†34. Show that the sequence \( \{u_n\} \) given by

\[
u_n = n^2 \quad \text{for all} \quad n \in \mathbb{N}
\]

diverges to infinity, by showing that

\[
\forall M \in \mathbb{N} \quad \exists N \in \mathbb{N} \quad \forall n > N \quad u_n > M.
\]

35. A function \( f(x) \) is called continuous at \( x = a \) iff:

\[
\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in (a - \delta, a + \delta) \quad |f(x) - f(a)| < \epsilon.
\]

a) Complete: “A function \( f(x) \) is not continuous at \( x = a \) iff: ......”

b) * Show that the function \( f(x) = |x| \) is not continuous at \( x = 3 \).
The definition of $\lim_{x \to \infty} f(x) = \ell$ is

"$\forall \epsilon > 0 \ \exists M \in \mathbb{R} \ \forall x > M \ |f(x) - \ell| < \epsilon.$" \hfill (**)

a) Write down the negation of (**)(that is, $\lim_{x \to \infty} f(x) \neq \ell$) and simplify it so that the negation symbol does not appear.

b) By working directly from the definition prove that $\lim_{x \to \infty} \frac{2x^2 + 1}{x^2 + 2} = 2$.


Negation; proof by contradiction

a) (F&D Chapter 6 Q16) Prove that each of the following is irrational:

   a) $\sqrt[3]{4}$
   b) $1 + \sqrt{3}$
   c) $\sqrt{3} - \sqrt{2}$

b) Prove that $\log_{10} 3$ is irrational.

39. Prove that $\log_5 15$ is irrational.

40. (F&D Chapter 6 Q1) What is the common feature in showing:

   a) a “not all” statement to be true?
   b) an “all” statement to be false?

Give an example in each case.

41. (F&D Chapter 6 Q11) Are the following statements true or false? Prove your answers.

   a) Let $a, b, c$ be three integers. If $a$ divides $c$ and $b$ divides $c$, then either $a$ divides $b$ or $b$ divides $a$.
   b) If $a, b, c, d$ are real numbers with $a < b$ and $c < d$, then $ac < bd$.

42. (F&D Chapter 5 Q5)

   a) Prove that if $a$ and $b$ are rational numbers with $a \neq b$ then,

   $$a + \frac{1}{\sqrt{2}}(b - a)$$

   is irrational.

   b) Hence prove that between any two rational numbers there is an irrational number.

43. (F&D Chapter 7 Q14) A set of real numbers is called bounded if it does not “go to infinity”. More precisely, $S$ is bounded if there exist real numbers $M, N$ such that for all $s \in S, M < s < N$.

   (For example, the set of real numbers $x$ such that $1 < x^3 < 2$ is bounded, since for all $x \in S, 1 < x < 1.5$.)

   a) Give an example of a set which is not bounded.
   b) Prove that if $S$ is bounded and $T \subset S$, then $T$ is bounded.
   c) Prove that any finite set of real numbers is bounded.
   d) Prove that if $T$ is not bounded and $T \subset S$, then $S$ is not bounded.
e) Prove that if $S$ and $T$ are bounded, then $S \cap T$ is bounded.

f) If $S$ and $T$ are bounded, is $S \cup T$ always bounded? Prove your answer.

g) Let $S$ and $T$ be bounded, Let,

\[ U = \{ u \in \mathbb{R} : u = s + t \text{ for some } s \in S, t \in T \} \]

Show that $U$ is bounded. It might help to calculate some examples first, say,

$S = [0, 1], T = [2, 3]$ 

(The set $U$ is sometimes denotes $S + T$, since it is the set of all sums of something in $S$ with something in $T$.)

44. (F&D Chapter 7 Q15) A region in the plane is called convex if the line segment joining any two points in the region lies wholly inside the region. For example, an ellipse, a parallelogram, a triangle and a straight line are convex, but on annulus and a star-shaped region are not. In symbols, $R$ is convex if, for all $(x_1, y_1)$ and $(x_2, y_2)$ in $R$, $(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in R$ for all $\lambda \in [0, 1]$.

a) Prove that if $R$ and $S$ are convex, then $R \cap S$ is convex.

b) If $R$ and $S$ are convex, is $R \cup S$ always convex? Prove your answer.

c) Prove that if $R$ is convex, then the reflection of $R$ in the $x$-axis is convex.

d) If $R$ is convex, is the set,

\[ 2R = \{(x, y) : (x, y) = (2x', 2y') \text{ for some } (x', y') \in R\} \]

always convex? Prove your answer and illustrate with a diagram.

45. (F&D Chapter 6 Q24) Show that there do not exist three consecutive whole numbers such that the cube of the greatest equals the sum of the cubes of the other two.

**Mathematical induction**

46. (F&D Chapter 8 Q2) Show that, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots + (-1)^{n-1}\frac{1}{n}$ is always positive.

47. (F&D Chapter 8 Q5)

a) Prove, by mathematical induction, that if $n$ is a whole number then,

\[ n^3 + 3n^2 + 2n \]

is divisible by 6.

b) Prove the same result without mathematical induction by first factorising $n^3 + 3n^2 + 2n$.

48. (F&D Chapter 8 Q7)

a) Calculate,

\[ \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} \]

for a few small values of $n$.

b) Make a conjecture about a formula for this expression.

c) Prove your conjecture by mathematical induction.
49. (F&D Chapter 8 Q8) The following is a famous fallacy that uses the method of mathematical induction. Explain what is wrong with it.

**Theorem**
Everything is the same colour.

**Proof**
Let \( P(n) \) be the statement, “In every set of \( n \) things, all the things have the same colour”.
We will show that \( P(n) \) is true for all \( n = 1, 2, 3, \ldots \), so that every set consists of things of the same colour.
Now, \( P(1) \) is true, since in every set with only one thing in it, everything is obviously of the same colour.
Now, suppose \( P(n) \) is true.
Consider any set of \( n + 1 \) things.
Take an element of the set, \( a \). The \( n \) things other than \( a \) form a set of \( n \) things, so they are all the same colour (since \( P(n) \) is true).
Now take a set of \( n \) things out of the \( n + 1 \) which does include \( a \). These are also all the same colour, so \( a \) is the same colour as the rest.
Therefore \( P(n+1) \) is true.

50. Show that \( 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \) whenever \( n \in \mathbb{Z}^+ \).

51. Show that \( 1^2 + 3^2 + \cdots + (2n+1)^2 = \frac{1}{3}(n+1)(2n+1)(2n+3) \) whenever \( n \in \mathbb{N} \).

†52. Show that for all \( n \in \mathbb{N} \), \( 64 \mid 7^{2n} + 16n - 1 \).

53. Prove that for all \( n \in \mathbb{Z}^+ \), \( 21 \mid 4^{n+1} + 5^{2n-1} \).

*54. Prove by induction that, for \( n \in \mathbb{Z}^+ \),
\[
\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \geq \frac{2}{3}n\sqrt{n}.
\]

55. Prove by induction that if the sequence \( \{u_n\} \) is defined by
\[
\begin{align*}
u_0 &= 0 \\
u_1 &= 1 \\
u_n &= u_{n-1} + u_{n-2} \quad \text{for all } n \geq 2
\end{align*}
\]
then
\[
u_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)
\]
for all \( n \geq 0 \).

†56. Prove, using induction, that if the sequence \( \{u_n\} \) is defined by:
\[
\begin{align*}
u_1 &= 12 \\
u_2 &= 30 \\
u_n &= 5u_{n-1} - 6u_{n-2} \quad \text{for } n \geq 3,
\end{align*}
\]
then \( u_n = 3 \times 2^n + 2 \times 3^n \), for all \( n \geq 1 \).

57. a) Prove by induction that for \( n \in \mathbb{N} \), the \( n \)th derivative of \( e^{x^2} \) is a polynomial times \( e^{x^2} \). Can you say anything about these polynomials?
(Note: By convention, the \( 0 \)th derivative of \( f(x) \) is just \( f(x) \).)
b) Guess and prove a similar result concerning the derivatives of the function $e^{1/x}$.

†58. [V] Prove that there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ we have $3^n < n!$.

**Logic and truth tables**

59. Which of the following are propositions?
   a) The moon is made of green cheese.
   b) Read this carefully.
   c) Two is a prime number.
   d) Will the game be over soon?
   e) Next year interest rates will rise.
   f) Next year interest rates will fall.
   g) $x^2 - 4 = 0$

60. Using letters for the component propositions, translate the following compound statements into symbolic notation:
   a) If prices go up, then housing will be plentiful and expensive; but if housing is not expensive, then it will still be plentiful.
   b) Either going to bed or going swimming is a sufficient condition for changing clothes; however, changing clothes does not mean going swimming.
   c) Either it will rain or it will snow but not both.
   d) If Janet wins or if she loses, she will be tired.
   e) Either Janet will win or, if she loses, she will be tired.

