Brownian Bridge and Principal Component Analysis: Towards Removing the Curse of Dimensionality

Xiaoqun Wang[1] and Ian H. Sloan[2]

[1] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China*

[2] *School of Mathematics, University of New South Wales, Sydney 2052, Australia*

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Abstract High-dimensional integrals occur in a variety of areas, including mathematical finance. In the classical settings, multivariate integration problems suffer from the curse of dimensionality. To vanquish the curse of dimensionality, one may shrink the function class. Here we use weighted function spaces, in which groups of variables are associated with weights, in order to capture the different importance of each group of variables. For practical applications, the principal difficulty then is in choosing the "right" weights for a given problem or class of problems. We work in weighted reproducing kernel Hilbert spaces (with "general" rather than "product" weights). We first present a principle to find the good weights for a given problem. This general approach is then applied to a simplified high-dimensional problem from finance, in which the dimension is the number of discrete time steps for a price of a risky asset which follows geometric Brownian motion. A second focus of this paper is on the dimension reduction techniques, such as the Brownian bridge (BB) and principal component analysis (PCA). It turns out that the behavior of the model problem is dramatically improved when QMC is used in conjunction with the dimension reduction techniques: if the "right" weights are used in every case, then the integration error can be bounded independently of the dimension, whereas without BB or PCA the error bound depends exponentially on the dimension. Finally, for this model problem we show how to construct shifted lattice rules which, when used in conjunction with BB or PCA, yield integration errors converging as $O(n^{-3/4+\delta})$ or $O(n^{-1+\delta})$ (for arbitrary small $\delta > 0$) respectively, independently of the dimension. Thus in both cases well-designed algorithms can avoid the curse of dimensionality, with PCA having an advantage over BB with respect to the proved order of convergence.

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Email addresses: xwang@math.tsinghua.edu.cn, sloan@maths.unsw.edu.au.

1 Introduction

Many problems in practice can be formulated as high-dimensional integrals of the form (after suitable transformations)

$$I_s(f) = \int_{[0,1]^s} f(\mathbf{x})\, d\mathbf{x}.$$

The dimensionality can be in the hundreds or even in the thousands in some applications, for instance, in finance (see Paskov & Traub, 1995). Classical numerical methods (e.g. product rules) suffer from an extreme form of the curse of dimensionality: the computational cost required to achieve a given accuracy increases exponentially with the dimension. Monte Carlo (MC) or quasi-Monte Carlo (QMC) methods can break the curse of dimensionality. Both MC and QMC estimates of $I_s(f)$ take the form

$$Q_{n,s}(f) := \frac{1}{n} \sum_{k=1}^{n} f(\mathbf{x}_k),$$

with the points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $[0,1]^s$ chosen randomly in MC or deterministically in QMC. Perhaps the most remarkable property of MC is that its convergence rate $O(n^{-1/2})$ is independent of the dimension. However, this convergence is very slow.

For QMC, the Koksma-Hlawka inequality (see Niederreiter, 1992) asserts that a QMC algorithm based on a low discrepancy sequence has a deterministic error bound $O(n^{-1}(\log n)^s)$ for functions of bounded variation in the sense of Hardy and Krause, which is asymptotically better than MC. However, the QMC error bound depends exponentially on s, and this causes problems for large s. Note, for example, that even for s not too large, say $s = 20$, we have $n^{-1}(\log n)^s > n^{-1/2}$ for $n = 10^{90}$. Nevertheless, empirical investigations demonstrate that QMC is significantly more efficient than MC for some applications in finance and a convergence order close to $O(n^{-1})$ is often observed in high dimensions, see Paskov & Traub (1995) and others. This raises a question: Is a well-designed QMC algorithm a better way to break the curse of dimensionality (at least for some classes of problems)? This question has attracted much recent attention; for a non-technical summary, see Kuo & Sloan (2005).

Clearly, the Koksma-Hlawka inequality cannot give a reasonable explanation for the success of QMC. A rough qualitative explanation for the success of QMC is that finance problems often have low effective dimension (see Caflisch *et al.*, 1997; Wang & Fang, 2003), a property that is favorable for QMC point sets (which typically have better uniformity in initial dimensions and better one-dimensional projections). The reason for the low effective dimension of some finance problems is explored in Wang & Sloan (2005).

Another way of explaining the success of QMC is to make use of the theory of *weighted function spaces*, where the importance of each group of variables is controlled by certain weights (see Sloan & Woźniakowski, 1998). A number of works have been devoted to studying the tractability and strong tractability of multivariate integration, as well as to constructing algorithms that achieve the corresponding error bounds (see Hickernell & Woźniakowski, 2000; Sloan *et al.*, 2002; Wang, 2003 and the references therein). However, in the theoretical studies the weighted spaces are artificial, in that the weights characterizing the spaces are supposed to be given in advance. Little is known how to match practical problems to the theoretical framework. The principal difficulty for practical applications is in choosing

appropriate weights. Another difficulty (which we shall also encounter here) is that many practical problems have integrands that do not belong to the usual weighted function spaces.

As a first result, we present an explicit formula for the optimal weight for each subset of the variables, for a particular choice of function space. (The space is the "unanchored" Sobolev space with "general" weights introduced in Dick *et al.*, 2006 and Sloan *et al.*, 2004). The optimality is in the sense of minimizing an error bound of the form: upper bound on the "average" worst-case error times the seminorm of f for a given function f (see Section 3). While the explicit computation of the optimal weights is feasible only in special cases, and fails if f does not belong to the particular space, nevertheless the knowledge of the optimal choice does give valuable insight into the vexing question of how best to choose the weights for a given practical problem. And as we shall see, it allows us to demonstrate that the best choice of weights can depend on the precise way a problem is formulated.

Techniques which may change the structure of finance problems have been developed, such as the Brownian bridge (BB, see Caflisch *et al.*, 1997; Moskowitz & Caflisch, 1996) and the principal component analysis (PCA, see Acworth *et al.*, 1998). The essential effect of BB and PCA is to change the integrand in such a way that the first few variables capture a large part of the total variance. It is shown in Caflisch *et al.* (1997), Wang (2006) and Wang & Fang (2003) that for some finance problems the effective dimension is reduced by BB or PCA, and that empirically the efficiency of QMC is improved significantly. A rough qualitative explanation is that BB or PCA allows the good quality of the initial coordinates and the one-dimensional projections of a QMC point set to be exploited. It is also shown in Sobol & Kucherenko (2005) on a simple model example that BB changes the integrand such that the additive part dominates.

However, most results on dimension reduction are empirical and qualitative. Little is known theoretically about the computational complexity or possible convergence order of QMC used in conjunction with BB or PCA. An attempt was made in Larcher *et al.* (2004), where it was shown non-constructively on a simple example that certain QMC algorithm with BB can achieve an error bound $O(n^{-1/2})$ that is independent of the dimension.

The main purpose of this paper is to extend the study of dimension reduction techniques in several aspects: we consider a model related to real finance problem; moreover, we study not only BB, but also PCA; and we consider how the "weights" in a weighted Sobolev space should best be chosen when using BB or PCA. We show how BB and PCA with appropriate choices of weights can change the tractability property of the problem and avoid the curse of dimensionality. We then show how to *construct* shifted lattice rules that achieve the corresponding error bounds. By using the constructive algorithm with the right weights, we show that a convergence rate $O(n^{-3/4+\delta})$ or even $O(n^{-1+\delta})$ (for arbitrary small $\delta > 0$) is achieved, in conjunction with BB or PCA respectively, independently of the dimension.

This paper is organized as follows. In Section 2 we introduce the weighted function space setting and the concept of tractability. In Section 3 we establish a principle for choosing the weights. Then in Section 4 we define a special function class of multiplicative functions, for which the analysis is especially easy. In Section 5 we introduce the model finance problem, which is multiplicative in nature, and study the BB and PCA techniques applied to the model finance problem. Finally, conclusions are presented in Section 6.

2 Errors, tractability and weighted function spaces

Let H_s be a Hilbert space of functions defined on $[0,1]^s$ with norm $||\cdot||_{H_s}$. Define the worst-case error of the QMC algorithm $Q_{n,s}(f)$ by

$$e(P_n; H_s) := \sup\{|I_s(f) - Q_{n,s}(f)| : f \in H_s, ||f||_{H_s} \le 1\},$$

where $P_n := \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is the set of QMC points for $Q_{n,s}$. For $n = 0$, we define the initial error as

$$e(0; H_s) = \sup\{|I_s(f)| : f \in H_s,\ ||f||_{H_s} \le 1\} = ||I_s||.$$

Let $n(\varepsilon, H_s)$ be the smallest n for which there exists a QMC algorithm $Q_{n,s}(f)$ such that $e(P_n; H_s) \le \varepsilon\, e(0; H_s)$, where $\varepsilon \in (0,1)$. The number $n(\varepsilon, H_s)$ is a function of both ε and s. Multivariate integration is *tractable* (or more accurately, *QMC tractable*) if there are nonnegative numbers C, p and q such that

$$n(\varepsilon, H_s) \le C\varepsilon^{-p} s^q \qquad \forall \varepsilon \in (0,1) \quad \text{and} \quad \forall s \ge 1. \tag{1}$$

Multivariate integration is *strongly tractable* if $q = 0$. In this case the infimum of all p satisfying the bound is called the exponent of strong tractability. We may also talk about tractability or strong tractability for subsets of H_s, see Section 4.

