On the Approximation of Smooth Functions Using Generalized Digital Nets

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Abstract

In this paper we study an approximation algorithm which firstly approximates certain Walsh coefficients of the function under consideration and consequently uses a Walsh polynomial to approximate the function. A similar approach has previously been used for approximating periodic functions, using lattice rules (and Fourier polynomials), and for approximating functions in Walsh Korobov spaces, using digital nets. Here, the key ingredient is the use of generalized digital nets (which have recently been shown to achieve higher order convergence rates for the integration of smooth functions). This allows us to approximate functions with square integrable mixed partial derivatives of order $\alpha > 1$ in each variable. The approximation error is studied in the worst case setting in the $L_2$ norm. We also discuss tractability of our proposed approximation algorithm, investigate its computational complexity, and present numerical examples.

Key words: Approximation, digital nets, tractability

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1. Introduction

In this paper we study the approximation of functions in a certain normed Walsh space $W$. This function space consists of Walsh series $f : [0, 1]^s \rightarrow \mathbb{R}$, $f(x) = \sum_{k \in \mathbb{N}_0^s} \hat{f}(k) \operatorname{wal}_k(x)$, where $\operatorname{wal}_k$ denotes the $k$th Walsh function (see [2, 16, 28]), whose Walsh coefficients $\hat{f}(k)$ exhibit a certain convergence behavior. Here, $\mathbb{N}_0$ denotes the set of non-negative integers. The crux of this Walsh space is that it contains all functions which have square integrable mixed partial derivatives of order $\alpha$ in each variable. This space was introduced in [5] (see also [3]) in the context of numerical integration of smooth functions. In [5] it was also shown that so-called digital $(t, \alpha, \beta, n \times m, s)$-nets achieve the

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(almost) optimal rate of convergence of the integration error for functions in $W$, a feature which we will use in this paper for approximation.

Here we would like to approximate functions $f \in W$ by Walsh polynomials, to be more precise, if $f(x) = \sum_{k \in \mathbb{N}_0^s} \hat{f}(k) \text{wal}_k(x)$, then we first approximate the Walsh coefficients $\hat{f}(k)$, where $k$ is in a certain set $\mathcal{A}$, and then approximate $f$ by a Walsh polynomial based on the approximated Walsh coefficients. The set $\mathcal{A}$ is chosen to contain the most important, i.e., largest, Walsh coefficients of $f$ for functions in the unit ball of $W$. The approximation of the Walsh coefficients can be done via approximating an integral using a quasi-Monte Carlo rule, i.e.,

$$\hat{f}(k) = \int_{[0,1]^s} f(x) \text{wal}_k(x) \, dx \approx \frac{1}{N} \sum_{i=0}^{N-1} f(x_i) \text{wal}_k(x_i) =: \hat{f}(k),$$

where $x_0, \ldots, x_{N-1} \in [0,1)^s$ are suitably chosen quadrature points. We compute $\hat{f}(k)$, which approximates $\hat{f}(k)$, for all $k \in \mathcal{A}$ and then approximate $f$ by

$$A(f) := \sum_{k \in \mathcal{A}} \hat{f}(k) \text{wal}_k \approx f = \sum_{k \in \mathbb{N}_0^s} \hat{f}(k) \text{wal}_k.$$

(Later on we will add indices to $W$, $\mathcal{A}$ and $A$ to make their dependence on certain parameters explicit.) Provided that $\hat{f}(k)$ is a good approximation of $\hat{f}(k)$ and that the set $\mathcal{A}$ captures the largest Walsh coefficients of $f$, our approximation $A(f)$ will be a good approximation of $f$. This algorithm was already proposed by Korobov [11] and was thoroughly analyzed for the first time in [14] for lattice rules (see [20, 25] for more information on lattice rules), and was adapted for classical digital nets in [7] (see [20] for more information on digital nets). In a similar vain are results on trigonometric polynomial interpolation using lattice rules, which are given in [12, 13, 14, 15, 17, 32, 33]. Approximation algorithms based on classical digital $(t, m, s)$-nets are considered in [7, 32]. A multivariate fast discrete Walsh transform for function interpolation using digital nets is studied in [18].

The difference between our method and previous approaches lies in particular in the choice of the function space, which includes a fairly large class of functions. To be more precise, we do not assume periodicity of the functions as is done in the context of lattice rules and we also do not require fast converging Walsh coefficients as was the case in [7, 32]. Note that the function space $W$ contains smooth functions, as outlined in the beginning, as well as piecewise constant functions, as Walsh functions are piecewise constant. The approximation $A(f)$ of $f$ is a Walsh polynomial; as Walsh functions are piecewise constant, the Walsh polynomial $A(f)$ is also piecewise constant.

