Three dimensional geometry, ZOME, and the elusive tetrahedron

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July 2012
Overview

- A) Introduce basic ideas of affine and projective rational trigonometry, with some proofs of main laws
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History:
- N J Wildberger ()

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**History:** Aristotle, Plato, Euclid, Tartaglia, Euler, Gauss, Cayley, Menger, Monge, Wolstenholme, Hilbert, Dehn, G. Richardson (*The Trigonometry of the Tetrahedron*, Math. Gazette, 1902) ++
The elusive tetrahedron

\[ P_1 P_2 P_3 P_4 \]

In classical trigonometry one tries to determine the various relations between the 6 side lengths, the 12 face angles, the 6 dihedral edge angles (between faces), the 4 solid angles, and the volume. There are also other secondary quantities. It is probably fair to say this has nowhere been done completely. With rational trigonometry we start with quadrances \( Q_{ij} = Q(P_i, P_j) \equiv \overrightarrow{P_i P_j} \cdot \overrightarrow{P_i P_j} \) instead of distances, but the real innovation is replacing angles with spreads.
The basic quantities: quadrances and spreads

Affine rational trigonometry of a face + projective rational trigonometry of a tripod
Quadrance between points

We start with a dot/inner product on a vector/linear space, such as

$$ (x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2 $$

or in three dimensions

$$ (x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2. $$

The quadrance of a vector $\vec{v}$ is

$$ Q(\vec{v}) \equiv \vec{v} \cdot \vec{v}. $$

The quadrance between points $A_1$ and $A_2$ is

$$ Q(A_1, A_2) \equiv Q(\overrightarrow{A_1A_2}) $$

so that

$$ Q([x_1, y_1], [x_2, y_2]) = (x_2 - x_1)^2 + (y_2 - y_1)^2. $$
In rational trigonometry we avoid the transcendental aspects of angles and the circular functions and their inverse functions. The **spread** $s$ between vectors $\vec{v}$ and $\vec{w}$ is

$$s(\vec{v}, \vec{w}) \equiv 1 - \frac{(\vec{v} \cdot \vec{w})^2}{Q(\vec{v}) Q(\vec{w})} = 1 - \frac{(\vec{v} \cdot \vec{w})^2}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}.$$

So

$$s((x_1, y_1), (x_2, y_2)) = 1 - \frac{(x_1x_2 + y_1y_2)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \frac{(x_1y_2 - x_2y_1)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}.$$

This extends to a notion of spread between lines (not rays!!)
Triangle $\triangle A_1A_2A_3$ with quadrances $Q_1, Q_2, Q_3$ and spreads $s_1, s_2, s_3$:

**Cross law**

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3)$$

**Spread law**

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}$$

**Triple spread formula**

$$(s_1 + s_2 + s_3)^2 = 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4s_1s_2s_3$$
Proofs of main laws: Cross law

\[ Q_3 = Q \left( \overrightarrow{A_1A_2} \right) = \left( \overrightarrow{A_3A_2} - \overrightarrow{A_3A_1} \right) \cdot \left( \overrightarrow{A_3A_2} - \overrightarrow{A_3A_1} \right) \]

\[ = Q_1 + Q_2 - 2 \left( \overrightarrow{A_3A_2} \cdot \overrightarrow{A_3A_1} \right) \]

so

\[ s_3 = 1 - \frac{\left( \overrightarrow{A_3A_2} \cdot \overrightarrow{A_3A_1} \right)^2}{Q_1 Q_2} = 1 - \frac{(Q_1 + Q_2 - Q_3)^2}{4 Q_1 Q_2} \]

Thus

\[ (Q_1 + Q_2 - Q_3)^2 = 4 Q_1 Q_2 \left( 1 - s_3 \right) \]