In this paper, we consider the reproducing kernel Hilbert spaces $H(K_{s,\gamma})$ with the following reproducing kernel

$$K_{s,\gamma}(\mathbf{x}, \mathbf{y}) := \sum_{u \subseteq \{1,\ldots,s\}} K^u(\mathbf{x}_u, \mathbf{y}_u), \tag{2}$$

where

$$K^u(\mathbf{x}_u, \mathbf{y}_u) := \gamma_{s,u} \prod_{j \in u} \left(\frac{1}{2} B_2(|x_j - y_j|) + (x_j - \frac{1}{2})(y_j - \frac{1}{2}) \right), \tag{3}$$

and $B_2(x) = x^2 - x + 1/6$ is the Bernoulli polynomial of degree 2, $\gamma := \{\gamma_{s,u}\}$ is a sequence of arbitrary positive numbers (with the convention $\gamma_{s,\emptyset} = 1$ and $K^\emptyset = 1$). The number $\gamma_{s,u}$ is a "weight" that controls the importance of the group of variables indexed by the subset u. One can take $\gamma_{s,u} = 0$ as the limiting case of positive $\gamma_{s,u}$'s. The particular space described here (sometimes referred to as "unanchored") has some theoretical advantages over the original space used in Sloan & Woźniakowski (1998).

An important special case is when the weights $\gamma_{s,u}$ are of the product form

$$\gamma_{s,u} = \prod_{j \in u} \gamma_{s,j} \quad \text{for} \quad \text{some} \quad \{\gamma_{s,j}\}, \quad \emptyset \ne u \subseteq \{1, \ldots, s\}, . \tag{4}$$

In this case the kernel (2) can be written as

$$K_{s,\gamma}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{s} K_{\gamma_{s,j}}(x_j, y_j), \tag{5}$$

where

$$K_{\gamma_{s,j}}(x, y) = 1 + \gamma_{s,j} \left(\frac{1}{2} B_2(|x - y|) + (x - \frac{1}{2})(y - \frac{1}{2}) \right), \tag{6}$$

and the corresponding Hilbert space $H(K_{s,\gamma})$ is then a tensor-product space

$$H(K_{s,\gamma}) = H_{\gamma_{s,1}} \otimes H_{\gamma_{s,2}} \otimes \cdots \otimes H_{\gamma_{s,s}},$$

where the spaces $H_{\gamma_{s,j}}$ are Sobolev spaces of absolutely continuous real functions defined on $[0,1]$ whose first derivatives belong to $L_2([0,1])$. The reproducing kernel of $H_{\gamma_{s,j}}$ is $K_{\gamma_{s,j}}(x,y)$ given in (6), and the inner product in $H_{\gamma_{s,j}}$ is given by

$$(f,g) = \int_0^1 f(x)dx \int_0^1 g(x)dx + \frac{1}{\gamma_{s,j}} \int_0^1 f'(x)g'(x)dx.$$

This and other possible tensor-product reproducing kernel Hilbert spaces are discussed in Sloan & Woźniakowski (2002).

Now we return to the kernel (2) with the general weights. The square of the norm in the space $H(K_{s,\gamma})$ is given by

$$||f||^2_{H(K_{s,\gamma})} = \sum_{u \subseteq \{1,\ldots,s\}} \gamma_{s,u}^{-1} \int_{[0,1]^{|u|}} \left(\int_{[0,1]^{s-|u|}} \frac{\partial^{|u|} f}{\partial \mathbf{x}_u} d\mathbf{x}_{-u} \right)^2 d\mathbf{x}_u, \tag{7}$$

where for a subset $u \subseteq \{1,\ldots,s\}$, $|u|$ denotes its cardinality, the symbol $\mathbf{x}_u$ denotes the $|u|$-dimensional vector of components x_j with $j \in u$, and $\mathbf{x}_{-u}$ denotes the vector $\mathbf{x}_{\{1,\ldots,s\}-u}$. Due to the linearity of $I_s(f) - Q_{n,s}(f)$, we have the error bound

$$|I_s(f) - Q_{n,s}(f)| \le e(P_n; H(K_{s,\gamma}))\, ||f||_{H(K_{s,\gamma})} \qquad \forall f \in H(K_{s,\gamma}). \tag{8}$$

Note that the initial error in the space $H(K_{s,\gamma})$ is $e(0; H(K_{s,\gamma})) = 1$, and it is known, see Sloan *et al.* (2004), that the square worst-case error is given by

$$e^2(P_n; H(K_{s,\gamma})) = -1 + \frac{1}{n^2} \sum_{i,j=1}^{n} K_{s,\gamma}(\mathbf{x}_i, \mathbf{x}_j).$$

The next theorem is proved in Sloan *et al.* (2004), Theorem 1 (B), by an averaging argument.

Theorem 1 *Let $H(K_{s,\gamma})$ be the reproducing kernel Hilbert space with the reproducing kernel (2). There exists a QMC algorithm for which*

$$e(P_n; H(K_{s,\gamma})) \le \frac{1}{\sqrt{n}} \left(\sum_{\emptyset \ne u \subseteq \{1,\ldots,s\}} \gamma_{s,u} 6^{-|u|} \right)^{1/2}.$$

Therefore, if

$$\sup_{s=1,2,\ldots} \left(\sum_{\emptyset \ne u \subseteq \{1,\ldots,s\}} \gamma_{s,u} 6^{-|u|} \right) < \infty,$$

then the multivariate integration problem in $H(K_{s,\gamma})$ is strongly tractable, with the ε-exponent at most 2.

3 Choosing the weights

It follows from Theorem 1 and (8) that there exists a QMC algorithm for which, for all $f \in H(K_{s,\gamma})$,

$$|I_s(f) - Q_{n,s}(f)| \le \frac{1}{\sqrt{n}} \left(\sum_{\emptyset \ne u \subseteq \{1,\ldots,s\}} \gamma_{s,u} 6^{-|u|} \right)^{1/2} \left(\sum_{\emptyset \ne u \subseteq \{1,\ldots,s\}} \gamma_{s,u}^{-1} \, w_u^2(f) \right)^{1/2}, \tag{9}$$

where

$$w_u^2(f) := \int_{[0,1]^{|u|}} \left(\int_{[0,1]^{s-|u|}} \frac{\partial^{|u|} f}{\partial \mathbf{x}_u} d\mathbf{x}_{-u} \right)^2 d\mathbf{x}_u. \tag{10}$$

Note that in the last factor of (9) we have replaced the norm of f by the seminorm obtained by omitting the $u = \emptyset$ term in (7). That this is valid follows from applying the bound (8) to $f - I_s(f)$. Note also that $f \in H(K_{s,\gamma})$ only if $w_u(f) < \infty$ for all $u \subseteq \{1, \ldots, s\}$, a condition that is the same for all choices of weights $\{\gamma_{s,u}\}$.

For a given $f \in H(K_{s,\gamma})$, the right-hand side of the inequality (9) depends on the weights $\gamma_{s,u}$: smaller weights lead to a smaller sum $\sum_{\emptyset \ne u \subseteq \{1,\ldots,s\}} \gamma_{s,u} 6^{-|u|}$, but lead to a larger seminorm of f. A natural idea is to choose suitable weights to minimize the error bound (9), following the ideas of Dick *et al.* (2004) and Larcher *et al.* (2004). A major difference is that in our case the optimization problem turns out to be much simpler.

The following theorem gives a simple answer to the question of how to choose the weights $\gamma_{s,u}$ for a given s and a given function f on $[0,1]^s$ that satisfies $w_u(f) < \infty$ for all $u \subseteq \{1, \ldots, s\}$.

Theorem 2 *Let $H(K_{s,\gamma})$ be the reproducing kernel Hilbert space with the reproducing kernel (2). The error bound given by the right-hand side of (9) (which is the product of the QMC mean of the worst-case error in $H(K_{s,\gamma})$ from Theorem 1, and the seminorm of f in $H(K_{s,\gamma})$) is minimized by*

$$\gamma_{s,u} = c\, 6^{|u|/2} w_u(f), \quad \emptyset \ne u \subseteq \{1, \ldots, s\}, \tag{11}$$

where $c > 0$ is an arbitrary constant independent of u. Thus there exists a QMC algorithm $Q_{n,s}$ for which

$$|I_s(f) - Q_{n,s}(f)| \le \frac{1}{\sqrt{n}} \sum_{\emptyset \ne u \subseteq \{1,\ldots,s\}} 6^{-|u|/2} \, w_u(f). \tag{12}$$

Proof The proof of (11) follows immediately from equating to zero the partial derivatives of the square of the right-hand side of (9) with respect to $\gamma_{s,u}$, $\emptyset \ne u \subseteq \{1, \ldots, s\}$. The bound (12) follows by substitution of (11) in (9). □

Remark The simplicity of the present optimization argument is a (possibly surprising) consequence of the fact that we are using here the so-called "general weights" introduced by Hickernell and Wozniakowski (2000) (see also Sloan *et al.* 2004), in which a separate weight $\gamma_{s,u}$ is prescribed for each subset of the variables. In contrast, the papers of Dick *et al.* (2004) and Larcher *et al.* (2004) assumed the weights to be of the product form (4), making the resulting system of equations for the stationary points nonlinear.

Remark The occurrence of the arbitrary constant c in (11) is natural, in that the bound on the right-hand side of (9) is clearly unchanged if each weight $\gamma_{s,u}$ is multiplied by the same constant. Later (see Section 4) we shall exploit the freedom in the choice of the constant by giving to c the (unique) value that in a particular application makes $\gamma_{s,u}$ to be of the product form (4).

The formula (11) for the weights indicates that the "optimal" weights depend on the function at hand. Roughly speaking, it is clear from (11) and (10) that if $|\partial^{|u|} f/\partial \mathbf{x}_u|$ is small, then the weights $\gamma_{s,u}$ should be small, which is consistent with our intuition. The theoretical importance of the choice of weights (11) will become clear in Section 5.