For the investigation of the error we consider the $L_2$ norm of the worst case error, i.e., the supremum of $\|f - A(f)\|_{L_2}$ over all functions in the unit ball of $W$. We prove that this error converges with order $N^{-\alpha/4+\delta}$, for all $\delta > 0$, see Section 4. This is in accordance with previous studies of this type of algorithm [7, 14]. We also investigate the dependence of our approach on the dimension and show that there are explicit constructions of suitable quadrature points such as
that we can achieve error bounds independent of the dimension (i.e., strong
tractability) for certain choices of parameters. In order to obtain an algorithm
which does not depend on the dimension, it is also important to show that the
set \(A\) can be constructed in a number of operations which is independent of the
dimension. This is done in Section 6.

The paper is structured as follows. In Section 2, we outline the definition of
digital \((t, \alpha, \beta, n \times m, s)\)-nets and state some of their properties, give the basic
features of the function space under consideration, and show a few technical
results needed later on. The approximation algorithm used in this paper is
introduced in Section 3. We then study upper bounds on the approximation
error, measured in the \(L_2\) norm in the worst case setting, in Section 4, and study
(strong) tractability in Section 5. In Section 6 we study the complexity of the
algorithm, and in Section 7 we present numerical experiments.

2. Preliminaries

In this section we introduce digital nets, the Walsh space, and several tech-
nical lemmas which will be needed later on.

2.1. Digital nets

The construction principle of a digital net in the sense of [19, 20] (with a
slight modification from [5]) is based on linear algebra over finite fields and
works as follows.

**Definition 1.** Let \(q\) be a prime power and let \(n, m, s\) be natural numbers. Let
\(C_1, \ldots, C_s\) be \(n \times m\) matrices over the finite field \(\mathbb{F}_q\) of order \(q\). We construct
\(q^m\) points in \([0, 1)^s\) in the following way. For \(0 \leq l < q^m\) let \(l = l^{(0)} + l^{(1)} q +
\ldots + l^{(m-1)} q^{m-1}\) be the base \(q\) representation of \(l\). Consider an arbitrary but
fixed bijection \(\rho: \{0, 1, \ldots, q-1\} \to \mathbb{F}_q\) where \(\rho(0)\) is the zero element in
\(\mathbb{F}_q\). Identify \(l\) with the vector \(l := (\rho(l^{(0)}), \ldots, \rho(l^{(m-1)}))^\top \in \mathbb{F}_q^m\). For \(1 \leq j \leq s\),
we multiply the matrix \(C_j\) by \(l\), i.e.,
\[C_j \cdot l =: (y_j, 1(l), \ldots, y_j, n(l))^\top \in \mathbb{F}_q^m,\]
and set
\[x_l^{(j)} := \frac{\rho^{-1}(y_{j, 1}(l))}{q} + \ldots + \frac{\rho^{-1}(y_{j, n}(l))}{q^n}.\]
Finally, set \(x_l := (x_l^{(1)}, \ldots, x_l^{(s)})\). The point set consisting of the points
\(x_0, \ldots, x_{q^m-1}\) is called a digital net over \(\mathbb{F}_q\). The matrices \(C_1, \ldots, C_s\) are called
the generating matrices of the digital net.

As can be seen from Definition 1, the properties of the points of a digital
net (such as, e.g., their distribution in the unit cube) are determined by the
properties of the generating matrices \(C_1, \ldots, C_s\). These properties are, in the
currently most general form of digital nets as introduced in [5], described by
the additional parameters $t$, $\alpha$, and $\beta$, which is why those nets are referred to as digital $(t, \alpha, \beta, n \times m, s)$-nets. The exact role of the parameters $t$, $\alpha$, and $\beta$ is stated in the following definition.