61. Write a statement that represents the negation of each of the following:
   a) If the food is good, then the service is excellent.
   b) Either the food is good or the service is excellent.
   c) Either the food is good and the service is excellent, or else the price is high.
   d) Neither the food is good nor the service excellent.
   e) If the price is high, then the food is good and the service is excellent.
62. Construct a truth table for each of the following propositions.
   a) \( \sim ((p \land q) \rightarrow (p \land q)) \);
   b) \( p \rightarrow (p \rightarrow q) \);
   c) \( \sim p \rightarrow (p \rightarrow q) \);
   d) \( (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)) \).

   Which of the above propositions are tautologies? Are any contradictions?

63. Show, by using truth tables, that the following pairs of propositions are logically equivalent.
   a) \( \sim (p \lor q) \), \( \sim p \land \sim q \);
   b) \( \sim p \lor q \), \( p \rightarrow q \);
   c) \( \sim p \rightarrow (q \lor r) \), \( \sim q \rightarrow (\sim r \rightarrow p) \);
   d) \( p \lor (p \land q), p \).

64. Use standard logical equivalences to simplify each of the following logical expressions.
   a) \( (p \lor \sim q) \land \sim (p \land q) \)
   b) \( [p \rightarrow (q \lor \sim p)] \rightarrow (p \land q) \)
   c) \( (p \land q) \lor (\sim p \land q) \land (q \land r) \)
   d) \( (p \leftrightarrow q) \lor (\sim p \land q) \)

65. Use standard logical equivalences to show that \( \sim (p \lor \sim q) \rightarrow (q \rightarrow r) \) is logically equivalent to each of the following propositions
   a) \( q \rightarrow (p \lor r) \);
   b) \( (q \rightarrow p) \lor (q \rightarrow r) \);
   c) \( \sim (q \rightarrow p) \rightarrow (\sim q \lor r) \).

Valid and invalid arguments

66. Suppose that
   “If I do not do my homework, I will not pass.
   If I study hard, I will pass.
   I passed.”

   Did I do my homework or not? Did I study hard or not? Explain.

67. “Watson, I have uncovered the following facts:
   • If Mrs Smith is lying then Moriarty has not escaped.
   • Either Moriarty is dead or he is really Jones.
   • If Moriarty is really Jones then he has escaped.
   • I am convinced Mrs Smith is lying.”

   “Good Lord Holmes,” replied Dr Watson, “what can you make of all this?”
   “Elementary my dear Watson, Moriarty is dead!”

   Is Holmes correct? Justify your answer.
68. Let $e$ denote “Einstein is right”, $b$ denote “Bohr is right” $q$ denote “Quantum mechanics is right”, $w$ denote “The world is crazy” and consider the sentences:

(1) “Either Einstein or Bohr is right, but they are not both right”.
(2) “Einstein is right only if quantum mechanics is wrong, and the world is crazy if quantum mechanics is right”.

a) Write the two sentences (1) and (2) as symbolic expressions involving $e, b, q$ and $w$.

b) Suppose the world is not crazy. From the truth of (1) and (2) is it valid to deduce that Einstein is right? Explain.

69. Consider the following two statements:

(1) “If either Peta or Queenie has passed then either Roger and Peta have both passed, or Roger and Queenie have both passed”.

and

(2) “If either Peta has passed or Queenie has failed then Roger has passed.”

a) Write the two statements (1) and (2) in symbolic form by letting $p$ stand for “Peta has passed,” $q$ stand for “Queenie has passed” and $r$ stand for “Roger has passed”.

b) Suppose that the statement (1) is false. Deduce that Roger has failed.

c) Suppose that the statement (1) is false and that the statement (2) is true. Decide whether or not Peta has passed.
PROBLEM SET 4

Enumeration and Probability

1. How many strings of six non-zero decimal digits
   a) begin with two odd digits?
   b) consist of one even digit, followed by two odd digits, followed by three digits less than 7?
   c) have no digit occurring more than once?
   d) contain exactly three nines?
   e) contain fewer than three nines?
   f) contain exactly three nines, with no other digit repeated?
   g) have their last digit equal to twice their first digit?

2. a) Express in terms of factorials $P(21, 8), C(21, 8), \binom{321}{123}, C(2n, n)$.
   b) Calculate explicitly $P(7, 4), \binom{7}{4}, P(6, 3), C(4, 2), C(201, 199)$.

3. How many seven-letter words can be made from the English alphabet which contain
   a) exactly one vowel,
   b) exactly two vowels,
   c) exactly three vowels,
   d) at least three vowels?

4. Repeat the previous question if repeated letters are not permitted.

5. a) A set of eight Scrabble® tiles can be arranged to form the word SATURDAY. How many
       three-letter “words” can be formed with these tiles if no tile is to be used more than once?
   b) How many ten-letter words can be formed from the letters of PARRAMATTA? How many
       nine-letter words? How many eight-letter words?
   c) How many four-letter “words” come before “UNSW” if all four-letter words are listed in
       alphabetical order?

6. How many eight-letter words constructed from the English alphabet have
   a) exactly two Ls?
   b) at least two Ls?

7. Consider the following.
   “Problem: How many eight-card hands chosen from a standard pack have at least one suit
   missing?
   Solution: Throw out one entire suit (4 possibilities), then select 8 of the remaining 39 cards.
   The number of hands is $4C(39, 8)$.”
   a) What is wrong with the given solution?
   b) Solve the problem correctly.
8. a) Find the number of positive integers less than or equal to 200 which are divisible neither by 6 nor by 7.
   b) Find the number of positive integers less than or equal to 200 which are divisible neither by 6, nor by 7, nor by 8.

9. What is the probability that a hand of eight cards dealt from a shuffled pack contains:
   a) exactly three cards of the same value and the remaining cards all from the remaining suit (for example, ♥4, ♦4, ♠4 and five clubs not including the ♣4);
   b) exactly three cards in at least one of the suits;
   c) exactly three cards in exactly one of the suits. (Hint. First find the number of ways in which five cards can be chosen from three specified suits, with none of the three represented by three cards.)

10. For any positive integer \(n\) we define \(\phi(n)\) to be the number of positive integers which are less than or equal to \(n\) and relatively prime to \(n\). Given that \(p, q, r\) are primes, all different, use the inclusion/exclusion principle to find \(\phi(pq)\), \(\phi(p^2q)\) and \(\phi(pqr)\).

11. A hand of thirteen cards is dealt from a shuffled pack. Giving reasons for your answers, determine which of the following statements are definitely true and which are possibly false.
   a) The hand has at least four cards in the same suit.
   b) The hand has exactly four cards in some suit.
   c) The hand has at least five cards in some suit.
   d) The hand has exactly one suit containing four or more cards.

12. a) Prove that if nine rooks (castles) are placed on a chessboard in any position whatever, then at least two of the rooks attack each other.
   b) Prove that if fifteen bishops are placed on a chessboard then two of them attack each other.

13. Let \(A\) be the set \(\{1, 2, 3, 4, 5, 6, 7, 8\}\). Prove that if 5 integers are selected at random from \(A\), then at least one pair of these integers has sum 9.

14. In a square with sides of length 3cm, 10 points are chosen at random. Prove that there must be at least two of these points whose distance apart is less than or equal to \(\sqrt{2}\) cm.

15. If 31 cards are chosen from a pack, prove that there must be at least 3 of the same value, and there must be at least 8 in the same suit.

16. a) To each integer \(n\) we assign an ordered pair \(p(n)\) whose members are the remainders when \(n\) is divided by 3 and 4 respectively. For example, \(p(5)\) and \(p(17)\) are both equal to \((2, 1)\). If ten thousand integers are chosen at random, how many can you say for certain must have the same value for \(p\)?
   *b) Repeat part (a) with the divisors 3 and 4 replaced by 4 and 6.

17. Let \(S\) and \(T\) be finite sets with \(|S| > |T|\), and let \(f\) be a function from \(S\) to \(T\). Show that \(f\) is not one-to-one.

18. Prove that there were two people in Australia yesterday who met exactly the same number of other people in Australia yesterday.
19. Twenty hotel management students all guess the answers on the final examination, so it can be taken that all orders of students on the list of results are equally likely. The top student is given a mark of 100, the next 95, and so on, down to 5 for the last student and no two students get the same mark. Find the probability that Polly gets an HD, both Manuel and Sybil get a CR or better, and Basil fails.

†20. A die is rolled 21 times. Find the probability of obtaining a 1, two 2s, . . . and six 6s.

†21. For this problem assume that the 365 dates of the year are equally likely as birthdays.
   a) Find the probability that two people chosen at random have the same birthday.
   b) Find the probability that in a group of n people, at least two have the same birthday.
   c) How large does n have to be for the probability in (b) to be greater than \( \frac{1}{2} \)?
   d) Criticise the assumption made at the beginning of this question.

†22. Twenty cars to be bought by a company must be selected from up to four specific models. In how many ways may the purchase be made if
   a) no restrictions apply?
   b) at least two of each model must be purchased?
   c) at most three different models must be purchased?

23. How many outcomes are possible from the roll of four dice
   a) if the dice are distinguishable (for example, they are of different colours)?
   b) if the dice are not distinguishable?