Cross law.
Proofs of main laws: Spread law and quadrea

From the Cross law \((Q_1 + Q_2 - Q_3)^2 = 4Q_1 Q_2 (1 - s_3)\) we get

\[
4Q_1 Q_2 s_3 = 4Q_1 Q_2 - (Q_1 + Q_2 - Q_3)^2
= (Q_1 + Q_2 + Q_3)^2 - 2 (Q_1^2 + Q_2^2 + Q_3^2)
\equiv \quad \mathcal{A} = \text{quadrea of triangle} = 16 \times (\text{area})^2
\]

By symmetry

\[
4Q_1 Q_2 s_3 = 4Q_1 Q_3 s_2 = 4Q_2 Q_3 s_1 = \mathcal{A}
\]

so that

\[
\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} = \frac{\mathcal{A}}{4Q_1 Q_2 Q_3}.
\]

Spread law.
Proofs of main laws: Triple spread formula

From the Spread law

\[
\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} = \frac{1}{D}
\]

so that \( Q_1 = Ds_1 \), \( Q_2 = Ds_2 \) and \( Q_3 = Ds_3 \). Sub into the Cross law

\[
(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3)
\]

to get

\[
D^2 \left( s_1 + s_2 - s_3 \right)^2 = 4D^2s_1s_2 \left( 1 - s_3 \right)
\]

or

\[
\left( s_1 + s_2 - s_3 \right)^2 = 4s_1s_2 \left( 1 - s_3 \right).
\]

Thus

\[
\left( s_1 + s_2 + s_3 \right)^2 = 2 \left( s_1^2 + s_2^2 + s_3^2 \right) + 4s_1s_2s_3. \]
A projective point \( a = [x : y : z] \) is a one dimensional subspace of \( \mathbb{V}^3 \), a three dim vector space over a field (usually the rationals). A **projective line** \( L = (l : m : n) \) is a two dimensional subspace of \( \mathbb{V}^3 \), representing the plane \( lx + my + nz = 0 \).

The **projective quadrance** \( q(a_1, a_2) \) is defined to be the spread between the projective points \( a_1, a_2 \). If \( a_1 = [x_1 : y_1 : z_1] \) and \( a_2 = [x_2 : y_2 : z_2] \) then

\[
q(a_1, a_2) = 1 - \frac{(x_1 x_2 + y_1 y_2 + z_1 z_2)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}.
\]

The **projective spread** \( S(L_1, L_2) \) is the spread between the projective lines, or their normals. If \( L_1 = (l_1 : m_1 : m_1) \) and \( L_2 = (l_2 : m_2 : n_2) \) then

\[
S(L_1, L_2) = 1 - \frac{(l_1 l_2 + m_1 m_2 + n_1 n_2)^2}{(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2)}.
\]
A projective triangle

Three projective points \(a_1, a_2, a_3\) form a **projective triangle**, or **tripod**. The associated projective lines are \(L_1 = a_2 a_3, L_2 = a_1 a_3\) and \(L_3 = a_1 a_2\). Define the projective quadrances

\[
q_1 = q(a_2, a_3) \quad q_2 = q(a_1, a_3) \quad q_3 = q(a_1, a_2)
\]

and the projective spreads

\[
S_1 = S(L_2, L_3) \quad S_2 = S(L_1, L_3) \quad S_3 = S(L_1, L_2).
\]
It turns out that these laws hold also for (universal) hyperbolic geometry!

Projective spread law

\[
\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}.
\]

Projective cross law

\[
(S_3 q_1 q_2 - q_1 - q_2 - q_3 + 2)^2 = 4 (1 - q_1) (1 - q_2) (1 - q_3).
\]

Dual projective cross law

\[
(S_1 S_2 q_3 - S_1 - S_2 - S_3 + 2)^2 = 4 (1 - S_1) (1 - S_2) (1 - S_3).
\]
Projective quadrea:

\[
\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3} = \frac{S}{q_1 q_2 q_3}
\]