**Definition 2.** Let $n, m, \alpha \geq 1$ be natural numbers, let $0 < \beta \leq \min(1, \alpha m/n)$ be a real number such that $\beta n$ is an integer. Let $\mathbb{F}_q$ be the finite field of prime power order $q$ and let $C_1, \ldots, C_s \in \mathbb{F}_q^{n \times m}$ with $C_j = (\bar{c}_{j,1}, \ldots, \bar{c}_{j,n})^\top$. The digital net with generating matrices $C_1, \ldots, C_s$ is called a digital $(t, \alpha, \beta, n \times m, s)$-net for a natural number $t$, $0 \leq t \leq \beta n$, if the following condition is satisfied: For each choice of $1 \leq i, j \leq s$, and $t, \alpha, \beta, n \in \mathbb{N}$, with $i_{1,1} + \cdots + i_{1,\min(\nu_1, \alpha)} + \cdots + i_{s,1} + \cdots + i_{s,\min(\nu_s, \alpha)} \leq \beta n - t$

the vectors

$\bar{c}_{1,i_{1,\nu_1}}, \ldots, \bar{c}_{1,i_{1,\nu_1}}, \ldots, \bar{c}_{s,i_{s,\nu_s}}, \ldots, \bar{c}_{s,i_{s,\nu_s}}$

are linearly independent over $\mathbb{F}_q$.

The definition of classical digital $(t, m, s)$-nets is obtained by choosing $\alpha = \beta = 1$ and $m = n$ in Definition 2.

For a digital net $P$ over $\mathbb{F}_q$ with generating matrices $C_1, \ldots, C_s$ and for $k \in \mathbb{N}_0$ with $q$-adic representation $k = \sum_{i=0}^\infty k(i)q^i$, we write

$\text{tr}_n(k) := (\rho(k(0)), \ldots, \rho(k(n-1)))^\top \in \mathbb{F}_q^n$.

The dual net $D$ of $P$ is then defined as

$D := \{(k_1, \ldots, k_s) \in \mathbb{N}_0^s \setminus \{0\} : C_1^\top \cdot \text{tr}_n(k_1) + \cdots + C_s^\top \cdot \text{tr}_n(k_s) = \bar{0} \in \mathbb{F}_q\}$.

**Remark 1.** One way of explicitly constructing digital $(t, \alpha, \beta, n \times m, s)$-nets over $\mathbb{F}_q$ was stated in [5]. Given, for an integer $d \geq 1$, the generating matrices of a classical digital $(t', m, sd)$-net over $\mathbb{F}_q$, it is shown in [5, Theorem 4.11] that these can be used to construct generating matrices of a digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-net over $\mathbb{F}_q$, where $t$ satisfies

$t \leq \min \left( \min(\alpha, d) m, \min(\alpha, d)t' + \left[ \frac{s(d-1) \min(\alpha, d)}{2} \right] \right)$.

In [1], the implementation of digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-nets over $\mathbb{F}_q$ was discussed and consequently digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-nets were used for numerical experiments. Furthermore, so-called propagation rules for generalized digital nets are outlined in the recent paper [6], which in some cases show how to find generalized digital nets with lower values of $t$.

**Remark 2.** Given a digital $(t, \alpha, \beta, n \times m, s)$-net $P$ over $\mathbb{F}_q$, and a set $\emptyset \neq u \subseteq \{1, \ldots, s\} =: S$, we can easily define the projection of $P$ onto the coordinates in $u$, which will be denoted by $P_u$. Indeed, $P_u$ is defined by restricting the conditions stated in Definition 2 to those $C_j$ for which $j \in u$. Note that $P_u$ is a digital $(t_u, \alpha, \beta, n \times m, |u|)$-net over $\mathbb{F}_q$. In particular, $t_S = t$ and $t_u \leq t$ for any $u \subseteq S$.~

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If we consider projections of a digital \((t, \alpha, \min(1, \alpha/d), dm \times m, s)\)-net \(P\) over \(\mathbb{F}_q\) that was constructed from a classical digital \((t', m, sd)\)-net over \(\mathbb{F}_q\) as outlined in [5, Theorem 4.11], we can obtain more information on the parameter \(t_u\) of the projection \(P_u\). To make this observation more precise, we define a mapping \(\chi_d\) as follows: Given a non-empty set \(u \subseteq S\), \(u = \{u_1, \ldots, u_r\}\), and an integer \(d \geq 1\), we define

\[
\chi_d(u) := \bigcup_{j=1}^r \{(u_j - 1)d + 1, \ldots, u_j d\}.
\]

With this notation, we can formulate the following lemma.