†24. a) Find the coefficient of \( w^2x^5y^7z^9 \) in \((w + x + y + z)^{23}\).
   b) When \((w + x + y + z)^{23}\) is expanded and terms collected, how many different terms will there be?

†25. How many solutions are there to the equation
   \[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 40 \]
   if \( x_1, x_2, x_3, x_4, x_5, x_6 \) are non-negative integers,
   a) with no further assumptions?
   b) with each \( x_j \geq 3 \)?
   c) with each \( x_j \leq 10 \)?
   *d) with each \( x_j \leq 8 \)?
   e) if each \( x_j \) is even?
   f) if at least one \( x_j \) is odd?
   *g) if every \( x_j \) is odd?
   *h) with \( x_1 \leq 9, 5 \leq x_2 \leq 14 \) and \( 10 \leq x_3 \leq 19 \)?
Recurrence Relations

27. Write down the first four terms of the sequences defined recursively by
   a) \(a_n = \frac{2}{3}a_{n-1},\quad a_0 = 16;\)
   b) \(a_n = 3a_{n-1} - 2a_{n-2},\quad a_0 = 2,\quad a_1 = 5;\)
   c) \(a_n = 2(a_{n-1} + a_{n-2} + \cdots + a_1 + a_0),\quad a_0 = 1.\)

28. Write down a recurrence and initial conditions to describe each of the following sequences.
   a) \(\{2, 4, 8, 16, \ldots\}\)
   b) \(\{1, 3, 5, 7, \ldots\}\)
   c) \(\{3, -6, 12, -24, \ldots\}\)
   d) \(\{1, 3, 6, 10, 15, 21, 28, \ldots\}\)
   e) \(\{2, 2, 2, 2, \ldots\}\)
   f) \(\{1, 2, 3, 5, 8, 13, 21, 34, \ldots\}\)

†29. Let \(a_n\) be the number of ways to climb \(n\) steps, if the person climbing the stairs can only take two steps or three steps at a time.
   a) Write down a recurrence relation with initial conditions for \(a_n\).
   b) Find \(a_4\) and \(a_8\).

†30. Let \(a_n\) denote the number of bit strings of length \(n\) where no two consecutive zeros are allowed.
   a) Find \(a_1, a_2, a_3,\)
   b) Show that \(a_n\) satisfies the recurrence \(a_n = a_{n-1} + a_{n-2}\) for \(n \geq 3\).

31. A bank lends me $50,000 at 18% per year interest, compounded monthly, and I pay back $900 per month. (So at the end of each month the amount I owe is increased by \(\left(\frac{48}{100}\right)\)% and then reduced by $900.)

   If \(u_n\) is the amount still owing after \(n\) months, write down a recurrence relation for \(u_n\).

†32. a) Find the general solution of the first order recurrence \(a_n = 5a_{n-1}\).
   b) Find the solution of the first order recurrence \(a_n + 4a_{n-1} = 0\) subject to the initial condition \(a_1 = -12\).

33. Find the general solution of the following recurrence relations, each holding for \(n \geq 2\).
   a) \(a_n + a_{n-1} - 6a_{n-2} = 0.\)
   b) \(a_n = 3a_{n-1} - 2a_{n-2}.\)
   c) \(a_n = 6a_{n-1} - 9a_{n-2}.\)
   d) \(a_n - 2a_{n-1} - 4a_{n-2} = 0.\)

†34. Find the solution of the following recurrence relations (defined for \(n \geq 2\)) subject to the given initial conditions
   a) \(a_n + 2a_{n-1} - 15a_{n-2} = 0,\quad a_0 = 7,\quad a_1 = -3.\)
   b) \(a_n = 5a_{n-1} - 6a_{n-2},\quad a_0 = 5,\quad a_1 = 13.\)
c) \( a_n + 4a_{n-1} + 4a_{n-2} = 0 \), \( a_0 = 2 \), \( a_1 = 4 \).
d) \( a_n - 4a_{n-1} - 6a_{n-2} = 0 \), \( a_0 = 2 \), \( a_1 = 4 \).

35. Find the general solution of the following recurrence relations (defined for \( n \geq 2 \)).

a) \( a_n + a_{n-1} - 6a_{n-2} = 4^n \)
b) \( a_n = 3a_{n-1} - 2a_{n-2} + 2^n \)
c) \( a_n = 6a_{n-1} - 9a_{n-2} + 8n + 4 \)
d) \( a_n = 6a_{n-1} - 9a_{n-2} + 3^n \)

36. Find the solution of the recurrence

\[ a_n - 3a_{n-1} - 4a_{n-2} = 5(-1)^n \]
for \( n \geq 2 \), given that \( a_0 = 1 \), \( a_1 = 8 \).

37. Suppose we wish to tile a \( 2 \times n \) rectangular board with smaller tiles of size \( 1 \times 2 \) and \( 2 \times 2 \).

Let \( a_n \) be the number of ways in which this can be done.

a) Show that, for \( n > 2 \), \( a_n = a_{n-1} + 2a_{n-2} \) and find \( a_1, a_2 \).
b) Solve the recurrence to find a closed formula for \( a_n \).

38. An \( n \)-digit quaternary sequence is a string of \( n \) digits chosen from the numbers 0, 1, 2, 3.

Let \( a_n \) be the number \( n \)-digit of quaternary sequences with an even number of 0’s.

a) Show that, for \( n \geq 1 \),

\[ a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1} \]

and \( a_0 = 1, a_1 = 3 \).
b) Find a closed formula for \( a_n \).

39. Define a set \( S \) of words on the alphabet \( \{x, y, z\} \) by

\begin{align*}
\text{(B)} & \quad x, y \in S \\
\text{(R)} & \quad \text{If } w \in S \text{ then } wx, wy, wzx, wzy \in S.
\end{align*}

Let \( a_n \) be the number of words of length \( n \) in \( S \).

a) Find the first few values of \( a_n \).
b) Find a recurrence relation for \( a_n \).
c) Solve the recurrence to find a closed formula for \( a_n \).

40. We call a word on the alphabet \( \{x, y, z\} \) \( z \)-abundant if the letter \( z \) appears at least once in any two successive letters. (So, for example, the empty word is \( z \)-abundant and so is \( xzzxzy \) but \( xzyxzy \) is not.)

Let \( a_n \) be the number of \( z \)-abundant words of length \( n \).

a) Show that \( a_1 = 3 \) and that \( a_2 = 5 \).
b) Explain carefully why, for \( n \geq 2 \),

\[
a_n = a_{n-1} + 2a_{n-2}.
\]

(Hint: Consider the possible ways in which a \( z \)-abundant word can begin.)

c) Find an explicit formula for \( a_n \).

\*41. a) Show that the set \( \{1, 2, 3, \ldots, n\} \) can be partitioned into two non-empty sets in precisely \( 2^{n-1} - 1 \) ways.

b) Let \( s_n \) be the number of ways in which the set \( \{1, 2, 3, \ldots, n\} \) can be partitioned into three non-empty sets.

Show that \( s_1 = 0 \), \( s_2 = 0 \), \( s_3 = 1 \) and write down the six such partitions for \( n = 4 \).

c) Show that, for \( n \geq 2 \), the sequence \( s_n \) defined above satisfies \( s_n = 3s_{n-1} + 2^{n-2} - 1 \) and find a closed formula for \( s_n \).
Graphs

1. Draw the graph $G = (V, E, f)$ with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and edge–endpoint or incidence function $f : E \rightarrow \{\{x, y\} | x, y \in V\}$

<table>
<thead>
<tr>
<th>$e$</th>
<th>$f(e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>${v_3, v_4}$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>${v_1}$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>${v_1, v_5}$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>${v_2, v_5}$</td>
</tr>
<tr>
<td>$e_5$</td>
<td>${v_2}$</td>
</tr>
<tr>
<td>$e_6$</td>
<td>${v_5, v_2}$</td>
</tr>
<tr>
<td>$e_7$</td>
<td>${v_4, v_2}$</td>
</tr>
</tbody>
</table>

2. For the graph

Find

a) the edge–endpoint function (in a table),
b) the vertex degrees (in a table),
c) total vertex degree and total number of edges,
d) all loops,
e) all parallel edges,
f) edges incident on $v_3$,
g) vertices adjacent to $v_8$,
h) all isolated vertices.
3. Determine with reasons which of the following graphs are simple.

\[ G_1 \quad G_2 \quad G_3 \quad G_4 \quad G_5 \]

4. How many edges does a graph have if it has vertices of degree 4, 3, 3, 2, 2? Draw a simple graph with these vertex degrees. (Is it the only example?)

5. Determine whether or not there is a graph or simple graph for each of the following sequences of vertex degrees. Draw examples of those that exist. (Try to minimise the number of loops or parallel edges in non–simple examples.)

i) 4, 4, 3, 2, 2, 1. ii) 4, 4, 3, 3, 2, 1. iii) 5, 5, 3, 2, 2, 1.
iv) 5, 4, 3, 2, 2. v) 6, 5, 4, 2, 2, 1. vi) 6, 5, 4, 3, 2, 1, 1.