Using altitude quadrances \( h_1, h_2, h_3 \), the Spread law \( S_1 = h_2 / q_3 \) gives

\[
S = h_1 q_1 = h_2 q_2 = h_3 q_3
\]
The ZOME construction system was invented by Steve Baer and his wife Holly in 1992, with design work by Marc Pelletier and Paul Hildebrandt. Struts of three different colours (blue, red, yellow), and three different sizes, fit into white balls with 62 holes. The system allows construction of the Platonic solids and many related objects, including 3D models of the 120 and 600 cells. (There is also a more advanced system employing green struts).
By constructing a pentagon and some right triangles, we can determine the quadrances of the ZOME struts to be

\[
\begin{align*}
B_1 &= 1 & B_2 &= \tau & B_3 &= \tau^2 \\
R_1 &= \beta & R_2 &= \beta \tau & R_3 &= \beta \tau^2 \\
Y_1 &= 3/4 & Y_2 &= 3\tau/4 & Y_3 &= 3\tau^2/4.
\end{align*}
\]

where

\[
\tau = \frac{3 + \sqrt{5}}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^2 \approx 2.618 \quad \text{and} \quad \beta = \frac{5 + \sqrt{5}}{8} \approx 0.905.
\]
ZOME and spreads

The following spreads figure prominently in the regular pentagon, and are the zeroes of the 5-th spread polynomial $S_5(s) = s(5 - 20s + 16s^2)$:

$$\alpha = \frac{5 - \sqrt{5}}{8} \approx 0.345 \quad \beta = \frac{5 + \sqrt{5}}{8} \approx 0.905$$

Note that $\alpha \leftrightarrow 36^\circ, 144^\circ$ and $\beta \leftrightarrow 72^\circ, 108^\circ$. With ZOME, we can easily construct an equilateral triangle (with equal spreads $s = 3/4$) and the **sharp** and **flat** triangles.
Two coloured primitive ZOME triangles

\[ \tau = \frac{3 + \sqrt{5}}{2} \quad \text{and} \quad \bar{\tau} = \frac{3 - \sqrt{5}}{2} \]
Three coloured primitive ZOME triangles

(This is not quite a complete list of ZOME triangles: there are also a few more that require more than one strut per side).
A ZOME tetrahedra

Here is one of the 65 possible ZOME tetrahedron. It arises naturally when we build simultaneously both a dodecahedron and an icosahedron with the same center, using ZOME.

Here are the planar quadrances and spreads of this tetrahedron, formed by the four triangular faces.
Tartaglia’s volume formula (also Euler, Cayley-Menger)

\[ v^2 = \frac{1}{288} \det \begin{pmatrix} 2Q_{12} & Q_{12} + Q_{13} - Q_{23} & Q_{12} + Q_{14} - Q_{24} \\ Q_{12} + Q_{13} - Q_{23} & 2Q_{13} & Q_{13} + Q_{14} - Q_{34} \\ Q_{12} + Q_{14} - Q_{24} & Q_{13} + Q_{14} - Q_{34} & 2Q_{14} \end{pmatrix} \]

**Tartaglia’s formula** The **quadrume** of the tetrahedron \( \mathcal{V} = 144v^2 \) is given by

\[ \mathcal{V} = Q_{12}Q_{13}Q_{24} + Q_{12}Q_{13}Q_{34} + Q_{12}Q_{14}Q_{23} + Q_{12}Q_{14}Q_{34} + Q_{12}Q_{23}Q_{34} + Q_{13}Q_{24}Q_{34} + Q_{12}Q_{24}Q_{34} + Q_{14}Q_{23}Q_{24} + Q_{14}Q_{23}Q_{34} + Q_{13}Q_{14}Q_{23} + Q_{13}Q_{14}Q_{24} + Q_{13}Q_{23}Q_{24} - Q_{12}Q_{14}Q_{24} - Q_{23}Q_{24}Q_{34} - Q_{12}Q_{13}Q_{23} - Q_{13}Q_{14}Q_{34} - Q_{13}Q_{24}^2 - Q_{12}Q_{34}^2 - Q_{13}Q_{24} - Q_{12}Q_{34} - Q_{14}Q_{23} - Q_{14}Q_{23}^2. \]
The Alternate spreads theorem