**Lemma 1.** Let \(P\) be a digital \((t, \alpha, \min(1, \alpha/d), dm \times m, s)\)-net over \(\mathbb{F}_q\) that was constructed from a classical digital \((t', m, sd)\)-net \(Q\) over \(\mathbb{F}_q\) as outlined in [5, Theorem 4.11]. Moreover, let \(u\) be a non-empty subset of \(S\) and let \(\chi_d\) be defined as above. Then the quality parameter \(t_u\) of the projection \(P_u\) of \(P\) onto the coordinates in \(u\) satisfies the bound

\[
t_u \leq \min \left( \min (\alpha, d), t', \chi_d(u) \min (\alpha, d) + \left\lceil \frac{|u|}{2} \right\rceil \right),
\]

where \(t'_{\chi_d(u)}\) denotes the quality parameter of the projection of \(Q\) onto the coordinates in \(\chi_d(u)\).

**Proof.** Since \(P\) is constructed as outlined in [5, Theorem 4.11], it follows that the generating matrices of \(P_u\) are constructed from the generating matrices of \(Q_{\chi_d(u)}\). The rest of the proof follows by proceeding in exactly the same way as in [5, Section 4.4], see also [3, Section 4].

## 2.2. The Space \(W_{\alpha, s, \gamma}\)

We would now like to give the definition of the space of functions we are going to study in this paper. For this purpose, we first give the definition of Walsh functions.

Let, in the following, \(\mathbb{N}_0\) denote the set of non-negative and \(\mathbb{N}\) the set of positive integers. Let us, for the sake of simplicity (cf. Remark 3) from now on assume that \(q\) is a prime. Each \(h \in \mathbb{N}_0\) has a unique \(q\)-adic representation \(h = \sum_{i=0}^a h^{(i)} q^i\), \(h^{(i)} \in \{0, \ldots, q - 1\}\), where \(h^{(a)} \neq 0\). Each \(x \in [0, 1]\) has a \(q\)-adic representation \(x = \sum_{i=1}^\infty x^{(i)} q^{-i}\), \(x^{(i)} \in \{0, \ldots, q - 1\}\), which is unique in the sense that infinitely many of the \(x^{(i)}\) must differ from \(q - 1\). We define the \(h\)-th Walsh function in base \(q\), \(\text{wal}_h : [0, 1) \rightarrow \mathbb{C}\) by

\[
\text{wal}_h(x) := e^{2\pi i (x^{(1)} h^{(0)} + \cdots + x^{(a+1)} h^{(a)})/q}.
\]

For dimension \(s \geq 2\) and vectors \(h = (h_1, \ldots, h_s) \in \mathbb{N}_0^s\) and \(x = (x_1, \ldots, x_s) \in [0, 1)^s\) we define \(\text{wal}_h : [0, 1)^s \rightarrow \mathbb{C}\) by

\[
\text{wal}_h(x) := \prod_{j=1}^s \text{wal}_{h_j}(x_j).
\]
It follows from the definition above that Walsh functions are piecewise constant functions. For more information on Walsh functions, see, e.g., [2, 28].

When studying integration errors resulting from the approximation of an integral based on a digital or generalized digital net, it is often useful to consider the Walsh series of the integrand \( f \). In particular, for \( f \in L_2([0, 1]^s) \), the Walsh series of \( f \) is given by

\[
f(x) \sim \sum_{h \in \mathbb{N}_0^s} \hat{f}(h) \text{wal}_h(x),
\]

where the Walsh coefficients \( \hat{f}(h) \) are given by

\[
\hat{f}(h) = \int_{[0,1]^s} f(x) \text{wal}_h(x) dx.
\]

In general, the Walsh series given in Equation (1) need not converge to \( f \), however, for the space of Walsh series \( W_{\alpha,s,\gamma} \), which we define in the following, it does, see also [5]. For more details on the convergence of Walsh series, we refer to [5].

Let \( \alpha \in \mathbb{N}, \alpha \geq 2, \) and let \( s \geq 1 \) be a given dimension. Throughout the paper we assume that \( q \) denotes a fixed prime, all digital nets are assumed to be over the finite field \( \mathbb{F}_q \) of order \( q \), and all Walsh functions are also considered in the same base \( q \).

**Remark 3.** Note that the assumption that—for the rest of the paper—\( q \) be a prime is merely a technical restriction to make notation and the main arguments not overly complicated. Indeed, the results shown in this paper can be generalized to the case where \( q \) is a prime power, by using a more general definition of Walsh functions over groups, see, e.g., [5].