The purpose of choosing good weights is twofold. First, for a given function and a given QMC algorithm that is as good as average, the best error bound for integration is obtained if we use the weights given by (11). Second, this knowledge allows us to choose suitable weights in the constructive algorithms (in particular, in the construction of good lattice rules, see Section 5.4). The importance of choosing suitable product weights in the construction of good lattice rules is stressed in Wang & Sloan (2006).

4 A special function class

Consider functions of the product form

$$f(\mathbf{x}) = \prod_{j=1}^{s} g_{s,j}(x_j), \tag{13}$$

satisfying

$$W_{s,j}(f) := W_{s,j} := \left(\int_0^1 |g'_{s,j}(x)|^2 dx \right)^{1/2} < \infty. \tag{14}$$

(The quantity $W_{s,j}(f)$ depends on f. We may write it as $W_{s,j}$ if the underlying function is clear.) For simplicity, we assume that $\int_0^1 g_{s,j}(x)dx = 1$ (in the general case, a normalization procedure can be used, see Section 5.3). Define

$$\mathcal{F}_s := \Big\{ f : f(\mathbf{x}) = \textstyle\prod_{j=1}^{s} g_{s,j}(x_j),\ g_{s,j} \text{ is absolutely continuous on } [0,1],$$
$$\int_0^1 g_{s,j}(x)dx = 1,\ W_{s,j}(f) < \infty \Big\},\ s = 1, 2, \ldots,$$

so that $\mathcal{F}_s$ is a class of multiplicative, continuous functions.

Consider the reproducing kernel Hilbert space $H(K_{s,\gamma})$ with the kernel (2), where the weights $\gamma_{s,u}$ will be specified below. For a function $f \in \mathcal{F}_s$, since

$$\frac{\partial^{|u|} f(\mathbf{x})}{\partial \mathbf{x}_u} = \prod_{j \in u} g'_{s,j}(x_j) \prod_{i \notin u} g_{s,i}(x_i),$$

we have

$$w_u^2(f) = \int_{[0,1]^{|u|}} \left(\int_{[0,1]^{s-|u|}} \frac{\partial^{|u|} f}{\partial \mathbf{x}_u} d\mathbf{x}_{-u} \right)^2 d\mathbf{x}_u = \prod_{j \in u} \int_0^1 |g'_{s,j}(x_j)|^2 dx_j = \prod_{j \in u} W_{s,j}^2, \tag{15}$$

where we used $\int_0^1 g_{s,j}(x)dx = 1$. Therefore, from (11) the optimal weights (which minimize the error bound) are given by

$$\gamma_{s,u} = 6^{|u|/2} w_u(f) = 6^{|u|/2} \prod_{j\in u} W_{s,j}, \quad \emptyset \neq u \subseteq \{1,\ldots,s\},$$

where we have made the specific choice $c = 1$ for the arbitrary constant in (11). This special choice of c (and no other choice) makes the weights of the product form (4), with

$$\gamma_{s,j} := \sqrt{6}\, W_{s,j} \quad \text{for} \quad j = 1,\ldots,s.$$

The functions f given in (13) belong to the Sobolev space $H(K_{s,\gamma})$. Therefore, from (12) we have

$$\begin{aligned} |I_s(f) - Q_{n,s}(f)| &\leq \frac{1}{\sqrt{n}} \sum_{\emptyset\neq u\subseteq\{1,\ldots,s\}} \prod_{j\in u} 6^{-1/2} W_{s,j} = \frac{1}{\sqrt{n}} \left(\prod_{j=1}^{s} \left(1 + \frac{W_{s,j}}{\sqrt{6}}\right) - 1 \right) \\ &\leq \frac{1}{\sqrt{n}} \left(\exp\left(\frac{\sum_{j=1}^{s} W_{s,j}}{\sqrt{6}} \right) - 1 \right). \end{aligned} \tag{16}$$

For a given $\varepsilon > 0$, let $n(\varepsilon, \mathcal{F}_s)$ be the minimal number of points n such that there exists a QMC algorithm for which

$$|I_s(f) - Q_{n,s}(f)| \leq \varepsilon \qquad \forall f \in \mathcal{F}_s.$$

We may define tractability or strong tractability on the function class $\mathcal{F}_s$ in the same way as in (1) and the text that follows it. Let us define

$$\mathcal{F}_s(M) := \{f \in \mathcal{F}_s : \sum_{j=1}^{s} W_{s,j}(f) \leq M\}, \quad s = 1, 2, \ldots.$$

From the error bound just obtained we have

$$n(\varepsilon, \mathcal{F}_s(M)) \leq \frac{\exp\left(\frac{\sqrt{6}}{3} M\right)}{\varepsilon^2} \approx \frac{2.26256^M}{\varepsilon^2}.$$

Thus, the complexity of the integration problem in the class $\mathcal{F}_s(M)$ depends on the quantity M that bounds $\sum_{j=1}^{s} W_{s,j}$. If M is a finite number, then the number $n(\varepsilon, \mathcal{F}_s(M))$ can be bounded by a polynomial in ε^{-1} independently of s. Summarizing these results, we have the following theorem, which will be useful in next section.

Theorem 3 *For a function $f \in \mathcal{F}_s, s \in \mathbb{N}$, there exists a QMC algorithm such that*

$$|I_s(f) - Q_{n,s}(f)| \leq \frac{1}{\sqrt{n}} \left(\exp\left(\sum_{j=1}^{s} W_{s,j}(f)/\sqrt{6} \right) - 1 \right).$$

For $M < \infty$, the minimal number $n(\varepsilon, \mathcal{F}_s(M))$ of function evaluations to guarantee an integration error not exceeding $\varepsilon > 0$ in the class $\mathcal{F}_s(M)$ is bounded by

$$n(\varepsilon, \mathcal{F}_s(M)) \leq 2.26256^M \varepsilon^{-2},$$

independently of the dimension s. That is, the integration problem in the function classes $\mathcal{F}_s(M)$ is strongly tractable with exponent at most 2.

5 Model finance problem and dimension reduction

The main focus of the rest of this paper is on a model finance problem and dimension reduction techniques, such as BB and PCA, which are often used in QMC to reduce the effective dimension. We are interested in how BB or PCA change the theoretical error bound, the tractability property of the problems and the theoretical convergence order of QMC algorithms.

5.1 The model finance problem

Mathematical finance is a rich source of high-dimensional integration problems. In this subsection we describe a model finance problem related to geometric Brownian motion, which is of the multiplicative form assumed in Section 4. Then in the following subsections we discuss three path generation methods, and study the strong tractability of the model finance problem non-constructively. And finally, we study strong tractability and convergence order *constructively*.

We assume that the risky asset S_t follows (under the risk-neutral measure) geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{17}$$

where r is the risk-free interest rate, σ is the volatility, and B_t is the standard Brownian motion. Let

$$\mathcal{G}(S_{t_1}, \ldots, S_{t_s}) := \prod_{k=1}^{s} S_{t_k}^{1/s}, \tag{18}$$

be the geometric average of the asset's prices at the times $t_1, \ldots, t_s$, with $0 \le t_1 \le \ldots \le t_s = T$. We are interested in the expectation

$$\mathbb{E}\left[\mathcal{G}(S_{t_1}, \ldots, S_{t_s})\right], \tag{19}$$

where $\mathbb{E}[\cdot]$ is the expectation under the risk-neutral measure. Formally, this is a geometric average option (see Glasserman, 2004) with strike price $K = 0$. By definition, this expectation is an s-dimensional integral. As is well known, in this special case the integral can be transformed to a one-dimensional integral and then evaluated analytically. However, we do not do this here because we want to examine the influence of different dimension reduction techniques on the tractability property and the importance of the choice of weights in constructing good lattice rules.

To calculate the expectation (19) by simulation, we need to simulate the asset trajectories. It follows from (17) that

$$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma B_t\right). \tag{20}$$

Thus simulating asset prices reduces to simulating the Brownian motion path $B_{t_1}, \ldots, B_{t_s}$.

For simplicity, suppose that the asset prices are sampled at equally spaced times $t_j = j\Delta t, j = 1, \ldots, s$ with $\Delta t = T/s$. Note that the random vector $(B_{t_1}, \ldots, B_{t_s})^T$ is normally distributed with mean zero-vector and covariance matrix $\mathbf{V}$ given by

$$\mathbf{V} = (\min(t_i, t_j))_{i,j=1}^{s}.$$

Let $A = (a_{ij})_{s\times s}$ be a matrix satisfying $AA^T = \mathbf{V}$. Then the Brownian motion can be generated by

$$(B_{t_1}, \ldots, B_{t_s})^T = A\,(z_1, \ldots, z_s)^T, \tag{21}$$

where $z_j \sim N(0,1)$ are independent and identically distributed standard normal variables. Note that

$$\prod_{k=1}^{s} S_{t_k}^{1/s} = \exp\left(\frac{1}{s}\sum_{k=1}^{s}\log S_{t_k}\right) = \exp\left(m + \frac{\sigma}{s}\sum_{k=1}^{s} B_{t_k}\right) = \exp\left(m + \frac{\sigma}{s}\sum_{j=1}^{s} A_j z_j\right),$$

where

$$m = \log S_0 + \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)(T + T/s), \quad \text{and} \quad A_j = \sum_{k=1}^{s} a_{kj}. \tag{22}$$

Thus the expectation (19) can be written as

$$\mathbb{E}\left[\mathcal{G}(S_{t_1}, \ldots, S_{t_s})\right] = \int_{\mathbb{R}^s} \exp\left(m + \sum_{j=1}^{s}\Gamma_{s,j}\, z_j\right) p_s(z_1, \ldots, z_s)d\mathbf{z},$$

where

$$p_s(z_1, \ldots, z_s) = \frac{1}{(2\pi)^{s/2}}\exp\left(-\frac{1}{2}\sum_{j=1}^{s} z_j^2\right),$$

and the parameters $\Gamma_{s,j}$ are given by

$$\Gamma_{s,j} = \frac{\sigma}{s}\,A_j, \quad j = 1, \ldots, s. \tag{23}$$

This integral can be changed into a "standard" form (over the unit cube $[0,1]^s$):

$$\mathbb{E}\left[\mathcal{G}(S_{t_1}, \ldots, S_{t_s})\right] = \int_{[0,1]^s} \exp\left(m + \sum_{j=1}^{s}\Gamma_{s,j}\,\Phi^{-1}(x_j)\right) d\mathbf{x}, \tag{24}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$ and $\Phi^{-1}(\cdot)$ is its inverse. The resulting integrand is naturally weighted, in the sense that it is controlled by the parameters $\Gamma_{s,j}$, which in turn depend on the generating matrix A and the dimension s. The dimensionality is the number of time steps, which can in practice be quite large. The integrand is a multiplicative function, as in Section 4.