Relations between 12 planar face spreads $s_{i,j}$: Four Triple spread formulas, one for each face, e.g.

$$(s_{1,4} + s_{2,4} + s_{3,4})^2 = 2 (s_{1,4}^2 + s_{2,4}^2 + s_{3,4}^2) + 4s_{1,4}s_{2,4}s_{3,4}$$

Alternate spreads theorem

$$\frac{s_{2,4}}{s_{3,4}} \frac{s_{3,2}}{s_{4,3}} \frac{s_{4,3}}{s_{2,3}} = 1.$$ 

There are four such relations, and the product of them is identically $1 = 1$. 
The trigonometry of a tetrahedron

In a tetrahedron, the Projective cross law has a lovely reformulation that allows us to express the dihedral edge spreads $E_{ij}$ in terms of $Q_{kl}$.

\[
(E_{12} s_{1,3} s_{1,4} - s_{1,2} - s_{1,3} - s_{1,4} + 2)^2 = 4 \left(1 - s_{1,2}\right) \left(1 - s_{1,3}\right) \left(1 - s_{1,4}\right)
\]

\[
= \frac{(Q_{34} - Q_{13} - Q_{14})^2}{4 Q_{13} Q_{14}} \cdot \frac{(Q_{24} - Q_{12} - Q_{14})^2}{4 Q_{12} Q_{14}} \cdot \frac{(Q_{23} - Q_{12} - Q_{13})^2}{4 Q_{12} Q_{13}}
\]

\[
= \frac{(Q_{34} - Q_{13} - Q_{14})^2 \cdot (Q_{24} - Q_{12} - Q_{14})^2 \cdot (Q_{23} - Q_{12} - Q_{13})^2}{16 Q_{12}^2 Q_{13}^2 Q_{14}^2}.
\]
Dihedral spreads and quadrances

Recall quadrea
\[ A = (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2) \equiv A(Q_1, Q_2, Q_3). \]

Dihedral spreads theorem

\[ E_{12} = \frac{4Q_{12}V}{A_3A_4}. \]

By symmetry we have

\[ V = \frac{E_{12}A_3A_4}{4Q_{12}} = \frac{E_{13}A_2A_4}{4Q_{13}} = \frac{E_{14}A_2A_3}{4Q_{14}} \]
\[ = \frac{E_{23}A_1A_4}{4Q_{23}} = \frac{E_{24}A_1A_3}{4Q_{24}} = \frac{E_{34}A_1A_2}{4Q_{34}} \]

from which we get

\[ \frac{E_{12}E_{34}}{Q_{12}Q_{34}} = \frac{E_{13}E_{24}}{Q_{13}Q_{24}} = \frac{E_{14}E_{23}}{Q_{14}Q_{23}} = \frac{16V^2}{A_1A_2A_3A_4}. \]
Solid spreads

The four projective quadreas $S_1, S_2, S_3, S_4$ are the solid spreads of the tetrahedron. Recall

$$S_1 = E_{12}s_{1,3}s_{1,4} = E_{13}s_{1,2}s_{1,4} = E_{14}s_{1,2}s_{1,3}. $$

Solid spread theorem

$$S_1 = \frac{\mathcal{V}_{}}{4Q_{12}Q_{13}Q_{14}}. $$

Proof: Using the Dihedral spreads theorem, 

$$S_1 = E_{12}s_{1,3}s_{1,4} = \left( \frac{4Q_{12}\mathcal{V}}{A_3A_4} \right) s_{1,3}s_{1,4}$$

$$= \frac{4Q_{12}\mathcal{V}}{(4Q_{12}Q_{14}s_{1,3})(4Q_{12}Q_{13}s_{1,4})} s_{1,3}s_{1,4} = \frac{\mathcal{V}_{}}{4Q_{12}Q_{13}Q_{14}}. $$

As Corollaries:

$$\frac{S_1}{S_2} = \frac{Q_{23}Q_{24}}{Q_{13}Q_{14}} \quad \frac{S_1S_2}{S_3S_4} = \frac{Q_{34}^2}{Q_{12}^2}. $$
Solid spread law