6. Draw the graphs
i) \( K_6 \). ii) \( K_{2,3} \).

7. Find the total number of vertices and edges for the special simple graphs
   a) \( K_n \)
   b) \( K_{m,n} \)
   c) \( C_n \), (cyclic graph with \( n \) vertices)
   d) \( Q_n \) (\( n \)-cube graph)

8. How many subgraphs are there of the graph \( G \):

\[ G \]

9. Recall that two vertices in the complement \( \overline{G} \) are neighbours iff they are not neighbours in \( G \). Find the following
   i) \( \overline{K_n} \) ii) \( \overline{K_{m,n}} \) iii) \( \overline{C_n} \) for \( n = 5, 6 \)

10. If a simple graph \( G \) has \( n \) vertices and \( m \) edges how many edges does \( \overline{G} \) have?
Bipartite Graphs

11. Determine, with reasons, which of the following graphs is bipartite.

a)

b)

c)

d)

e)

f)

g)

12. For what values of $n$ are the following graphs bipartite?

i) $K_n$  ii) $C_n$  iii) $Q_n$
Adjacency Matrices

†13. Represent the following graphs by adjacency matrices

a)  
\[ \begin{bmatrix}
A & e_1 & B \\
e_2 & e_3 & e_5 \\
e_6 & e_4 & C
\end{bmatrix} \]

(Order vertices alphabetically)

b)  \( K_4 \)

c)  \( K_{2,3} \).

†14. Draw the graph with adjacency matrices

i)  
\[ \begin{bmatrix}
0 & 1 & 2 \\
1 & 1 & 0 \\
2 & 0 & 0
\end{bmatrix} \]

ii)  
\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
2 & 0 & 2 & 1 \\
0 & 2 & 0 & 3 \\
1 & 1 & 3 & 2
\end{bmatrix}
\]

Isomorphism

†15. Determine, with reasons, whether or not the following pairs of graphs are isomorphic.

a)  
\[
\begin{bmatrix}
a & b \\
c & d \\
e & f
\end{bmatrix} \quad \begin{bmatrix}
x & y \\
z & t \\
u & u
\end{bmatrix}
\]

b)  
\[
\begin{bmatrix}
a & b \\
c & d \\
e & f
\end{bmatrix} \quad \begin{bmatrix}
x & y \\
z & v \\	v & t
\end{bmatrix}
\]

c)  
\[
\begin{bmatrix}
a & b \\
c & e \\
e & f
\end{bmatrix} \quad \begin{bmatrix}
x & y \\
z & t \\	v & u
\end{bmatrix}
\]
16. A simple graph $G$ is self-complementary if it is isomorphic to its complement $\overline{G}$ (see Q. 9).

a) Show if $G$ is self–complementary then $G$ has $4k$ or $4k + 1$ vertices for some integer $k$. Is the converse true?

*b) Find all self–complementary graphs with four or five vertices.

** c) (Challenge) Find a self–complementary graph with 8 vertices.

*17. Find all non–isomorphic simple graphs with

a) 2 vertices,

b) 3 vertices,

c) 4 vertices.
Connectivity

18. Find the connected components (draw separately) of the graph

\[ \text{Adjacency Matrix and Paths} \]

19. What is the number of walks of length \( n \) between any two adjacent vertices of \( K_4 \) for \( n = 2, 3, 4, 5 \)?

\[ \dagger \]

20. a) Find the adjacency matrix \( M \) of the graph \( G \)

\[ \text{(ordering the vertices as } A, B, C, D) \].

b) Find \( M^2, M^3 \).

c) How many walks of length 2 and 3 are there between \( B \) and \( C \)? Write them down.

d) Let \( N = I_4 + M + M^2 + M^3 \).

i) The (3, 4) entry of \( N \) is 20. What does this mean in terms of walks?

ii) No entries of \( N \) are zero. What does this mean?
Euler and Hamilton Paths and Circuits.

21. Determine, with reasons, whether or not there is

α) an Euler path, which is not a circuit,
β) an Euler circuit,
γ) a Hamilton path, which is not a circuit,
δ) a Hamilton circuit,

in the following graphs. Give an example, if one exists, in each case.

a)

b)

c)

d)

e)

f)

22. For what values of \( n \) does \( K_n, C_n, Q_n \) have an Euler circuit?

23. For what values of \( n \) does \( K_n, C_n, Q_n \) have an Euler path which is not a circuit?

24. For what values of \( m, n \) does \( K_{m,n} \) have

a) an Euler circuit,
b) an Euler path which is not a circuit,
c) a Hamilton circuit.
25. Show each $Q_n$ has a Hamilton circuit.

[Hint: Construct one recursively, constructing Hamiltonian paths between adjacent vertices of $Q_n$. Construct such a path for $Q_{n+1}$ from one such path for $Q_n$].

26. The Knight’s Tour Puzzle asks if it is possible to find a sequence of 64 knight’s moves so that a knight on a chessboard visits all the different squares and ends up on the starting point.

   a) Formulate the problem in terms of graphs.
   b) Can the puzzle be solved?
   c) Can you find an explicit solution?
   d) What happens on a $3 \times 3$ chessboard?

Planar Graphs

27. Show that the following graphs are planar by redrawing them as planar maps.
28. For each of

a) Find the degree of each of the regions indicated by an asterisk in each map.
b) What is the sum of the degrees of the regions in each map?
c) Give the dual of each of the planar maps, drawing it on a separate diagram.
d) Verify Euler’s formula in the maps.

29. A connected planar graph has 11 vertices; 5 have degree 1, 5 have degree 4, and 1 has degree 5.

a) How many edges are there?
b) How many regions are there?
c) Give an example of such a graph, drawing it as a planar map.

30. a) Show if $G$ is a connected planar simple graph with $v$ vertices and $e$ edges with $v \geq 3$ then $e \leq 3v - 6$.
   
   b) Further show if $G$ has no circuits of length 3 then $e \leq 2v - 4$.

31. Apply the last question to show the following graphs are non-planar.

- a)
- b)

32. Use the results of Q30 to decide which $Q_n$ are planar.
33. Use Kuratowski’s Theorem to show that the following graphs are not planar.

a)

\[
\begin{array}{c}
A \\
B \\
C \\
D \\
E \\
F \\
G \\
H \\
\end{array}
\]

b)

\[
\begin{array}{c}
A \\
B \\
C \\
D \\
E \\
F \\
G \\
\end{array}
\]

c)

\[
\begin{array}{c}
A \\
B \\
C \\
D \\
E \\
F \\
G \\
\end{array}
\]

d)

\[
\begin{array}{c}
A \\
B \\
C \\
D \\
E \\
F \\
G \\
\end{array}
\]

34. Show the converses of Q30 a) and b) are false by considering the following examples. (Hint: Kuratowski’s Theorem.)

a)

\[
\begin{array}{c}
A \\
D \\
E \\
F \\
C \\
\end{array}
\]

b)

\[
\begin{array}{c}
G \\
H \\
F \\
D \\
E \\
\end{array}
\]
**Trees**

35. A tree $T$ has 8 vertices, **at least** two of which have degree 3.
   a) How many edges are there?
   b) What are the possible vertex degrees for $T$ in non-increasing order?
   c) What are the possible forms for $T$ up to isomorphism?

*36. Prove that a tree has at least one vertex of degree 1.

*37. Use Q36 to prove by induction that a tree with $n$ vertices has $n - 1$ edges.

†38. a) Use Kruskal’s algorithm to find a minimal spanning tree for the following weighted simple graph.

   ![Graph](image)

   b) Use Dijkstra’s algorithm to construct a tree giving shortest paths from $A$ to each of the other vertices in the weighted graph in part (a).

39. a) Use Kruskal’s algorithm to find a minimal spanning tree for the following weighted graph. Then find a second minimal spanning tree for the graph. How many other minimal spanning trees are there?

   ![Graph](image)

   b) Use Dijkstra’s algorithm to construct a tree giving shortest paths from $A$ to each of the other vertices in the weighted graph in part (a).

40. Use Dijkstra’s algorithm to find a shortest path from the vertex $v_0$ to the vertex $v_1$ in the following weighted graph.

   ![Graph](image)
ANSWERS TO SELECTED PROBLEMS

Important Note
Here are some answers (not solutions) and some hints to the problems. These are NOT intended as complete solutions and rarely are any reasons given. To obtain full marks in test and examination questions FULL reasoning must be given, and your work should be clearly and logically set out.

Problem Set 1.

1. $A = B$.

2. Yes.

5. a) 4; b) 16; c) 65536.

6. a) $T$; b) $F$; c) $T$; d) $F$; e) $F$; f) $T$; g) $T$

8. Yes.

10. a) No; b) No; c) Yes; d) Yes; e) $T$.

11. 6.

12. a) True; b) False.

13. a) $P(A) \cup P(B) \subseteq P(A \cup B)$; b) The sets are equal.

14. There is no containment relation.

15. a) $A - B$; b) $A$; c) $U$.

16. $B$.

17. $A \cup B; A \cap B$.

20. b) \{2, 3, 4, 8, 9, 10, 14, 15, 16\}.