5.2 The BB and PCA constructions

In theory, any matrix A satisfying $AA^T = \mathbf{V}$ can be used as the generating matrix in (21), but it is well known that the resulting integrands can have quite different dimension structure, thus the efficiency of QMC algorithms may differ widely for different choices of A (but MC algorithms with or without dimension reduction are equivalent). We will consider three constructions of the Brownian motion, with our focus on the associated generating matrices A and the parameters $\Gamma_{s,j}$ characterizing the weighted property.

- The **standard construction** generates the Brownian motion sequentially: given $B_0 = 0$,

$$B_{t_j} = B_{t_{j-1}} + \sqrt{\Delta t}\, z_j,\ z_j \sim N(0,1),\ j = 1, \ldots, s.$$

This method takes $O(s)$ operations to generate a path. The corresponding generating matrix A in (21) is the *Cholesky decomposition* of the covariance matrix $\mathbf{V}$, which takes the form

$$A = A^{\mathrm{STD}} := \sqrt{\Delta t} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

- **The BB construction** (see Caflisch *et al.*, 1997; Moskowitz and Caflisch, 1996). Given a past value B_{t_i} and a future value B_{t_k}, the value B_{t_j} (with $t_i < t_j < t_k$) can be simulated according to the BB formula:

$$B_{t_j} = (1-\rho)B_{t_i} + \rho B_{t_k} + \sqrt{\rho(1-\rho)(k-i)\Delta t}\, z, \quad z \sim N(0,1),$$

where $\rho = (j-i)/(k-i)$. Assume the number of time steps $s = 2^\ell$ (ℓ is a non-negative integer). Given $B_0 = 0$, the Brownian motion is generated at times in order $T, T/2, T/4, 3T/4, \ldots$

$$\begin{aligned} B_T &= \sqrt{T}\, z_1; \\ B_{T/2} &= \frac{1}{2}(B_0 + B_T) + \sqrt{\frac{T}{4}}\, z_2 = \frac{\sqrt{T}}{2}\, z_1 + \frac{\sqrt{T}}{2}\, z_2; \\ B_{T/4} &= \frac{1}{2}(B_0 + B_{T/2}) + \sqrt{\frac{T}{8}}\, z_3 = \frac{\sqrt{T}}{4}\, z_1 + \frac{\sqrt{T}}{4}\, z_2 + \sqrt{\frac{T}{8}}\, z_3; \\ B_{3T/4} &= \frac{1}{2}(B_{T/2} + B_T) + \sqrt{\frac{T}{8}}\, z_4 = \frac{3\sqrt{T}}{4}\, z_1 + \frac{\sqrt{T}}{4}\, z_2 + \sqrt{\frac{T}{8}}\, z_4; \\ &\vdots \end{aligned}$$

The BB construction takes $O(s)$ operations to generate a path. The generating matrix A^{BB} associated with the BB construction can also be obtained explicitly, see Wang (2006).

- **The construction based on PCA** (see Acworth *et al.*, 1998) takes the generating matrix to be

$$A^{\mathrm{PCA}} = W D^{1/2},$$

where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ is a diagonal matrix of eigenvalues of the covariance matrix $\mathbf{V}$ arranged in a non-increasing order and the columns of W are the corresponding eigenvectors of unit-length. The PCA takes $O(s^3)$ operations to generate a path.

The following results are taken from Wang (2006), which give the explicit expressions for the parameters $\Gamma_{s,j}$ involved in the integrand in (24) under three path generations.

Theorem 4 *Assume that $t_j = j\Delta t,\ j = 1, \ldots, s$, with $\Delta t = T/s$. Let $\Gamma_{s,j}^{\mathrm{STD}}, \Gamma_{s,j}^{\mathrm{BB}}$ and $\Gamma_{s,j}^{\mathrm{PCA}}$ be the parameters in (24) associated with the standard, BB and PCA constructions of the Brownian motion, respectively.*

- *For the standard construction we have*

$$\Gamma_{s,j}^{\mathrm{STD}} = \frac{(s-j+1)}{s\sqrt{s}}\sigma\sqrt{T}, \quad j = 1, \ldots, s.$$

- *For the BB construction we have for $s = 2^\ell$ that*

$$\Gamma_{s,j}^{\mathrm{BB}} = \begin{cases} \frac{1}{2}(1+2^{-\ell})\sigma\sqrt{T}, & \text{if } j = 1, \\ 2^{-(3q+1)/2}\sigma\sqrt{T}, & \text{if } j = 2^{q-1}+1, \ldots, 2^q; \quad q = 1, \ldots, \ell. \end{cases}$$

- *For the PCA construction we have*

$$\Gamma_{s,j}^{\mathrm{PCA}} = \frac{\sigma\sqrt{T}}{2s\sqrt{s(2s+1)}}\frac{\cos h_j}{\sin^2 h_j} \quad \text{with} \quad h_j = \frac{(2j-1)\pi}{2(2s+1)}, \quad j = 1, \ldots, s.$$

This theorem indicates that the parameters $\Gamma_{s,j}$ associated with different constructions have quite different behaviors (for a given dimension s): in the standard construction the parameter $\Gamma_{s,j}^{\mathrm{STD}}$ decreases slowly as j increases, while in BB construction, $\Gamma_{s,j}^{\mathrm{BB}}$ decreases very fast, and in PCA construction $\Gamma_{s,j}^{\mathrm{PCA}}$ decreases even faster. For a simple illustration, we set $s = 4$ and $\sigma = 1, T = 1$, for the three constructions we have

$$\begin{array}{llll} \Gamma_{4,1}^{\mathrm{STD}} = 1/2, & \Gamma_{4,2}^{\mathrm{STD}} = 3/8, & \Gamma_{4,3}^{\mathrm{STD}} = 1/4, & \Gamma_{4,4}^{\mathrm{STD}} = 1/4; \\ \Gamma_{4,1}^{\mathrm{BB}} = 5/8, & \Gamma_{4,2}^{\mathrm{BB}} = 1/4, & \Gamma_{4,3}^{\mathrm{BB}} = \sqrt{2}/16, & \Gamma_{4,4}^{\mathrm{BB}} = \sqrt{2}/16; \\ \Gamma_{4,1}^{\mathrm{PCA}} = 0.6804, & \Gamma_{4,2}^{\mathrm{PCA}} = 0.0072, & \Gamma_{4,3}^{\mathrm{PCA}} = 0.0023, & \Gamma_{4,4}^{\mathrm{PCA}} = 0.0008. \end{array}$$

It is shown in Wang (2006) how the parameters $\Gamma_{s,j}$ affect the effective dimension and reflect the importance of each variable. Since these parameters reflect the different weighted properties of the underlying functions, they should be useful in choosing suitable weights. The explicit expressions for the weights in Theorem 4 are important in the following study. They open the way to proving theoretically the tractability property and the potential convergence orders of the underlying problems.

Note that for all three constructions the parameters $\Gamma_{s,j}$ are positive and are *uniformly bounded*: $\Gamma_{s,j} \le \Gamma$ for all s and j, where Γ is a constant. In fact, based on the explicit expressions for the parameters $\Gamma_{s,j}$ in Theorem 4, it is straightforward to show that

$$\Gamma_{s,j}^{\mathrm{STD}} \le \sigma\sqrt{T}, \quad j = 1, \ldots, s;\ s = 1, 2, ...;$$

$$\Gamma_{s,j}^{\mathrm{BB}} \le \sigma\sqrt{T}, \quad j = 1, \ldots, s;\ s = 2^\ell, \ell = 1, 2, ...;$$

$$\Gamma_{s,j}^{\mathrm{PCA}} \le \frac{3\sqrt{3}}{\pi}\sigma\sqrt{T}; \quad j = 1, \ldots, s;\ s = 1, 2, ...$$

The fact that these parameters are uniformly bounded (with respect to j and s) is important in the analysis below.

5.3 Strong tractability property of dimension reduction algorithms

Intuitively, BB and PCA aggregate a large number of the original variables in a small number of important variables, and thus may change the problem to be more tractable.