The function space under consideration in this paper is the space \( W_{\alpha,s,\gamma} \subseteq L_2([0, 1]^s) \) as introduced in [5]. Here \( \gamma = (\gamma_j)_{j=1}^\infty \) is a sequence of positive, non-increasing weights, with \( \gamma_1 \leq 1 \), which are introduced to model the importance of different variables for our approximation problem, see [26]. Given a positive integer \( h \) with base \( q \) expansion \( h = h^{(1)} q^{a_1-1} + h^{(2)} q^{a_2-1} + \cdots + h^{(v)} q^{a_v-1} \), \( 1 \leq a_v < \cdots < a_1, v \geq 1 \), we define \( \mu_\alpha(h) := a_1 + \cdots + a_{\min(v,\alpha)} \). Furthermore, \( \mu_\alpha(0) := 0 \).

For \( h \in \mathbb{N}_0 \) and a weight \( \gamma > 0 \), we define a function

\[
r_{\alpha,\gamma}(h) := \begin{cases} 1 & \text{if } h = 0, \\ \gamma q^{-\mu_\alpha(h)} & \text{otherwise}. \end{cases}
\]

If we consider a vector \( h \in \mathbb{N}_0^s \), we set

\[
r_{\alpha,s,\gamma}(h) := \prod_{j=1}^s r_{\alpha,\gamma_j}(h_j).
\]
The space \( W_{\alpha,s,\gamma} \) consists of all Walsh series \( f = \sum_{h \in \mathbb{N}_0^s} \hat{f}(h) \text{wal}_h \) for which the norm
\[
\|f\|_{W_{\alpha,s,\gamma}} := \sup_{h \in \mathbb{N}_0^s} r_{\alpha,s,\gamma}(h) \left| \hat{f}(h) \right|
\]
is finite. It follows immediately that for any \( f \in W_{\alpha,s,\gamma} \), and any \( h \in \mathbb{N}_0^s \),
\[
\left| \hat{f}(h) \right| \leq \|f\|_{W_{\alpha,s,\gamma}} r_{\alpha,s,\gamma}(h). \tag{2}
\]

For \( \alpha \geq 2 \), the following property was shown in [5]: Let \( f : [0,1]^s \rightarrow \mathbb{R} \) be such that all mixed partial derivatives up to order \( \alpha \) in each variable are square integrable, then \( f \in W_{\alpha,s,\gamma} \). Furthermore an inequality using a Sobolev type norm and the norm (2) was shown in [5], see also [3, 4]. Consequently, the results we are going to establish in the following for functions in \( W_{\alpha,s,\gamma} \) also apply automatically to smooth functions.

Numerical integration of functions \( f \in W_{\alpha,s,\gamma} \) has been analyzed in [5] using quasi-Monte Carlo algorithms \( q^{-m} \sum_{h=0}^{q^{n-1}} f(x_h) \), where the quadrature points \( x_0, \ldots, x_{q^{n-1}} \) stem from a digital \((t, \alpha, \beta, n \times m, s)\)-net. These quadrature algorithms achieve the almost optimal rate of convergence of the integration error of order \( q^{m(-\alpha+\delta)} \), for any \( \delta > 0 \). We will make use of this result subsequently. The assumption \( \alpha > 1 \) is needed to ensure that the sum of the absolute values of the Walsh coefficients converges. The case \( \alpha = 1 \) requires a different analysis, which was carried out in [8] for numerical integration.

2.3. Technical Lemmas

In this section, we present a few technical results needed for the proofs of the main results.

Lemma 2. Let \( l, \alpha \geq 1 \) be positive integers. Then,
\[
|\{ h \in \mathbb{N} : \mu_{\alpha}(h) \leq l \}| \leq q^l.
\]

Proof. The result follows by observing that if \( \mu_{\alpha}(h) \leq l \), then \( h \leq q^{l-1}(q-1) \).
\[ \square \]

Lemma 3. Let \( \alpha > 1 \). Then there exists a finite constant \( C_{\alpha} \) such that
\[
\sum_{h=1}^{\infty} r_{\alpha,1}(h) \leq C_{\alpha}.
\]

Proof. An explicit bound for this sum was obtained in the proof of Lemma 2.10 in [3]. See also Lemma 4.2 in [9].
\[ \square \]

Remark 4. In the following, we will denote by \( \xi(\alpha) \) the value of the sum \( \sum_{h=1}^{\infty} r_{\alpha,1}(h) \). By Lemma 3, we know that \( \xi(\alpha) \) is finite whenever \( \alpha > 1 \). Note, furthermore, that \( \xi(\alpha) \geq 1 \) for \( \alpha > 1 \).
The following function will occur frequently in this paper. For \( \alpha, \eta \in \mathbb{N} \) let

\[
\bar{u}(\alpha, \eta) = \begin{cases} 
1 & \text{if } \eta \geq \alpha, \\
\alpha/\eta & \text{if } \alpha > \eta.
\end{cases}
\]