22. a) \{1, 2, \ldots, 21\}; b) \{90, \ldots, 105\}; c) $\emptyset$.

23. a) \{0, 4\}; b) \(\left(\frac{90}{30}\right), 10\]; c) \{1\}.

24. a) No; b) Yes; c) No; d) No; e) Yes.

25. a) 3; b) 4; c) $-4$; d) $-3$.

27. a) 1 - 1, not onto; b) bijection; c) onto, not 1 - 1; d) not 1 - 1 and not onto.

28. a) One-to-one and onto; b) Neither one-to-one nor onto.

29. a) $f(x) \geq 2$; b) No; c) No, eg. $f(0) = f(4) = 6$.

31. $f$ is not 1 - 1 but is onto.

32. a) $\sqrt{4x^2 - 12x + 11}$; b) $g \circ f(x) = x$. 
33. b) Yes.

35. a) \( f^{-1}(x) = \frac{x-3}{5} \).

36. a) \([-1, 17] ; \) b) \([-3, -1] \cup [1, 3] \).

37. b) \((-1, 17] ; \) b) \([-3, -1] \cup [1, 3] \).

38. a) i) The sets are equal. ii) \( f(A \cap B) \subseteq f(A) \cap f(B) \).

39. 40430.

42. \( \frac{9}{2} - \frac{4}{n-1} - \frac{1}{n} \).

44. \( \frac{N+1}{2N} \).

**Problem Set 2.**

1. a) 2, 5; b) -11, 10; c) 77, 0.

2. \( T, F, T, F \).

3. \( T, F, T, F, F, F \).

4. \( 3^2 \cdot 13, \; 11 \cdot 13, \; 2^2 \cdot 3 \cdot 17^2, \; 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \).

5. a) \( 2^2 \cdot 3^3 \cdot 5^2, \; 2^5 \cdot 3^5 \cdot 5^3 \); b) \( 3, \; 2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \); c) 3, not defined.

6. 1, 3, 2.

8. a) \( n = 6, 4; \) b) \( 5^8 \equiv -1, 6; \) c) \( -2, 10. \)

9. a) \( 6 = -1 \cdot 12 + 1 \cdot 18; \) b) \( 3 = 29 \cdot 11 - 16 \cdot 201; \) c) \( 1 = -8 \cdot 13 + 5 \cdot 21; \)
   d) \( 1 = -13 \cdot 83 + 30 \cdot 36; \) e) \( 2 = 5 \cdot 22 - 2 \cdot 54; \) f) \( 7 = 39 \cdot 112 - 7 \cdot 623. \)

10. a) \( x \equiv 4 \) (mod 7); b) no solution; c) \( x \equiv 6 \) (mod 11);
   d) \( x \equiv 502 \) (mod 1001); e) \( x \equiv 99, 206, 313 \) (mod 321) or \( x \equiv 99 \) (mod 107)
   f) \( x \equiv 200, 751, 1302, 1853, 2404 \) (mod 2755) or \( x \equiv 200 \) (mod 551)
   g) \( x \equiv 29, 70, 111, 152, 193, 234, 275, 316, 357, 398, 439, 480, 521 \) (mod 533) or
   \( x \equiv 29 \) (mod 41)

13. c) \( R_3 = \{(3, 2), \; (3, 4), \; (5, 2), \; (5, 4), \; (5, 6)\} \).
14. $R_3$
   
   a)
   
   ![Graph of $R_3$]
   
   b) $M_{R_3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

17. \{\{0, 3, 6\}, \{1, 4\}, \{2, 5\}\}

18. b) \{\{0, 5\}, \{1, 4, 6\}, \{2, 3, 7, 8\}\}

21. $S = \{0, 6\} \cup \{1, 5, 7, 11\} \cup \{2, 4, 8, 10\} \cup \{3, 9\}$. 

24. b) 54, 60, 72 c) 2, 9 d) no e) no 
   f) \{36, 54, 72\} g) none h) \{2, 4, 6, 12\} i) 12

Problem Set 3.

15. a) Converse is false. It is known that $2^p - 1$ is prime for the primes
   $p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127$ and more than 20 other (larger) values of $p$. These
   are called Mersenne primes.

20. Converse is false.

21. b) True.
   c) True.

59. a, c, e and f

60. a) $p =$"prices go up", 
   $q =$"housing will be plentiful";
   $r =$"housing will be expensive"
   $[p \rightarrow q \land r] \land [\sim r \rightarrow q]$

   e) $p =$"Janet wins"; $q =$"Janet loses"; $r =$ "Janet will be tired"
   $p \lor (q \rightarrow r)$.

61. a) The food is good but the service is poor.
   b) The food is poor and so is the service.

62. a) Contradiction.
b) Contingency.

c) Tautology.

d) Tautology

64. a) \( \sim q \)
b) \( p \)
c) \( p \land q \land r \)
d) \( \sim p \lor q \).

66. I did my homework, but there is no way of deciding whether or not I studied.

67. Holmes is correct.

68. We cannot deduce whether Einstein is right or wrong.

69. c) Peta must have failed.

**Problem Set 4.**

1. a) \( 5^2 \times 9^4 \); b) \( 4 \times 5^2 \times 6^3 \); c) \( P(9, 6) = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \); d) \( C(6, 3) \times 8^3 \);
e) \( 8^6 + 6 \times 8^5 + C(6, 2)8^4 \); f) \( C(6, 3)P(8, 3) \); g) \( 4 \times 9^4 \).

2. a) \( 21!/13!, 21!/8!13!, 321!/123!198!, (2n)!/(n!)^2 \); b) \( 840, 35, 120, 6, 20100 \).

3. a) \( 7 \times 5 \times 21^6 \); b) \( C(7, 2) \times 5^2 \times 21^5 \); c) \( C(7, 3) \times 5^3 \times 21^4 \);
d) \( 26^7 - 21^7 - 7 \times 5 \times 21^6 - C(7, 2) \times 5^2 \times 21^5 \).

4. a) \( C(7, 1)P(5, 1)P(21, 6) \); b) \( C(7, 2)P(5, 2)P(21, 5) \); c) \( C(7, 3)P(5, 3)P(21, 4) \);
d) \( C(7, 3)P(5, 3)P(21, 4) + C(7, 4)P(5, 4)P(21, 3) + C(7, 5)P(5, 5)P(21, 2) \).

5. a) \( 228 \); b) \( 37800; 37800; 22260 \); c) \( 20 \times 26^3 + 13 \times 26^2 + 18 \times 26 + 22 = 360798 \).

6. a) \( C(8, 2) \times 25^6 \); b) \( 26^6 - 25^8 - 8 \times 25^7 \).

7. b) \( 4C(39, 8) - 6C(26, 8) + 4C(13, 8) \).

8. a) \( 143 \); b) \( 128 \).

9. a) \( 13C(4, 3)C(12, 5)/C(52, 8) \);
b) \( 4C(13, 3)C(39, 5) - 6C(13, 3)C(26, 2)C(52, 8) \);
c) \( 4C(13, 3) \times (C(39, 5) - 3C(13, 3)C(26, 2))/C(52, 8) \).

10. \( \phi(pq) = (p - 1)(q - 1), \phi(p^2q) = p(p - 1)(q - 1), \phi(pqr) = (p - 1)(q - 1)(r - 1) \).

16. a) \( 834 \); b) \( 834 \) again (NOT 417).

19. \( 4 \times 7 \times 6 \times 9 \times 16!/20! \).

20. \( 21!/(1!2!3!4!5!6!6^{21}) \).

21. a) \( \frac{1}{365} \); b) \( 1 - P(365, n)/365^n \); c) \( n \geq 23 \).
22. a) $C(23,3)$; b) $C(15,3)$; c) $C(23,3) - C(19,3)$.

23. a) $6^4$; b) $C(9,5)$.

24. a) $23!/2!5!7!9!$; b) $C(26,3)$.

25. a) $C(45,5)$; b) $C(27,5)$;

c) $C(45,5) - C(6,1)C(34,5) + C(6,2)C(23,5) - C(6,3)C(12,5)$

d) $C(13,5)$; e) $C(25,5)$; f) $C(45,5) - C(25,5)$;

g) $C(22,5)$; h) $C(30,5) - 3C(20,5) + 3C(10,5)$.

28. (There are other correct answers.)

a) $a_n = 2a_{n-1}$, $a_0 = 2$; b) $a_n = a_{n-1} + 2$, $a_0 = 1$; c) $a_n = -2a_{n-1}$, $a_0 = 3$;

d) $a_n = a_{n-1} + n$, $a_1 = 1$; e) $a_n = a_{n-1}$, $a_0 = 2$; f) $a_n = a_{n-1} + a_{n-2}$, $a_0 = 1$, $a_1 = 2$.

29. a) $a_n = a_{n-2} + a_{n-3}$ for $n \geq 4$, $a_1 = 0$, $a_2 = 1$, $a_3 = 1$;  

b) 1, 4.

30. 2, 3, 5.