We have seen in Section 5.1 that the problem (17), (18), (19) leads to the integration of $f(\mathbf{x})$ over the unit cube $[0,1]^s$, where $f(\mathbf{x})$ has the form (see (24))

$$f(\mathbf{x}) = e^m \prod_{j=1}^{s} g_{s,j}(x_j) \quad \text{with} \quad g_{s,j}(x_j) = \exp\left(\Gamma_{s,j}\Phi^{-1}(x_j)\right).$$

The parameters $\Gamma_{s,j}$ are given in Theorem 4 for three constructions. Note that from (22) it follows that e^m is uniformly bounded with respect to s (and is independent of $\mathbf{x}$ and the method of path generation), so it is ignored below in the study of tractability. We are interested in how the integration error of $f(\mathbf{x})$ depends on the dimension s and on the number of points n.

We note that $I_s(f)$ is bounded uniformly in s (ignoring the e^m factor). In fact,

$$I_s(f) = \prod_{j=1}^{s} \int_0^1 \exp(\Gamma_{s,j}\Phi^{-1}(x))dx = \prod_{j=1}^{s} \exp\left(\frac{1}{2}\Gamma_{s,j}^2\right) = \exp\left(\frac{1}{2}\sum_{j=1}^{s}\Gamma_{s,j}^2\right).$$

Moreover, for any construction we have $\lim_{s\to\infty}\sum_{j=1}^{s}\Gamma_{s,j}^2 = \sigma^2 T/3$ (see Wang, 2006).

For the function $f(\mathbf{x})$ above, ignoring the e^m factor, we have (see (14))

$$\begin{aligned} W_{s,j}^2(f) := \int_0^1 |g'_{s,j}(x)|^2 dx &\geq \Gamma_{s,j}^2 \int_{\frac{1}{2}}^1 \exp(2\Gamma_{s,j}\Phi^{-1}(x))\,|(\Phi^{-1})'(x)|^2 dx \\ &\geq \Gamma_{s,j}^2 \int_{\frac{1}{2}}^1 |(\Phi^{-1})'(x)|^2\, dx \;=\; \infty. \end{aligned}$$

Here we use the fact that $\exp(2\Gamma_{s,j}\Phi^{-1}(x)) \geq 1$, since $\Gamma_{s,j} > 0$ for the three constructions. From (7) and based on the discussions in Section 4, we have

$$||f||_{H(K_{s,\gamma})}^2 = \sum_{u\subseteq\{1,\ldots,s\}} \gamma_{s,u}^{-1} \prod_{j\in u} W_{s,j}^2(f).$$

Thus $||f||_{H(K_{s,\gamma})} = \infty$ for an arbitrary sequence of weights $\gamma = \{\gamma_{s,u}\}$. Therefore, the function $f(\mathbf{x})$ cannot belong to the Sobolev space $H(K_{s,\gamma})$. The trouble is caused by the function Φ^{-1}. Of course the difficulty is not special to our particular problem: it is well known (see Sloan, 2002) that the process of transformation to the unit cube by the transformation $z_j = \Phi^{-1}(x_j)$ in (24) places most integrands arising in financial mathematics outside the commonly used reproducing kernel Hilbert spaces.

Since our purpose is mainly to study the impact of dimension reduction techniques, we choose, similarly as in Larcher *et al.* (2004) and Sloan (2002), to replace Φ^{-1} by some more convenient function $\Psi : [0,1] \to \mathbf{R}$, which is in some sense "close" to Φ^{-1}. Here we define

$$\Psi(x) = \Psi_\tau(x) := \begin{cases} \Phi^{-1}(\tau), & \text{if } x \in [0,\tau), \\ \Phi^{-1}(x), & \text{if } x \in [\tau, 1-\tau], \\ \Phi^{-1}(1-\tau), & \text{if } x \in (1-\tau, 1], \end{cases} \tag{25}$$

for some parameter $\tau \in (0, 1/2)$. Clearly, $\Psi(x)$ is bounded, $|\Psi(x)| \le \Phi^{-1}(\tau)$ and $\int_0^1 |\Psi'(x)|^2 dx < \infty$; moreover, $|\Psi(x)| \le |\Phi^{-1}(x)|$ for $x \in [0, 1]$. Note that if τ is close to 0, then the approximation of $\Psi(x)$ to $\Phi^{-1}(x)$ is good, but the bound on $\Psi(x)$ becomes large. Here we take τ to be a fixed parameter. The introduction of this admittedly artificial parameter is forced upon us because here we cannot (as is often done in theoretical settings) simply ignore the fact that the functions from the finance applications are not in the function space.

Let

$$\mu_{s,j} := \int_0^1 \exp(\Gamma_{s,j}\Psi(x))dx, \quad j = 1, \ldots, s.$$

Instead of considering the original function $f(\mathbf{x})$, we consider a closely related function $\widetilde{f}(\mathbf{x})$:

$$\widetilde{f}(\mathbf{x}) := \prod_{j=1}^{s} \mu_{s,j}^{-1} \exp(\Gamma_{s,j}\Psi(x_j)) = \prod_{j=1}^{s} \widetilde{g}_{s,j}(x_j) \tag{26}$$

where

$$\widetilde{g}_{s,j}(x) = \mu_{s,j}^{-1} \exp(\Gamma_{s,j}\Psi(x)). \tag{27}$$

It is obvious that $\int_0^1 \widetilde{g}_{s,j}(x)dx = 1$ (thus $I_s(\widetilde{f}) = 1$). The function $\widetilde{f}$, like the function f, depends on the path generation of the Brownian motion, since the parameters $\Gamma_{s,j}$ do.

Given a path generation method (standard, BB or PCA), we define a function class with a single element

$$\widetilde{\mathcal{F}}_s = \left\{ \widetilde{f}(\mathbf{x}) \mid \widetilde{f}(\mathbf{x}) = \prod_{j=1}^{s} \widetilde{g}_{s,j}(x_j) \right\}, \quad s = 1, 2, \ldots, \tag{28}$$

with $\widetilde{g}_{s,j}$ given by (27). We may write $\widetilde{\mathcal{F}}_s^{\mathrm{BB}}$ or $\widetilde{\mathcal{F}}_s^{\mathrm{PCA}}$ to stress the dependence on the path generation BB or PCA. In the case of BB, for s not being a power of 2, define $\widetilde{\mathcal{F}}_s^{\mathrm{BB}} = \widetilde{\mathcal{F}}_{2^k}^{\mathrm{BB}}$ for $s = 2^k + 1, \ldots, 2^{k+1} - 1, k = 1, 2, \ldots$.

For a given path generation method, we may estimate the associated quantity $W_{s,j}(\widetilde{f})$ using (14). We have from (27) that

$$W_{s,j}(\widetilde{f}) := W_{s,j} := \left(\int_0^1 |\widetilde{g}'_{s,j}(x)|^2 dx \right)^{1/2} = \frac{\Gamma_{s,j}}{\mu_{s,j}} \left(\int_0^1 \exp(2\Gamma_{s,j}\Psi(x)) \, |\Psi'(x)|^2 dx \right)^{1/2}. \tag{29}$$

We stress that the quantity $W_{s,j}(\widetilde{f})$ is dependent on the path generation through the parameter $\Gamma_{s,j}$. Since $\Psi(x)$ is bounded, and the parameters $\Gamma_{s,j}$ are *uniformly* bounded (for all j and s), it follows that $\exp(2\Gamma_{s,j}\Psi(x))$ is bounded. Moreover, since $\int_0^1 |\Psi'(x)|^2 dx < \infty$ and $\mu_{s,j} \ge 1$, we have

$$W_{s,j} \le C_1 \, \Gamma_{s,j} < \infty, \tag{30}$$

for some constant C_1 (depending on the path generation).

We choose the weights to be (see Section 4)

$$\gamma_{s,u} = 6^{|u|/2} \prod_{j \in u} W_{s,j}, \quad \emptyset \ne u \subseteq \{1, \ldots, s\},$$

which is equivalent to choosing the weights to be of the product form (4) with

$$\gamma_{s,j} := \sqrt{6} \, W_{s,j} \quad \text{for} \quad j = 1, \ldots, s.$$

Then the functions $\widetilde{f}(\mathbf{x})$ belong to the weighted Sobolev space $H(K_{s,\gamma})$ equipped with these weights (clearly, these weights depend on the path generation method). Therefore, $\widetilde{\mathcal{F}}_s \subset H(K_{s,\gamma})$. Now we are ready to apply Theorem 3.

Theorem 5 *Let $\widetilde{\mathcal{F}}_s$ be the function class defined in (28) associated with a path generation (standard, BB or PCA). Then there exists a QMC algorithm, such that*

$$|I_s(\widetilde{f}) - Q_{n,s}(\widetilde{f})| \le \frac{1}{\sqrt{n}} \left(\exp\left(\sum_{j=1}^{s} W_{s,j}/\sqrt{6} \right) - 1 \right), \quad s = 1, 2, \ldots.$$

The error bound is related to the magnitude of the sum $\sum_{j=1}^{s} W_{s,j}$, which depends on $\widetilde{f}$ and on the path generation. It turns out that this sum has quite different behavior under different path generation due to the different weighted property of the underlying function.

Now we study this sum for different path generations. We need a lower bound on $W_{s,j}$. Since $|\Psi(x)| \le |\Phi^{-1}(x)|$ for $x \in [0,1]$, and $\Gamma_{s,j}$ are positive and are uniformly bounded (see the last paragraph of Section 5.2), we have

$$\begin{aligned} \mu_{s,j} &= \int_0^1 \exp(\Gamma_{s,j}\Psi(x))dx \le \int_0^1 \exp(\Gamma_{s,j}\,|\Psi(x)|)dx \\ &\le \int_0^1 \exp(\Gamma_{s,j}\,|\Phi^{-1}(x)|)dx \le 2\,e^{\frac{1}{2}\Gamma_{s,j}^2} \le 2\,e^{\frac{1}{2}\Gamma^2}, \end{aligned}$$

where Γ is the upper bound of all $\Gamma_{s,j}$. Thus from (29) we have a lower bound for $W_{s,j}$:

$$W_{s,j} \ge \frac{\Gamma_{s,j}}{2\,e^{\frac{1}{2}\Gamma^2}} \left(\int_0^1 |\Psi'(x)|^2 dx \right)^{1/2} =: C_2\,\Gamma_{s,j}.$$

Combining this bound with (30), we see that $W_{s,j}$ and $\Gamma_{s,j}$ are in the same order.