**Lemma 4.** For any \( h \in \mathbb{N}_0 \) and any \( \alpha, \eta \geq 1 \), we have

\[ \mu_\alpha(h) \leq \bar{u}(\alpha, \eta) \mu_\eta(h). \]

**Proof.** Let \( h = h^{(1)} q^{a_1 - 1} + \cdots + h^{(v)} q^{a_v - 1} \) with \( 0 < h^{(1)}, \ldots, h^{(v)} < q \) and \( a_1 > a_2 > \cdots > a_v > 0 \). If \( \eta \geq \alpha \), then we have \( \mu_\alpha(h) \leq \mu_\eta(h) \) for any \( h \in \mathbb{N}_0 \) as \( \min(\alpha, v) \leq \min(\eta, v) \). Otherwise, we have

\[
\frac{\alpha}{\eta} = \frac{a_1 + \cdots + a_{\min(\alpha, v)}}{\alpha} \leq \frac{a_1 + \cdots + a_{\min(\eta, v)}}{\eta} = \frac{\mu_\eta(h)}{\eta}.
\]

\[ \square \]

We need two further technical results. The following estimate is similar to Formula (23) in [22] and Lemma 2 in [7].

**Lemma 5.** For any \( \alpha \geq 2 \) and \( h, k \in \mathbb{N}_0^v \), we have

\[ r_{\alpha, s, \gamma}(h \oplus k) \leq r_{\alpha, s, \gamma}(k) r_{\alpha, s, \gamma}(h)^{-1}. \]

**Proof.** Note that it suffices to prove the result for the case \( s = 1 \). If either \( h = 0 \) or \( k = 0 \), the result is immediate. In the case where both \( h \) and \( k \) are positive, we have

\[
r_{\alpha, \gamma}(k \oplus h) r_{\alpha, \gamma}(k) = \gamma \left( \frac{q^{-\mu_\alpha(h) + \mu_\alpha(k)}}{q^{\mu_\alpha(k \oplus h)}} \right).
\]

We now show that \( \mu_\alpha(k) - \mu_\alpha(h) \leq \mu_\alpha(k \oplus h) \), which will complete the proof. To this end, let us consider the base \( q \) expansion of \( k \) and \( h \),

\[
k = k^{(1)} q^{a_1 - 1} + \cdots + k^{(v)} q^{a_v - 1},
\]

\[
h = h^{(1)} q^{b_1 - 1} + \cdots + h^{(w)} q^{b_w - 1},
\]

where \( v, w \geq 1 \), \( k^{(1)}, \ldots, k^{(v)}, h^{(1)}, \ldots, h^{(w)} \in \{1, \ldots, q - 1\} \), and

\[
1 \leq a_v < \cdots < a_1,
\]

\[
1 \leq b_w < \cdots < b_1.
\]

We distinguish two cases:

**Case 1:** Let us first assume \( \{a_1, \ldots, a_v\} \cap \{b_1, \ldots, b_w\} = \emptyset \). In this case,

\[
k \oplus h = \sum_{i=1}^{v} k^{(i)} q^{a_i - 1} + \sum_{j=1}^{w} h^{(j)} q^{b_j - 1},
\]

hence

\[
\mu_\alpha(k \oplus h) \geq \mu_\alpha(k) \geq \mu_\alpha(k) - \mu_\alpha(h).
\]
Case 2: Let us now assume that $\{a_1, \ldots, a_v\} \cap \{b_1, \ldots, b_w\} = \{a_i\}_{i \in I}$, where $I \subseteq \{1, \ldots, \min(v, w)\}$. We then have

$$k \oplus h = \sum_{i \in I} (k^{(i)} \oplus h^{(i)})q^{a_i - 1} + \sum_{i \in (1, \ldots, v) \setminus I} k^{(i)}q^{a_i - 1} + \sum_{j \in (1, \ldots, w) \setminus I} h^{(j)}q^{b_j - 1}. $$

Hence,

$$\mu_\alpha(k \oplus h) \geq \mu_\alpha \left( \sum_{i \in I} k^{(i)}q^{a_i - 1} \right) \geq \mu_\alpha(k) - \sum_{i \in I \cap \{1, \ldots, \alpha\}} a_i \geq \mu_\alpha(k) - \mu_\alpha(h).$$

We also need the following lemma.