31. $u_n = \left(1 + \frac{18}{1201}\right) u_{n-1} - 900$, for $n \geq 1$, $u_0 = 50,000$.

32. a) $a_n = A(5^n)$; b) $a_n = 3(-4)^n$.

33. a) $a_n = A(2^n) + B(-3)^n$; b) $a_n = A + B(2^n)$; c) $a_n = A(3^n) + Bn(3^n)$;

d) $a_n = A(1 + \sqrt{5})^n + B(1 - \sqrt{5})^n$.

34. a) $a_n = 4(3^n) + 3(-5)^n$; b) $a_n = 2^{n+1} + 3^{n+1}$; c) $a_n = 2(-2)^n - 4n(-2)^n$;

d) $a_n = (2 + \sqrt{10})^n + (2 - \sqrt{10})^n$.

35. a) $a_n = A(2^n) + B(-3)^n + \frac{8}{3} 4^n$; b) $a_n = A + B(2^n) + n(2^{n+1})$;

c) $a_n = A(3^n) + Bn(3^n) + 2n + 7$; d) $a_n = A(3^n) + Bn(3^n) + \frac{1}{3} n^2 3^n$.

36. $a_n = 2(4^n) + (-1)^{n+1} + n(-1)^n$.

37. a) $a_1 = 1$, $a_2 = 3$; b) $a_n = \frac{1}{5} \left(2^{n+1} + (-1)^n\right)$.

38. b) $a_n = 2^{n-1} + \frac{1}{2} 4^n$

39. a) 0, 2, 4, 12, 32, 88; b) $a_n = 2a_{n-1} + 2a_{n-2}$; c) $\frac{1}{\sqrt{3}} \left((1 + \sqrt{3})^n - (1 - \sqrt{3})^n\right)$.

40. c) $a_n = \frac{1}{3} \left(2^{n+2} + (-1)^{n+1}\right)$

41. c) $\frac{1}{2} \left(3^{n-1} + 1 - 2^n\right)$

Problem Set 5.

2. c) Total Vertex degree = 32, total number of edges = 16.

d) Loops are $e_{10}, e_{14}$.

e) Sets of parallel edges are $\{e_7, e_8\}$, $\{e_9, e_{12}\}$, $\{e_{15}, e_{16}\}$.
f) Edges incident to $v_3$ are $e_4, e_6, e_9, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$.
g) Vertices adjacent to $v_8$ are $v_2, v_3, v_7$.
h) $v_9$ is the only isolated vertex.

3. $G_1, G_3, G_4$ are nonsimple. The rest are simple.

4. Number of edges = 7.

![Graph](attachment://graph.png)

It is the only simple graph with these degrees. (The two degree 2 vertices cannot be adjacent.)

5. a) $\exists$ simple examples

![Graph](attachment://graph2.png)

b) There is no graph with these vertex degrees.

c) d) e) f) There is no simple graph with these vertex degrees.
7. \(|V(G)|, |E(G)|\) are respectively
   a) \(n, \binom{n}{2}\)
   b) \(m + n, mn\)
   c) \(n, n\)
   d) \(2^n, n2^{n-1}\)

8. \(35 = 15 + 12 + 6 + 1 + 1\)

9. a) \(\overline{K}_n \cong E_n\), ie, the graph with \(n\) vertices and no edges.
   b) \(\overline{K}_{m,n} \cong K_m \oplus K_n\) (disjoint union)

10. \(\frac{n(n-1)}{2} - m\)

11. \(Y = \text{Bipartite}, N = \text{Not Bipartite}\).
    a) \(Y\)   b) \(N\)   c) \(Y\)   d) \(N\)   e) \(Y\)   f) \(N\)   g) \(N\).

12. a) \(n = 2\)   b) \(n \geq 3\) and even   c) all \(n \geq 0\).

13. a) \[
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]
   b) \[
\begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}
\]
   c) \[
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

14. a)

15. a) Isomorphic eg. \(f : V(G_1) \to V(G_2)\)
    \[
    f : \begin{array}{cccc}
a & b & c & d \\
t & u & y & x \\
\end{array}
    \]

b) Isomorphic eg. \(f : V(G_1) \to V(G_2)\)
    \[
    f : \begin{array}{cccccc}
a & b & c & d & e & f \\
v & z & y & t & u & x \\
\end{array}
    \]
c) Not isomorphic.

d) Not isomorphic.

e) Isomorphic eg. \( f : V(G_1) \rightarrow V(G_2) \)

\[
\begin{array}{cccccccc}
| f | & a & b & c & d & e & f & g \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>t</td>
<td>u</td>
<td>v</td>
<td>w</td>
<td>x</td>
<td>z</td>
<td></td>
</tr>
</tbody>
</table>
\end{array}
\]

f) Not isomorphic.

16. b) Up to isomorphism there is/are

\( \alpha \) Only 1 self–complementary graph on 4 vertices

\[
\begin{array}{c}
\text{Graph A}
\end{array}
\]

\( \beta \) 2 self–complementary graphs on 5 vertices

\[
\begin{array}{c}
\text{Graph B}
\end{array}
\]

\[
\begin{array}{c}
\text{Graph C}
\end{array}
\]

17. a) \( n = 2 \): 2 possible

b) \( n = 3 \): 4 possible

c) \( n = 4 \): 11 possible

18. There are 4 components with vertex sets \( \{A, B, H, L\}, \{C, F\}, \{D, G, I, M\}, \{E, J, K, N\} \).

19. a) 2 b) 7 c) 20 d) 61.

20. a) \( M = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 2 \\
0 & 1 & 2 & 0
\end{pmatrix} \)

b) \( M^2 = \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 3 & 3 & 2 \\
1 & 3 & 6 & 3 \\
1 & 2 & 3 & 5
\end{pmatrix} \), \( M^3 = \begin{pmatrix}
0 & 3 & 3 & 2 \\
3 & 5 & 10 & 9 \\
3 & 10 & 15 & 15 \\
2 & 9 & 15 & 8
\end{pmatrix} \)

c) 3 and 10 respectively.

Walks of length 2 \( B \) to \( C \) are

\( BDe_4C \)

\( BDe_5C \)

\( BCe_3C \)
Walks of length 3 from \( B \) to \( C \) are

\[
BABC \\
BDBC \\
BCBC \\
BCe_4De_4C \\
BCe_4De_5C \\
BCe_5De_4C \\
BCe_5De_5C \\
BDe_4Ce_3C \\
BDe_5Ce_3C \\
BCe_3Ce_3C
\]

d) i) This means there are 20 walks of length \( \leq 3 \) from \( C \) to \( D \).
   ii) This means \( G \) is connected.

21. a) \( \alpha \) \( N \)
    \( \beta \) Y eg. \( ABCDEFGICHA \)
    \( \gamma \) \( N \)
    \( \delta \) \( N \)

   b) \( \alpha \) \( N \)
    \( \beta \) \( N \)
    \( \gamma \) \( Y \)
    \( \delta \) \( N \)

   c) \( \alpha \) Y eg \( CBAECDEGe_1BFGe_2BD \).
    \( \beta \) \( N \)
    \( \gamma \) Y eg \( DCEABFG \)
    \( \delta \) \( N \)

   d) \( \alpha \) Y eg \( BAECADCFDEF \)
    \( \beta \) \( N \)
    \( \gamma \) Y eg \( ABCDFE \)
    \( \delta \) Y eg \( ABCDFEA. \)

   e) \( \alpha \) Y eg \( BAEBDFC \)
    \( \beta \) \( N \)
    \( \gamma \) Y eg \( EABCDF \)
    \( \delta \) \( N \)

   f) \( \alpha \) Y eg \( GBAEDBCGFCEG \)
    \( \beta \) \( N \)
    \( \gamma \) Y eg \( ABDECGF \)
    \( \delta \) \( N. \)

22. a) \( n \geq 1 \) and odd
   b) All \( n \geq 3 \)
   c) All \( n \geq 0 \) and even.

23. a) \( n = 2 \)
   b) Never
   c) \( n = 1. \)
24.  a) $m, n$ both even  
   b) $m = 2$ and $n$ odd or $n = 2$ and $m$ odd  
   c) $m = n \geq 2$.  

28.  i) a) 2 b) 8  
     ii) a) 1 b) 8  
     iii) a) 4 b) 12  
     iv) a) 3 b) 12  
     v) a) 6 b) 6  
     vi) a) 9 b) 12  

29.  a) 15 b) 6  

32.  Only for $n = 0, 1, 2, 3$.  

36.  a) 7 b) 3, 3, 1, 1, 1, 1 and 3, 3, 2, 1, 1, 1  
     c) There are six possible answers.
QUESTIONS  \(\text{ (Time allowed: 20 minutes)}\)

1. \(3\) marks
   
   Prove that \(\{10k + 7 \mid k \in \mathbb{Z}\}\) is a proper subset of \(\{5m - 8 \mid m \in \mathbb{Z}\}\).

2. \(3\) marks
   
   Let \(f\) be a function from \(A\) to \(B\) and \(g\) a function from \(B\) to \(C\). Show that if the composite function \(g \circ f\) is one–to–one (injective), then \(f\) is one–to–one (injective).