Lemma 6 *For a given method of path generation (standard, BB or PCA), there exist constants C_1 and C_2 such that*

$$C_2\,\Gamma_{s,j} \le W_{s,j} \le C_1\,\Gamma_{s,j}, \tag{31}$$

where the parameters $\Gamma_{s,j}$ are given in Theorem 4 for three path generations.

Under the standard construction, from Theorem 4 we have

$$\sum_{j=1}^{s} \Gamma_{s,j}^{\mathrm{STD}} = \sigma\sqrt{T} \sum_{j=1}^{s} \frac{(s-j+1)}{s\sqrt{s}} = \sigma\sqrt{T}\,\frac{(s+1)}{2\sqrt{s}}.$$

Thus

$$\frac{1}{2}\sigma\sqrt{sT} \le \sum_{j=1}^{s} \Gamma_{s,j}^{\mathrm{STD}} \le \sigma\sqrt{sT}.$$

It then follows from Lemma 6 that under the standard construction

$$\frac{C_2}{2}\sigma\sqrt{sT} \le \sum_{j=1}^{s} W_{s,j} \le C_1\,\sigma\sqrt{sT}.$$

Therefore, under the standard construction the sum $\sum_{j=1}^{s} W_{s,j}$ is in the order $O(\sqrt{s})$. Based on Theorem 5, the error bound is $O(e^{c\sqrt{s}}\, n^{-1/2})$, which is exponentially dependent on $\sqrt{s}$. Thus *the standard construction does not avoid the curse of dimensionality.*

We show below that the use of BB or PCA can overcome this problem. Under the BB construction, the parameters $\Gamma_{s,j}^{\mathrm{BB}}$ are given in Theorem 4. We can calculate

$$\lim_{\ell\to\infty} \sum_{j=1}^{2^\ell} \Gamma_{2^\ell,j}^{\mathrm{BB}} = \frac{3-\sqrt{2}}{4-2\sqrt{2}}\,\sigma\sqrt{T}.$$

So from (31) there exists a constant M^{BB}, independent of the dimension s, such that

$$\sup_{\ell=1,2,\dots} \sum_{j=1}^{2^\ell} W_{2^\ell,j} \le M^{\mathrm{BB}}.$$

This indicates that $\widetilde{\mathcal{F}}_s^{\mathrm{BB}} = \widetilde{\mathcal{F}}_s^{\mathrm{BB}}(M^{\mathrm{BB}}) \subset \mathcal{F}_s(M^{\mathrm{BB}})$, where $\mathcal{F}_s(M^{\mathrm{BB}})$ is defined in Section 4.

Finally, under the PCA construction the parameters $\Gamma_{s,j}$ are given by (see Theorem 4)

$$\Gamma_{s,j}^{\mathrm{PCA}} := \frac{\sigma\sqrt{T}}{2s\sqrt{s(2s+1)}}\,\frac{\cos h_j}{\sin^2 h_j},\ h_j = \frac{(2j-1)\pi}{2(2s+1)}, \quad j=1,\dots,s, \tag{32}$$

thus

$$\sum_{j=1}^{s} \Gamma_{s,j}^{\mathrm{PCA}} = \frac{\sigma\sqrt{T}}{2s\sqrt{s(2s+1)}} \sum_{j=1}^{s} F(h_j),$$

where

$$F(x) := \frac{\cos x}{\sin^2 x}, \quad 0 < x \le \frac{\pi}{2}.$$

It is easily seen that

$$F(x) = \frac{1}{x^2} - \frac{1}{6} + O(x^2).$$

Since $G(x) := F(x) - \frac{1}{x^2}$ extends to a continuous function on $[0, \frac{\pi}{2}]$, it is easily seen that

$$\left| \frac{\sigma\sqrt{T}}{2s\sqrt{s(2s+1)}} \sum_{j=1}^{s} G(h_j) \right| \le \frac{\sigma\sqrt{T}}{2s\sqrt{s(2s+1)}}\, s \max_{x\in[0,\frac{\pi}{2}]} |G(x)|,$$

the right-hand side of which converges to zero as $s \to \infty$. Thus

$$\lim_{s\to\infty} \sum_{j=1}^{s} \Gamma_{s,j}^{\mathrm{PCA}} = \frac{\sigma\sqrt{T}}{2} \lim_{s\to\infty} \frac{1}{s\sqrt{s(2s+1)}} \sum_{j=1}^{s} \frac{1}{h_j^2} = \frac{\sigma\sqrt{T}}{\sqrt{2}}.$$

Thus the sum of $\Gamma_{s,j}^{\mathrm{PCA}}$ is uniformly bounded in s, like that of BB, but with a smaller value for the limit.

It follows from (31) that there exists a constant M^{PCA}, which is independent of the dimension s, such that

$$\sup_{s=1,2,\dots} \sum_{j=1}^{s} W_{s,j} \le M^{\mathrm{PCA}}.$$

This indicates that $\widetilde{\mathcal{F}}_s^{\mathrm{PCA}} = \widetilde{\mathcal{F}}_s^{\mathrm{PCA}}(M^{\mathrm{PCA}}) \subset \mathcal{F}_s(M^{\mathrm{PCA}})$.

Summarizing the results above, we have the following theorem.

Theorem 7 *Let $\widetilde{\mathcal{F}}_s$ be the function class defined in (28) that is associated with a path generation (standard, BB or PCA), let $\widetilde{f}$ denote the corresponding function given by (26). Then*

- *There exists a QMC algorithm, such that*

$$|I_s(\widetilde{f}) - Q_{n,s}(\widetilde{f})| \le \exp\left(M^*/\sqrt{6}\right) n^{-1/2}, \quad s = 1, 2, \ldots.$$

  where $M^* = M^{\mathrm{BB}}$ and $\widetilde{\mathcal{F}}_s = \widetilde{\mathcal{F}}_s^{\mathrm{BB}}$, if the Brownian motion is generated by BB; or $M^* = M^{\mathrm{PCA}}$ and $\widetilde{\mathcal{F}}_s = \widetilde{\mathcal{F}}_s^{\mathrm{PCA}}$, if the Brownian motion is generated by PCA. Both M^{BB} and M^{PCA} are independent of the dimension s, i.e., we have strong tractability of integration on the classes $\widetilde{\mathcal{F}}_s^{\mathrm{BB}}$ and $\widetilde{\mathcal{F}}_s^{\mathrm{PCA}}$.

- *If the Brownian motion is generated by the standard method, then the corresponding $M^* = O(e^{c\sqrt{s}})$ and the error bound is $O(e^{c\sqrt{s}}\, n^{-1/2})$, which is exponentially dependent on the dimension s.*

5.4 Higher convergence order — a constructive approach

The results in the previous subsections are non-constructive, that is, we do not know which QMC algorithm achieves the corresponding error bound; moreover, the convergence order in Theorem 7 is only $O(n^{-1/2})$, which is not optimal. It is natural to ask if the convergence order can be improved, and also to ask whether we can construct QMC algorithms that achieve the improved error bounds. The answer to both questions is: yes.

Given a path generation method, consider the function class $\widetilde{\mathcal{F}}_s$ defined in (28). For $\widetilde{f} \in \widetilde{\mathcal{F}}_s$, we have shown that

$$W_{s,j}(\widetilde{f}) := W_{s,j} := \left(\int_0^1 |\widetilde{g}'_{s,j}(x)|^2 dx\right)^{1/2} < \infty.$$

Based on the results of Section 4, it is optimal to choose the weights to be

$$\gamma_{s,u} = 6^{|u|/2} \prod_{j\in u} W_{s,j}, \quad \emptyset \ne u \subseteq \{1, \ldots, s\}, \tag{33}$$

or equivalently, choose the weights to be of the product form (4) with

$$\gamma_{s,j} := \sqrt{6}\, W_{s,j} \quad \text{for} \quad j = 1, \ldots, s. \tag{34}$$

Then the function $\widetilde{f}(\mathbf{x})$ belongs to the weighted Sobolev space $H(K_{s,\gamma})$ equipped with the weights $\gamma_{s,u}$ (or equivalently, with the product weights $\gamma_{s,j}$). In this case the kernel $K_{s,\gamma}(\mathbf{x}, \mathbf{y})$ has the product form (5) due to the product form of the weights. The weights $\gamma_{s,u}$ or $\gamma_{s,j}$ depend on $\widetilde{f}$, and thus depend on the path generation method. From (7) and (15), the squared norm of $\widetilde{f}(\mathbf{x})$ can be calculated:

$$||\widetilde{f}||^2_{H(K_{s,\gamma})} = \sum_{u\subseteq\{1,\ldots,s\}} \gamma_{s,u}^{-1} \prod_{j\in u} W_{s,j}^2 = \prod_{j=1}^{s} \left(1 + \frac{W_{s,j}}{\sqrt{6}}\right) \le \exp\left(\frac{\sum_{j=1}^{s} W_{s,j}}{\sqrt{6}}\right).$$

If the Brownian motion is generated by BB or PCA, then we have shown that

$$\sup_{s=1,2,\ldots} \sum_{j=1}^{s} W_{s,j} \leq M^*,$$

where $M^* = M^{\mathrm{BB}}$ or $M^* = M^{\mathrm{PCA}}$ (see previous subsection). Thus in both cases the norm $||\widetilde{f}||_{H(K_{s,\gamma})}$ is uniformly bounded in s. This fact is useful below.