As another corollary of $S_1 = \frac{\gamma}{4Q_{12}Q_{13}Q_{14}}$ we get

\[
\frac{S_1}{Q_{23}Q_{24}Q_{34}} = \frac{S_2}{Q_{13}Q_{14}Q_{34}} = \frac{S_3}{Q_{12}Q_{14}Q_{24}} = \frac{S_4}{Q_{12}Q_{13}Q_{23}} = \frac{\gamma}{4Q_{12}Q_{13}Q_{14}Q_{23}Q_{24}Q_{34}}.
\]

In addition to $E_{12} = \frac{4Q_{12}\gamma}{A_3A_4}$ we have also the shorter result

\[
1 - E_{12} = \left( \frac{Q_{12}^2 + 2Q_{12}Q_{34} + Q_{13}Q_{14} + Q_{23}Q_{24}}{-Q_{12}Q_{13} - Q_{12}Q_{14} - Q_{12}Q_{23} - Q_{12}Q_{24} - Q_{13}Q_{24} - Q_{14}Q_{23}} \right)^2.
\]
The right tetrahedron

The special case of a right tetrahedron, where say $S_1 = 1$, deserves attention, since it is the three dimensional analog of a right triangle.

**Right Pythagoras theorem** If the tetrahedron has solid spread $S_1 = 1$ then

$$A_1 = A_2 + A_3 + A_4.$$

**Right solid spreads** If the tetrahedron has solid spread $S_1 = 1$ then

$$(S_2 + S_3 + S_4 - 1)^2 = 4S_2S_3S_4.$$

This relation is precisely the *Triple cross formula*, i.e. the quantities $C_1 = 1 - S_1$, $C_2 = 1 - S_2$ and $C_3 = 1 - S_3$ satisfy the Triple spread formula!
Dihedral spread relation

If \( a \equiv 1 - E_{12} \quad b \equiv 1 - E_{13} \quad c \equiv 1 - E_{14} \)
and \( x \equiv 1 - E_{34} \quad y \equiv 1 - E_{24} \quad z \equiv 1 - E_{23} \) and

\[
\begin{align*}
t & \equiv - \frac{1}{2} (E_{12}E_{34} + E_{13}E_{24} + E_{14}E_{23}) - 1 \\
f & \equiv - \frac{1}{2} (E_{12}E_{34} + E_{13}E_{24} - E_{14}E_{23}) \\
g & \equiv - \frac{1}{2} (E_{12}E_{34} - E_{13}E_{24} + E_{14}E_{23}) \\
h & \equiv - \frac{1}{2} (-E_{12}E_{34} + E_{13}E_{24} + E_{14}E_{23}) \\
u & \equiv \frac{t^2 + abxy + bcyz + acxz - (abc + ayz + bxz + cxy)}{2} \\
v & \equiv \frac{u^2 - (abxyf^2 + acxzg^2 + bcyzh^2)}{2}
\end{align*}
\]

then

\[
v^2 = abcxyz (af^2g^2x + bf^2h^2y + cg^2h^2z + 2fghu)\].
Suppose that
\[ Q_{12} = 1 \quad Q_{13} = 2 \quad Q_{14} = 3 \quad Q_{23} = 2 \quad Q_{24} = 5 \quad Q_{34} = 4 \]

Then
\[ E_{12} = \frac{68}{77} \quad E_{13} = \frac{136}{161} \quad E_{14} = \frac{204}{253} \quad E_{23} = \frac{136}{217} \quad E_{24} = \frac{340}{341} \]

If \( E_{12} \) was unknown, it would satisfy the polynomial relation
\[
0 = (77E - 68)(28E - 17)(1713481E^2 - 3537072E + 6071312) \\
\quad \times (5929E^2 - 7623E + 2423)(5929E^2 - 2772E + 16724).
\]