3. \(4\) marks
   
   Prove that if \(k > 1\) then
   
   \[
   \frac{1}{(k - 1)^2} - \frac{1}{(k + 1)^2} = \frac{4k}{(k^2 - 1)^2}.
   \]

   Hence simplify
   
   \[
   \sum_{k=2}^{n} \frac{k}{(k^2 - 1)^2}.
   \]
Note: The use of a calculator is NOT permitted in this test

Note: To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw.

QUESTIONS (Time allowed: 20 minutes)

1. (3 marks)
   Let
   \[ A_1 = \{ a \} , \quad A_2 = A_1 \cup \{ A_1 \} \quad \text{and} \quad A_3 = A_2 \cup \{ A_2 \} . \]
   (i) List the elements of \( A_3 \).
   (ii) Are the following statements true or false? Give clear reasons.
       (i) \( \{ \{ a \} \} \in A_3 \);
       (ii) \( \{ \{ a \} \} \subseteq A_3 \).

2. (3 marks)
   Let \( f \) be a function from \( A \) to \( B \) and \( g \) a function from \( B \) to \( C \). Show that if the composite function \( g \circ f \) is one–to–one (injective), then \( f \) is one–to–one (injective).

3. (4 marks)
   Prove that if \( k > 1 \) then
   \[ \frac{1}{k-1} + \frac{6}{k} - \frac{7}{k+1} = \frac{8k-6}{k(k^2-1)} . \]
   Hence simplify
   \[ \sum_{k=2}^{n} \frac{4k-3}{k(k^2-1)} . \]
QUESTIONS  (Time allowed: 20 minutes)

1. (3 marks)
   
   Let
   
   \[ A_1 = \{ a, b, c \}, \quad A_2 = P(A_1) \quad \text{and} \quad A_3 = P(A_2). \]

   (i) How many elements are there in \( A_3 \)?
   
   (ii) Are the following statements true or false? Give clear reasons.
      
      (i) \( A_2 \in A_3 \);
      (ii) \( A_2 \subseteq A_3 \).

2. (3 marks)
   
   A function \( f: \mathbb{R} \to \mathbb{R} \) is defined by
   
   \[ f(x) = 2x^3 + 3x^2 - 4 . \]

   Find the range of \( f \). Is \( f \) one–to–one (injective)? Is \( f \) onto (surjective)? Is \( f \) a bijection? Give reasons for all your answers.

3. (4 marks)
   
   Prove that if \( k > 1 \) then
   
   \[ \frac{5}{k - 1} \cdot \frac{3}{k} \cdot \frac{2}{k + 2} = \frac{9k + 6}{(k - 1)k(k + 2)} . \]

   Hence simplify
   
   \[ \sum_{k=2}^{n} \frac{3k + 2}{(k - 1)k(k + 2)} . \]
QUESTIONS  

1. (3 marks)
   Let \( A = \{ n, s, w \} \).
   (i) Write in full the set \( P(A) \).
   (ii) How many elements are there in the set \( P(A \cup P(A)) \)?

2. (3 marks)
   Let \( f \) be a function from \( A \) to \( B \) and let \( g \) be a function from \( B \) to \( C \).
   (i) State what it means to say that \( f \) is a one–to–one function, and what it means to say that \( g \) is a one–to–one function.
   (ii) Prove that if both \( f \) and \( g \) are one–to–one functions, then the composite function \( g \circ f \) is also one–to–one.

3. (4 marks)
   Use the laws of set algebra to simplify
   \[
   (A \cap B^c) \cup (A^c \cap B^c)^c.
   \]
   Show your working and give a reason for each step.
QUESTIONS  (Time allowed: 20 minutes)

1. (3 marks)
   If \( A = \{ a \} \), \( B = \{ b, c \} \) and \( C = \{ d, e, f, g, h, i, j \} \), find
   (i) \( P(A \times B) \);
   (ii) \( |P(B \times C)| \).

2. (3 marks)
   Let \( f \) and \( g \) be functions from \( \mathbb{R} \) to \( \mathbb{R} \) defined by
   \[ f(x) = x^2 \quad \text{and} \quad g(x) = e^x. \]
   (i) Write down formulae for the composite functions \( f \circ g \) and \( g \circ f \).
   (ii) Is \( f \circ g \) a one-to-one function? Is \( g \circ f \) a one-to-one function? Give reasons for your answers.

3. (4 marks)
   Use the formula
   \[ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \]
   to prove that
   \[ \tan k \tan(k - 1) = \frac{\tan k - \tan(k - 1)}{\tan 1} - 1. \]
   Hence simplify
   \[ \sum_{k=1}^{n} \tan k \tan(k - 1). \]
QUESTIONS  

1. (3 marks)
   Solve the following congruences, or explain why they have no solution:
   (i) $28x \equiv 3 \pmod{66}$;
   (ii) $29x \equiv 3 \pmod{67}$.

2. (3 marks)
   Consider the divisibility relation on the set
   \[ S = \{ 2, 6, 7, 14, 15, 30, 70, 105, 210 \} . \]
   It is given that this relation is a partial order on $S$ (do not prove it!).
   (i) Draw the Hasse diagram for this partial order.
   (ii) Find all maximal elements and all minimal elements of $S$.
   (iii) Does $S$ have a greatest element? Does $S$ have a least element? If so, write them down; if not, explain why not.

3. (4 marks)
   A relation $\sim$ is defined on $\mathbb{Z}^+$ by
   \[ x \sim y \text{ if and only if } y = 3^k x \text{ for some integer } k \]
   (note that $k$ may be positive, negative or zero). Prove that $\sim$ is an equivalence relation.
QUESTIONS  \( \text{(Time allowed: 20 minutes)} \)

1. \( (3 \text{ marks}) \)
   Solve the congruence \( 20x \equiv 16 \pmod{92} \). Give your answer as
   (i) a congruence to the smallest possible modulus;
   (ii) a congruence modulo 92.

2. \( (3 \text{ marks}) \)
   Let \( a, b \) and \( c \) be integers. Prove that if \( a^2 \mid b \) and \( b^3 \mid c \) then \( a^4b^5 \mid c^3 \). Be sure to set out your answer clearly and logically.

3. \( (4 \text{ marks}) \)
   A relation \( \sim \) is defined on the set of all real numbers by
   \[
   x \sim y \quad \text{if and only if} \quad \sin x = \sin y .
   \]
   You may assume that \( \sim \) is an equivalence relation.
   (i) Find the equivalence class of 0.
   (ii) For any \( a \in \mathbb{R} \), find the equivalence class of \( a \).
1. (3 marks)
   Solve the congruence $45x \equiv 15 \pmod{78}$. Give your answer as
   (i) a congruence to the smallest possible modulus;
   (ii) a congruence modulo 78.

2. (3 marks)
   Let $a$ and $m$ be integers. Prove that if $a \mid m$ and $a + 1 \mid m$ then $a(a + 1) \mid m$. Be sure to set out
   your answer clearly and logically.

3. (4 marks)
   A relation $\preceq$ is defined on the set of all positive integers by
   $$x \preceq y \quad \text{if and only if} \quad y = 3^k x \text{ for some non-negative integer } k.$$ 
   Prove that $\preceq$ is a partial order.
QUESTION 1 (3 marks)
Solve the following congruences, or explain why they have no solution:
(i) $79x \equiv 5 \pmod{98}$;
(ii) $78x \equiv 5 \pmod{99}$.

QUESTION 2 (3 marks)
Let $x, y$ and $m$ be integers. Prove that if $m \mid 4x + y$ and $m \mid 7x + 2y$ then $m \mid x$ and $m \mid y$.

QUESTION 3 (4 marks)
Let $F$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. A relation $\preceq$ is defined on $F$ by

$$f \preceq g \iff f(x) \leq g(x) \quad \text{for all } x \in \mathbb{R}.$$ 

Prove that $\preceq$ is a partial order.
QUESTIONS  \hspace{1cm} (Time allowed: 20 minutes)

1.  \hspace{1cm} (3 marks)
   Solve the following congruences, or explain why they have no solution:
   (i)  $25x \equiv 3 \pmod{109}$;
   (ii) $25x \equiv 3 \pmod{110}$.

2.  \hspace{1cm} (3 marks)
   (i) Find the prime factorisation of 6500, and of 1120.
   (ii) Hence write down, in factorised form, $\gcd(6500, 1120)$ and $\text{lcm}(6500, 1120)$.

3.  \hspace{1cm} (4 marks)
   We write $\mathbb{R}^+$ for the set of positive real numbers. A relation $\sim$ is defined on $\mathbb{R}^+$ by
   \[ x \sim y \quad \text{if and only if} \quad x - y \text{ is an integer} \, . \]
   Prove that $\sim$ is an equivalence relation.
Questions (Time allowed: 20 minutes)

1. (3 marks)
   (i) Construct truth tables for the two propositional calculus formulae
   \[ \sim p \rightarrow (q \land p) \quad \text{and} \quad (p \land q) \rightarrow q. \]
   (ii) Giving reasons for your answer, determine whether the first formula logically implies the second, or the second logically implies the first, or both, or neither.