To study the error bound for $\widetilde{f} \in \widetilde{\mathcal{F}}_s(M)$, we only need to study the worst-case error of the weighted Sobolev space $H(K_{s,\gamma})$, since $\widetilde{\mathcal{F}}_s \subset H(K_{s,\gamma})$. The theory of shift-invariant kernels is useful, see Hickernell & Woźniakowski (2000). The shift-invariant kernel corresponding to $K_{s,\gamma}(\mathbf{x},\mathbf{y})$ is (see Wang *et al.*, 2004)

$$K^{\mathrm{sh}}_{s,\gamma}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} \left(1 + \gamma_{s,j} B_2(\{x_j - y_j\})\right).$$

This is a reproducing kernel of a special weighted Korobov space (see Dick *et al*, 2004). It is shown in Hickernell & Woźniakowski (2002) that for a point set $P = \{\mathbf{x}_k\} \subset [0,1]^s$, there exists a shift $\Delta \in [0,1)^s$ such that

$$e(P + \Delta; H(K_{s,\gamma})) \leq e(P; H(K^{\mathrm{sh}}_{s,\gamma})),$$

where $P + \Delta$ is the shifted point set $P + \Delta := \{\{\mathbf{x}_k + \Delta\} : \mathbf{x}_k \in P, k = 1, \ldots, n\}, \Delta \in \mathbb{R}^s$ and $e(P; H(K^{\mathrm{sh}}_{s,\gamma}))$ is the worst-case error in f the Korobov space $H(K^{\mathrm{sh}}_{s,\gamma})$. In other words, $e(P; H(K^{\mathrm{sh}}_{s,\gamma}))$ gives an upper bound on the value of $e(P+\Delta; H(K_{s,\gamma}))$ with a good choice of Δ and measures the average performance of $e(P + \Delta; H(K_{s,\gamma}))$ with respect to the random shifts Δ.

Let n be a prime number. We will construct a good lattice point set of the form

$$P(\mathbf{z}) := \left\{ \left\{ \frac{k\mathbf{z}}{n} \right\} : k = 0, 1, \ldots, n-1 \right\},$$

where $\mathbf{z}$ is a generating vector of integers with no common factor with n. Here the braces around a vector indicate that we take the fractional part of each component of the vector. We refer the reader to Niederreiter (1992) and Sloan & Joe (1994) for the theory of good lattice rules. In dimension $s \geq 3$ one normally finds lattice rules by computer searches based on some criterion of "goodness". There are a number of such criteria available. Here we use the worst-case error of the weighted Korobov space $H(K^{\mathrm{sh}}_{s,\gamma})$ as the criterion. We use the following component-by-component construction. The main benefits of this construction are its extensibility in dimension s and (strong) tractability under appropriate conditions on the weights, see Dick *et al.* (2006) and Sloan *et al.* (2002). The main difference here is that we need to choose the right weights according to the path generation to reflect the different weighted character of the underlying integrands. The weights also depend on the problem dimension s. The importance of choosing weights in constructing good lattice rules is stressed in Wang & Sloan (2006). Here we focus on what can be achieved by correctly choosing the weights in BB and PCA cases.

Component-by-component construction

Given a path generation (standard, BB or PCA), suppose that the weights $\gamma_{s,j}$ are chosen as indicated in (34) (depending on the path generation). Let n be a prime number. The generator $\mathbf{z}$ is found as follows:

1. Set z_1, the first component of $\mathbf{z}$, to 1;

2. For $d = 2, 3, \ldots, s$ and $z_1, \ldots, z_{d-1}$ fixed, sequentially find $z_d \in \{1, 2, \ldots, n-1\}$, such that

$$e^2(P(z_1, \ldots, z_d); H(K^{\rm sh}_{d,\gamma})) = -1 + \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^{d} \left(1 + \gamma_{s,j} B_2\left(\left\{\frac{k z_j}{n}\right\}\right)\right)$$

is minimized.

The next theorem is proved in Kuo (2003).

Theorem 8 *Suppose $H(K_{s,\gamma})$ is the weighted Sobolev space with kernel (2) and (3) equipped with product weights $\gamma_{s,u} = \Pi_{j=1}^{s} \gamma_{s,j}$. Let n be a prime number and $\mathbf{z}$ be the generator found by the component-by-component algorithm. Then*

- *There exist a shift $\Delta \in [0,1)^s$, such that*

$$e(P(\mathbf{z}) + \Delta; H(K_{s,\gamma})) \le C_s(\delta)\, n^{-1+\delta} \qquad \forall \delta \in (0, 1/2],$$

  where $C_s(\delta)$ is a function of s and δ.

- *If for some $\delta \in (0, 1/2]$ the weights $\gamma_{s,j}$ satisfy*

$$\sup_{s=1,2,\ldots} \sum_{j=1}^{s} \gamma_{s,j}^{\frac{1}{2(1-\delta)}} < \infty, \tag{35}$$

  then $C_s(\delta)$ is uniformly bounded in s and the worst-case error $e(P(\mathbf{z}) + \Delta; H(K_{s,\gamma}))$ converges as $O(n^{-1+\delta})$ with the implied constant independent of s (but dependent of δ).

Note that based on Lemma 6 we have

$$\gamma_{s,j} = \sqrt{6}\, W_{s,j} = O(\Gamma_{s,j}),$$

thus the condition (35) is equivalent to the following one (in terms of $\Gamma_{s,j}$)

$$\sup_{s=1,2,\ldots} \sum_{j=1}^{s} \Gamma_{s,j}^{\frac{1}{2(1-\delta)}} < \infty. \tag{36}$$

Obviously, the condition (36) is not satisfied for $\delta \in (0, 1/2]$ if the Brownian motion is generated by the standard method, since $\sum_{j=1}^{s} \Gamma^{\rm STD}_{s,j} = O(\sqrt{s})$.

Now we check the condition (36) when the BB or PCA algorithm is applied, and find for which δ is it satisfied. Under the BB construction, for $s = 2^\ell$ we have from Theorem 4 that

$$\begin{aligned}
\sum_{j=1}^{s} \left(\Gamma^{\rm BB}_{s,j}\right)^\beta &= \left(\Gamma^{\rm BB}_{s,1}\right)^\beta + (\sigma\sqrt{T})^\beta \sum_{q=1}^{\ell} 2^{q-1} (2^{-(3q+1)/2})^\beta \\
&= \left(\Gamma^{\rm BB}_{s,1}\right)^\beta + (\sigma\sqrt{T})^\beta 2^{-(2+\beta)/2)} \sum_{q=1}^{\ell} 2^{(2-3\beta)q/2}.
\end{aligned}$$

Clearly,

$$\sup_{s=1,2,\ldots} \sum_{j=1}^{s} \left(\Gamma_{s,j}^{\mathrm{BB}}\right)^{\beta} < \infty \quad \Leftrightarrow \quad \beta > 2/3.$$

Equivalently,

$$\sup_{s=1,2,\ldots} \sum_{j=1}^{s} \left(\Gamma_{s,j}^{\mathrm{BB}}\right)^{\frac{1}{2(1-\delta)}} < \infty \quad \Leftrightarrow \quad \delta \in (1/4, 1).$$

Therefore, under the BB construction, the condition (36) is satisfied for $\delta \in (1/4, 1)$. Since we are only interested in $\delta \in (0, 1/2]$, based on Theorem 8 we have

$$e(P(\mathbf{z}) + \Delta; H(K_{s,\gamma})) = O(n^{-1+\delta}) \qquad \forall \delta \in (1/4, 1/2]$$

with the implied constant independent of the dimension. Equivalently,

$$e(P(\mathbf{z}) + \Delta; H(K_{s,\gamma})) = O(n^{-3/4+\delta}) \qquad \forall \delta \in (0, 1/4],$$

which implies that for $\widetilde{f} \in \widetilde{\mathcal{F}}_s^{\mathrm{BB}}$

$$|I_s(\widetilde{f}) - Q_{n,s}(\widetilde{f})| \le e(P(\mathbf{z}) + \Delta; H(K_{s,\gamma}))\, ||\widetilde{f}||_{H(K_{s,\gamma})} = O(n^{-3/4+\delta}) \qquad \forall \delta \in (0, 1/4],$$

with the implied constant independent of the dimension s (but dependent on δ), where we used the fact that $||\widetilde{f}||_{H(K_{s,\gamma})}$ is uniformly bounded in s under the BB construction.

Under the PCA construction, we can show that

$$\sup_{s=1,2,\ldots} \sum_{j=1}^{s} \left(\Gamma_{s,j}^{\mathrm{PCA}}\right)^{\frac{1}{2(1-\delta)}} < \infty \qquad \forall \delta \in (0, 1/2]. \tag{37}$$

Indeed, using the same notations as in the previous subsection we have from (32) that

$$\Gamma_{s,j}^{\mathrm{PCA}} = C(s)\left(G(h_j) + h_j^{-2}\right) \quad \text{with} \quad C(s) = \frac{\sigma\sqrt{T}}{2s\sqrt{s(2s+1)}},\ h_j = \frac{(2j-1)\pi}{2(2s+1)}.$$

For $\delta \in (0, 1/2]$, we have $\frac{1}{2(1-\delta)} \in (1/2, 1]$. From Jensen's inequality

$$\begin{aligned} \left(\Gamma_{s,j}^{\mathrm{PCA}}\right)^{\frac{1}{2(1-\delta)}} &= D(s,\delta)\, \left(|G(h_j)| + h_j^{-2}\right)^{\frac{1}{2(1-\delta)}} \\ &\le D(s,\delta)\, |G(h_j)|^{\frac{1}{2(1-\delta)}} + D(s,\delta)\, h_j^{-\frac{1}{1-\delta}}, \end{aligned}$$

where $D(s,\delta) = C(s)^{\frac{1}{2(1-\delta)}}$. By essentially the same argument as in the preceding subsection we come to the assertion (37). Therefore, for $\widetilde{f} \in \widetilde{\mathcal{F}}_s^{\mathrm{PCA}}$

$$|I_s(\widetilde{f}) - Q_{n,s}(\widetilde{f})| = O(n^{-1+\delta}) \qquad \forall \delta \in (0, 1/2],$$

with the implied constant depending on δ but independent of the dimension s.