Lemma 6. Let $f \in W_{\alpha,s,\gamma}$ and $h \in \mathbb{N}_0^s$. Then,

$$\hat{f}_{\text{wal}} h \in W_{\alpha,s,\gamma}.$$

Proof. For short we write $g_h = \hat{f}_{\text{wal}} h$. Then the $k$-th Walsh coefficient of $g_h$, $k \in \mathbb{N}_0^s$, is given by

$$\hat{g_h}(k) = \int_{[0,1]^s} g_h(x) \text{wal}_k(x) \, dx$$

$$= \int_{[0,1]^s} f(x) \text{wal}_h(x) \text{wal}_k(x) \, dx$$

$$= \int_{[0,1]^s} \left( \sum_{l \in \mathbb{N}_0^s} \hat{f}(l) \text{wal}_l(x) \right) \text{wal}_h(x) \text{wal}_k(x) \, dx$$

$$= \sum_{l \in \mathbb{N}_0^s} \hat{f}(l) \int_{[0,1]^s} \text{wal}_l(x) \text{wal}_k(x) \, dx$$

$$= \hat{f}(h \oplus k).$$

Therefore,

$$\|g_h\|_{W_{\alpha,s,\gamma}} = \sup_{k \in \mathbb{N}_0^s} \frac{|\hat{g_h}(k)|}{r_{\alpha,s,\gamma}(k)}$$

$$= \sup_{k \in \mathbb{N}_0^s} \frac{|\hat{f}(h \oplus k)|}{r_{\alpha,s,\gamma}(k)}$$

$$\leq \frac{1}{r_{\alpha,s,\gamma}(h)} \sup_{k \in \mathbb{N}_0^s} \frac{|\hat{f}(h \oplus k)|}{r_{\alpha,s,\gamma}(h \oplus k)}$$

$$= \frac{1}{r_{\alpha,s,\gamma}(h)} \|f\|_{W_{\alpha,s,\gamma}},$$

where we used Lemma 5. The result follows by the assumption that $f \in W_{\alpha,s,\gamma}$.

$\square$
3. The Approximation Algorithm

For the approximation algorithm we study in this paper, we choose a real number $M \geq q/\gamma_1$ (see Remark 7 below for an explanation of this particular lower bound for $M$), a positive integer $\eta$ and define a set

$$\mathcal{A}(s, M, \eta) := \{ h \in \mathbb{N}_0^s : r_{\eta, s, \gamma}(h)^{-1} \leq M \}.$$ 

**Remark 5.** Notice that we allow $\eta$ to be different from $\alpha$ (which is the reason why we need the function $\bar{u}$ below). In the next section we analyze the worst case error, at which stage we also give recommendations on how to best choose the parameters $M$ and $\eta$. These recommendations are based on the upper bound on the worst case error.

In the following lemma, we summarize a few properties of the set $\mathcal{A}(s, M, \eta)$, cf. [14, Lemma 1].

**Lemma 7.** Given $M \geq q/\gamma_1$, and $s, \eta \geq 1$, let $\mathcal{A}(s, M, \eta)$ be defined as above. Then we have

(a) If $h = (h_1, \ldots, h_s) \in \mathcal{A}(s, M, \eta)$, then $\mu_{\eta}(h_j) \leq \log_q(\gamma_j M)$ for all $j \in \{1, \ldots, s\}$ for which $h_j > 0$.

(b) $\mathcal{A}(s + 1, M, \eta) = \{(h, 0) : h \in \mathcal{A}(s, M, \eta)\} \cup \bigcup_{h_{s+1} = 1}^{\infty} \left\{ (h, h_{s+1}) : h \in \mathcal{A}\left(s, \frac{\gamma_{s+1} M}{q \mu_{\eta}(h_{s+1})}, \eta\right) \right\}.$

(c) $|\mathcal{A}(s + 1, M, \eta)| = |\mathcal{A}(s, M, \eta)| + \sum_{h_{s+1} = 1}^{\infty} \left| \mathcal{A}\left(s, \frac{\gamma_{s+1} M}{q \mu_{\eta}(h_{s+1})}, \eta\right) \right|.$

(d) $|\mathcal{A}(s, M, \eta)| \leq M \prod_{j=1}^{s} (1 + \xi(\eta)\gamma_j).$

**Proof.** Regarding (a), if $h \in \mathcal{A}(s, M, \eta)$, then necessarily $r_{\eta, s+1}(h_{s+1})^{-1} \leq M$ for all $j \in \{1, \ldots, s\}$, which leads to $\mu_{\eta}(h_j) \leq \log_q(\gamma_j M)$ if $h_j \neq 0$.