2. (3 marks)
   Prove that \( \log_6 11 \) is irrational.

3. (4 marks)
   Show that
   \[ q(n) = 11n^2 + 32n \]
   is a prime number for two integer values of \( n \), and is composite for all other integer values of \( n \).
Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw. In a question asking for proofs, your answers must be fully and clearly explained, carefully set out and neatly written, and logically correct.

QUESTIONS  (Time allowed: 20 minutes)

1. (3 marks)
   (i) Construct truth tables for the two propositional calculus formulae

   \[(p \rightarrow (\sim q)) \land r \quad \text{and} \quad q \rightarrow ((\sim p) \land r) .\]

   (ii) Giving reasons for your answer, determine whether the first formula logically implies the second, or the second logically implies the first, or both, or neither.

2. (3 marks)
   You are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow and any necessary algebraic details.

   **Theorem.** If \( m \) and \( n \) are positive integers then \( m! n! < (m + n)! \).

   **Basic ideas:** \( m! = 1 \times 2 \times \cdots \times m \), and \( 1 < m + 1, 2 < m + 2, \ldots, n < m + n. \)

3. (4 marks)
   Prove that if \( a \) is any positive real number then the equation

   \[ a x = \cos \pi x \]

   has exactly one solution \( x \) such that \( 0 \leq x \leq 1. \)
Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw. In a question asking for proofs, your answers must be fully and clearly explained, carefully set out and neatly written, and logically correct.

QUESTIONS (Time allowed: 20 minutes)

1. (3 marks)
   Use standard logical equivalences to show that
   \[ p \rightarrow (\neg(q \land (\neg p))) \]
   is a tautology. Show all your working and give a reason for every step. You may assume that \( u \rightarrow v \) is logically equivalent to \( (\neg u) \lor v \).

2. (3 marks)
   You are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow and any necessary algebraic details.
   
   **Theorem.** If \( n \) is a positive integer then
   \[ (1 \times 2) + (2 \times 5) + (3 \times 8) + \cdots + n(3n - 1) = n^2(n + 1) . \]
   
   **Basic ideas:** \( n^2(n + 1) + (n + 1)(3n + 2) = (n + 1)^2(n + 2) \).

3. (4 marks)
   Prove that if \( x \) is a real number and \( 2x^2 - 3 = 0 \) then \( x \) is irrational.
Note: The use of a calculator is NOT permitted in this test.

Note. To obtain full marks for this test you must give reasons for the steps you take and the conclusions you draw. In a question asking for proofs, your answers must be fully and clearly explained, carefully set out and neatly written, and logically correct.

**QUESTIONS** *(Time allowed: 20 minutes)*

1. **(3 marks)**
   (i) Express the following argument in symbolic form using logical connectives. Be careful to define any notation you introduce.

   “If I earn some money then I will go for a holiday this summer.

   I will either go for a holiday or work this summer.

   Therefore, if I don’t go for a holiday this summer then I will not have earned any money and will be working.”

   (ii) Show that the above argument is logically valid.

2. **(3 marks)**
   You are given a theorem, together with the basic ideas needed to prove it. Write up a detailed proof of the theorem. Your answer must be written in complete sentences, with correct spelling and grammar. It must include a suitable introduction and conclusion; reasons for all statements made; correct logical flow and any necessary algebraic details.

   **Theorem.** Between any two different rational numbers there is another rational number.

   **Basic ideas:** if \( x \) and \( y \) are rational then so is \( \frac{x + y}{2} \).

3. **(4 marks)**
   Prove that if \( n \) is a positive integer then \( 4^{2n} + 10n - 1 \) is a multiple of 25.
1. (3 marks)
Use standard logical equivalences to show that
\[(p \rightarrow q) \land (q \rightarrow (\sim p \lor r))\]
is logically equivalent to \[p \rightarrow (q \land r)\]. Show all your working and give a reason for every step.
You may assume that \[u \rightarrow v\] is logically equivalent to \[(\sim u) \lor v\].

2. (3 marks)
Prove that if \(n\) is a positive integer then
\[(n + 1)(n + 2) \cdots (2n) = 2^n \times 1 \times 3 \times 5 \times \cdots \times (2n - 1)\].

3. (4 marks)
Prove that if \(n\) is any positive integer than \(\sqrt{4n - 2}\) is irrational.
QUESTIONS  (Time allowed: 20 minutes)

1.  (2 marks)
    How many eight–letter words can be formed from the English alphabet
    (i) if no letter may be used twice?
    (ii) if one letter is to be used exactly twice and each other letter is to be
         used at most once?
        (DISCRETE is an example of such a word.)

2.  (2 marks)
    An ordinary six–sided die is rolled ten times. Find the probability that five different faces come
    up twice each.

3.  (3 marks)
    Find a particular solution of the recurrence relation
    \[ a_n - 8a_{n-1} + 15a_{n-2} = 21 \times 2^n. \]

4.  (3 marks)
    How many hands of 13 cards can be chosen from a standard pack which have exactly five cards
    in some suit?
Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test your solutions must be fully and clearly explained. Answers should be left in terms of powers, factorials and $P$ and $C$ notation where appropriate.

**QUESTIONS**  (Time allowed: 20 minutes)

1. (1 mark)
   Find the number of seven-letter words which can be constructed from the English alphabet and which contain at least six consonants.

2. (2 marks)
   Find the probability that a hand of eight cards dealt from a standard 52-card pack contains exactly three kings.

3. (3 marks)
   (i) How many solutions has the equation
   \[ x_1 + x_2 + x_3 + x_4 + x_5 = 16 \]
   in which $x_1, x_2, x_3, x_4, x_5$ are non-negative integers? Explain your reasoning.
   (ii) How many of these solutions satisfy the conditions $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 2$ and $x_5 \geq 1$? Explain.

4. (4 marks)
   Find the general solution of the recurrence relation
   \[ a_n - 16a_{n-2} = 9 \times 2^n. \]
Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test your solutions must be fully and clearly explained. Answers should be left in terms of powers, factorials and $P$ and $C$ notation where appropriate.

QUESTIONS  
(Time allowed: 20 minutes)

1. (2 marks)
   How many solutions has the equation
   \[ x_1 + x_2 + x_3 + x_4 + x_5 = 66 \]
   (i) if $x_1, x_2, x_3, x_4, x_5$ are non-negative integers?
   (ii) if $x_1, x_2, x_3, x_4, x_5$ are even numbers?

2. (2 marks)
   In a survey, viewers are given a list of 20 TV programmes. They are asked to label their three favourites 1, 2 and 3, and to put a tick against those they have heard of (if any) from the remaining 17. In how many ways can the form be filled out? (Assume that everyone has three favourite programmes to nominate.)

3. (3 marks)
   Find the solution of the recurrence relation
   \[ a_n - 10a_{n-1} + 21a_{n-2} = 0 \]
   subject to the initial conditions $a_0 = -1$ and $a_1 = 1$.

4. (3 marks)
   There are 50 houses along one side of Discrete St. A survey shows that 26 of these houses have MATH1081 students living in them (what a coincidence). Prove that there are two MATH1081 students who live exactly five houses apart in Discrete St.
Note: The use of a calculator is NOT permitted in this test

Note. To obtain full marks for this test your solutions must be fully and clearly explained. Answers should be left in terms of powers, factorials and $P$ and $C$ notation where appropriate.

**QUESTIONS**  (Time allowed: 20 minutes)

1. (2 marks)
   How many ten–letter words can be constructed from the English alphabet which have no repeated letters and contain at least eight consonants?

2. (2 marks)
   How many thirteen–card hands can be selected from a standard pack which contain exactly four spades or exactly four diamonds?

3. (3 marks)
   Find the general solution of the recurrence relation
   \[ a_n + 3a_{n-1} - 10a_{n-2} = 2^n. \]

4. (3 marks)
   A course has seven elective topics, and students must complete exactly three of them in order to pass the course. Show that if 200 students passed the course, at least six of them must have completed the same electives as each other.
QUESTIONS  (Time allowed: 20 minutes)

1. (2 marks)
   Students in a certain subject have 6 class hours per week. The timetable schedules 4 possible
timeslots each day (Monday to Friday). In how many ways can a student timetable classes for
this subject
   (i) if any choice of 6 class hours is allowed?
   (ii) if the student must choose two hours on one day and one hour on every other day?

2. (2 marks)
   How many solutions has the equation
   \[ x_1 + x_2 + x_3 + x_4 + x_5 = 55 \]
   if \( x_1, x_2, x_3, x_4, x_5 \) are non–negative integers
   (i) with no further restrictions?
   (ii) and every \( x_k \) is an odd number?

3. (3 marks)
   Find the solution of the recurrence relation
   \[ a_n - 7a_{n-1} + 10a_{n-2} = 0 \]
   which satisfies the initial conditions \( a_0 = 5 \) and \( a_1 = 1 \).

4. (3 marks)
   How many eleven–letter words (constructed from the English alphabet) contain the subword FRED?