Summarizing the results above, we have the following theorem.

Theorem 9 *Let $\widetilde{f}$ be the function defined by (26) corresponding to the BB and PCA constructions, respectively. Let n be a prime number and let $\mathbf{z}^{\rm BB}$ and $\mathbf{z}^{\rm PCA}$ be the associated generators found by the component-by-component algorithm with the weights chosen as indicated in (33) or (34) for the BB and PCA cases, respectively. Then there exist shifts $\Delta^{\rm BB}$ and $\Delta^{\rm PCA} \in [0,1)^s$, such that the QMC algorithm based on the shifted lattice point set $P(\mathbf{z}^{\rm BB}) + \Delta^{\rm BB}$ or $P(\mathbf{z}^{\rm PCA}) + \Delta^{\rm PCA}$ respectively satisfies the following:*

- *If the Brownian motion is generated by BB, then for $\widetilde{f} \in \widetilde{\mathcal{F}}_s^{\rm BB}$*

$$|I_s(\widetilde{f}) - Q_{n,s}(\widetilde{f})| = O(n^{-3/4+\delta}) \qquad \forall \delta \in (0, 1/4],\ s = 1, 2, \ldots.$$

  with the implied constant depending on δ but independent of the dimension s.

- *If the Brownian motion is generated by PCA, then for $\widetilde{f} \in \widetilde{\mathcal{F}}_s^{\rm PCA}$*

$$|I_s(\widetilde{f}) - Q_{n,s}(\widetilde{f})| = O(n^{-1+\delta}) \qquad \forall \delta \in (0, 1/2],\ s = 1, 2, \ldots.$$

  with the implied constant independent of the dimension s. That is, the shifted lattice rule based on $P(\mathbf{z}^{\rm PCA}) + \Delta^{\rm PCA}$, in connection with PCA, achieves for the function $\widetilde{f} \in \widetilde{\mathcal{F}}_s^{\rm PCA}$ an error bound which is independent of the dimension and for which the convergence order is optimal.

Theorem 9 shows what can be achieved by choosing the right weights in BB or PCA: shifted lattice rules can be found by component-by-component algorithm in each case, when they are used in conjunction with BB or PCA, the convergence is much faster than that of MC, with the convergence order approaching the size $O(n^{-3/4})$ for functions of the class $\widetilde{\mathcal{F}}_s^{\rm BB}$ or $O(n^{-1})$ for the class $\widetilde{\mathcal{F}}_s^{\rm PCA}$, independently of the dimension. Therefore, the curse of dimensionality is removed in both cases. The possible optimal convergence rate which can be achieved by BB or PCA is different, showing the different power of the BB and PCA algorithms for this kind of problems.

Remark In Theorem 9, one may use the Korobov lattice rule with a generator of the form

$$\mathbf{z} = (1, a, \ldots, a^{s-1}) \pmod n$$

for some suitable integer a found under the same criterion of goodness, and using the same weights as in the component-by-component algorithm. The usefulness of the Korobov rule in weighted spaces is justified in Wang *et al.* (2004). Based on the results of Wang *et al.* (2004), similar error bounds to those in Theorem 9 can be established for shifted Korobov rule (in conjunction with BB or PCA), but with the implied constant now depending at worst *polynomially* on the dimension s. Thus the extreme form of the curse of dimensionality is avoided here too.

Remark The result in Theorem 3 for multiplicative functions can be generalized to more general functions, say to the functions which are the sums of the functions of the form (13). As an example, consider a problem which is related to (19), but with an arithmetic rather than geometric average,

$$\mathbb{E}\left[\mathcal{A}(S_{t_1}, \ldots, S_{t_s})\right],$$

where

$$\mathcal{A}(S_{t_1}, \ldots, S_{t_s}) := \frac{1}{s} \sum_{k=1}^{s} S_{t_k},$$

is the arithmetic average of the stock prices at the times $t_1, \ldots, t_s$. This is related to the pricing of arithmetic Asian options with zero-strike price. Note that

$$\mathbb{E}\left[\mathcal{A}(S_{t_1}, \ldots, S_{t_s})\right] = \frac{1}{s} \sum_{k=1}^{s} \mathbb{E}\left[S_{t_k}\right]. \tag{38}$$

The value S_{t_k} is given in (20) and the Brownian motion is generated by (21). Each expectation $\mathbb{E}\left[S_{t_k}\right]$ in the right hand side of (38), writing as a multi-dimensional integral of a multiplicative function, can be analyzed as before:

$$\mathbb{E}\left[S_{t_k}\right] = S_0 e^{(r-\sigma^2/2)t_k} \int_{[0,1]^s} \exp\left(\sigma \sum_{j=1}^{s} a_{kj} \Phi^{-1}(x_j)\right) d\mathbf{x},$$

where a_{kj} is the (k, j)-element of the generating matrix A in (21), whose expressions under different path generations can be found in Wang (2006). The underlying functions have similar multiplicative structure to the one associated with the geometric average case, except that the parameters $\Gamma_{s,j}$ in 24) are replaced by a_{kj}. The factor $1/s$ and the sum over s in (38) have some cancellation effect. Therefore, similar results to those Theorems 7 and 9 can be established. We omit the details. This indicates that the results for BB and PCA in this section have some generality in finance.

Note that the introduction of a non-zero strike price into the option pricing destroys the simple multiplicative nature of the integrands, and complicates the theoretical discussion. The influence of a non-zero strike price on the dimension structure has been studied empirically in Wang (2006). In that paper it is found that when using BB and PCA the dimension structure (e.g. the effective dimensions) of the underlying functions for the case of non-zero strike prices has no significant difference from the case of zero strike price. It is also found there that from a numerical point of view, the difference between the pricing of geometric Asian options (analytically tractable) and of arithmetic Asian options (analytically intractable) is small.

6 Conclusions

Much previous work has been devoted to studying the tractability and strong tractability for weighted function spaces for which the weights are supposed to be given in advance, without taking into account the particular problem at hand. The first focus of this paper is to identify suitable weights for practical problems, and to bridge the gulf between theory and practice. The purpose of choosing good weights is twofold: one is to get a better bound for the integration error; the other is to use these good weights in a constructive algorithm for lattice rules, such that the resulting rules have good performance on the given problem. We presented a general method on how to choose the right weights, with the aim of minimizing the associated "average error bound". Because we are working with "general" weights, the arguments are much simpler than those in Dick *et al.* (2004) and Larcher *et al.* (2004).

The second focus of this paper is on dimension reduction techniques in a simple finance model. The general approach for choosing weights is applied to a model of a multiplicative high-dimensional problem from finance with BB or PCA algorithm. We demonstrated theoretically how BB and PCA can lead to strong tractability, whereas the standard construction for Brownian motion leads to intractability, with the error bound depending exponentially on the dimension. Moreover, by identifying suitable weights in BB or PCA, we showed what can be achieved in each case with this choice of weights: shifted lattice rules can be constructed by a component-by-component construction, such that when they are used in conjunction with BB or PCA, the algorithms can achieve a convergence order $O(n^{-3/4+\delta})$ or $O(n^{-1+\delta})$ respectively for arbitrary small $\delta > 0$, with constants independent of the dimension. These results indicate that well-designed algorithms and well chosen weights can avoid the curse of dimensionality, highlighting the potential possibility of BB or PCA, in conjunction with "good" lattice rules. This explains the superiority of BB and PCA over the standard construction for some applications and gives further insight into the BB and PCA algorithms. The investigation also indicates the importance of choosing weights in the construction of good lattice rules. The basic insight is that different path generations lead to different weighted properties of the functions, and these lead to different choices of weights in constructing lattice rules, and finally these affect the tractability property and the convergence order.

We studied only a very simple finance problem, with a zero strike price, with a simplification necessitated by the fact that the inverse of the standard normal cumulative distribution function $\Phi^{-1}(x)$ does not have a square-integrable first derivative. We gained insight on dimension reduction techniques and on the choices of weights in constructing good lattice rules. It seem very likely that our general conclusions will be also valid for more general problems (say, the bond valuation problem considered in Wang 2006). We must be mindful of the possible pitfalls of BB and PCA, since they do not take into account the nature of the payoff function. Moreover, other classes of problems and more complicated models are worth studying (such as these with stochastic volatilities). For example, for American option pricing, the theoretical impact of BB and PCA is as yet unknown (an empirical study of BB in American option pricing in presented in Chaudhary 2005). It is important to know for which kind of practical problems we have intractability, tractability or strong tractability, whether an $O(n^{-1+\delta})$ or better convergence rate can hold, how to construct algorithms that achieve a good convergence order, and what is the role of dimension reduction algorithms, i.e., whether they can (or cannot) reduce the computational complexity or remove the curse of dimensionality.

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