So perhaps there are "linked tetrahedra" that share 5 dihedral spreads?
Our example tetrahedron

\[
\begin{align*}
Q_{12} &= \beta \tau \\
Q_{23} &= Q_{24} = \beta \\
Q_{13} &= 3\tau / 4 \\
Q_{14} &= 3\tau / 4 \\
Q_{34} &= 1 \\
\mathcal{V} &= \frac{9}{4} + \sqrt{5} \\
A_1 &= \frac{3}{2} + \frac{1}{2} \sqrt{5} \\
A_2 &= A_3 = A_4 = \frac{7}{2} + \frac{3}{2} \sqrt{5} \\
E_{12} &= E_{23} = E_{24} = \beta \\
E_{13} &= E_{14} = \frac{3}{4} \\
E_{34} &= 1 \\
S_1 &= \frac{2}{9} - \frac{2}{45} \sqrt{5} \\
S_2 &= \frac{2}{5} + \frac{2}{25} \sqrt{5} \\
S_3 = S_4 &= \frac{1}{3} + \frac{2}{15} \sqrt{5}
\end{align*}
\]
Theorem (Crelle 1821) The circumdiameter quadrance $D$ is

$$D = \frac{\mathcal{A}^*}{\mathcal{V}}$$

where $\mathcal{A}^* = A(Q_{12} Q_{34}, Q_{13} Q_{24}, Q_{14} Q_{23})$.

Example In our tetrahedron,

$$D = \frac{\mathcal{A}^*}{\mathcal{V}} = \left(\frac{45}{8} + \frac{5}{2}\sqrt{5}\right) / \mathcal{V} = \frac{5}{2}.$$
Some ZOME challenges

(Picture thanks to Alejandra Plaice).
The formulas in this talk are sufficient to study the 65 possible ZOME tetrahedra.

Each can now be analysed metrically.
A challenge: analyse metrically every ZOME tetrahedron.

A harder theoretical challenge: what further relations are there between these various quantities (and others)? What are the relations between the relations? What are the possibilities when we extend ZOME to the green struts?!
Two key examples

Here are the two essential triangles that are studied by all students, essentially the only ones that are accessible to elementary classical trigonometric, but now expressed using rational trigonometry. Note the absence of any transcendental quantities—these are fictions induced by a myopic and distorted view.

What are the three dimensional analogs??
The right isosceles tetrahedron

Suppose that we have a right tetrahedron with $Q_{12} = Q_{13} = Q_{14} = 1$ while $Q_{23} = Q_{24} = Q_{34} = 2$. Then the quadrume is $V = 4$. The quadreas are

$$A_1 = 12 \text{ and } A_2 = A_3 = A_4 = 4$$

The edge dihedral spreads are

$$E_{12} = E_{13} = E_{14} = 1 \text{ and } E_{23} = E_{34} = E_{24} = \frac{2}{3}.$$ 

The solid spreads are

$$S_1 = 1 \text{ and } S_2 = S_3 = S_4 = \frac{1}{4}.$$ 

Note that

$$\left(S_2 + S_3 + S_4 - 1\right)^2 = 4S_2S_3S_4.$$
The regular tetrahedron

Suppose that all quadrances 
\( Q_{ij} = 1 \). Then the quadreas of the faces are all 
\[ A = 3 \]
and the quadrume is 
\[ V = 2. \]
The faces spreads are all 
\[ s_{ij} = \frac{3}{4} \]
the edge spreads are all 
\[ E_{ij} = \frac{8}{9} \]
and the solid spreads are all 
\[ S_i = \frac{1}{2}. \]
Rational trigonometry allows us to \textit{metrically describe the regular tetrahedron}—in a way that I hope Pythagoras and Euclid would approve!

ARTICLES:
Universal Hyperbolic Geometry III: First steps in projective triangle geometry *KoG* (2011)

YOUTUBE VIDEOS (*njwildberger*):
- **WildTrig** (Rational Trigonometry)
- **UnivHypGeom** (Universal Hyperbolic Geometry)
- Hyperbolic geometry is projective relativistic geometry
- Triangle geometry old and new: An introduction to hyperbolic triangle geometry

THANK YOU!