Concerning (b), suppose that $(h, h_{s+1}) \in \mathcal{A}(s + 1, M, \eta)$. Then we have $r_{s, \eta, \gamma_1}(h) \leq r_{\eta, s+1}(h_{s+1}) M$, which simplifies to the condition $h \in \mathcal{A}(s, M, \eta)$ if $h_{s+1} = 0$ and $h \in \mathcal{A}\left(s, \frac{\gamma_{s+1} M}{q \mu_{\eta}(h_{s+1})}, \eta\right)$ otherwise.

Item (c) is a direct consequence of (b).

To prove (d), we proceed by induction on $s$. For $s = 1$, we have, by (a),

$$|\mathcal{A}(1, M, \eta)| = 1 + \left| \{ h \in \mathbb{N} : \mu_{\eta}(h) \leq \left\lfloor \log_q(\gamma_1 M) \right\rfloor \} \right|.$$
Applying Lemma 2, this yields

\[ |A(1, M, \eta)| \leq 1 + M \gamma_1 \leq M(1 + \xi(\eta)\gamma_1). \]

Suppose now we have already shown the result for fixed \( s \). Then we get for \( s+1 \), using (c), the induction assumption, and Lemma 3,

\[
|A(s+1, M, \eta)| = |A(s, M, \eta)| + \sum_{h_{s+1}=1}^{\infty} A \left( s, \frac{\gamma_{s+1} M}{q^{\mu_0(h_{s+1})}}, \eta \right) \\
\leq M \prod_{j=1}^{s} (1 + \xi(\eta)\gamma_j) + \sum_{h_{s+1}=1}^{\infty} \gamma_{s+1} M \prod_{j=1}^{s} (1 + \xi(\eta)\gamma_j) \\
= M \left( \prod_{j=1}^{s} (1 + \xi(\eta)\gamma_j) \right) (1 + \xi(\eta)\gamma_{s+1}).
\]

Given a function \( f \in W_{\alpha, s, \gamma} \), we approximate \( f \) by an algorithm \( A_{N, \eta, s, M} \) using \( N \) function evaluations,

\[
(A_{N, \eta, s, M}(f))(x) := \sum_{h \in A(s, M, \eta)} \left( \frac{1}{N} \sum_{n=0}^{N-1} f(x_n)\text{wal}_h(x_n) \right) \text{wal}_h(x), \quad x \in [0, 1)^s,
\]  

(3)

where the points \( x_0, \ldots, x_{N-1} \) stem from a digital \((t, \alpha, \beta, n \times m, s)\)-net over \( \mathbb{F}_q \) with \( N = q^m \) points. An approximation algorithm of this form was first introduced in [11], analyzed in detail in [14], and also used in [7].

4. Error analysis

Given a point set \( P \) of \( N \) points in \([0, 1)^s\), we study the quality of our approximation algorithm in terms of the worst case error measured in the \( L_2 \)-norm, i.e., we consider the error

\[
e_{N, \alpha, \eta, s, M}^{\text{app}}(P) = \sup_{f \in W_{\alpha, s, \gamma}, \|f\|_{W_{\alpha, s, \gamma}} \leq 1} \|f - A_{N, \eta, s, M}(f)\|_{L_2}.
\]

For a fixed \( f \) in the unit ball of \( W_{\alpha, s, \gamma} \), we have by Parseval’s identity

\[
\|f - A_{N, \eta, s, M}(f)\|^2_{L_2} = \sum_{h \notin A(s, M, \eta)} |\hat{f}(h)|^2 \\
+ \sum_{h \in A(s, M, \eta)} \left| \int_{[0,1]^s} f(x)\text{wal}_h(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n)\text{wal}_h(x_n) \right|^2.
\]  

(4)
Let us analyze the first term in (4), \( \sum_{h \notin A(s, M, \eta)} \left| \hat{f}(h) \right|^2 \). By (2) and due to the fact that \( f \) is in the unit ball of \( W_{\alpha,s,\gamma} \), we obtain

\[
\sum_{h \notin A(s, M, \eta)} \left| \hat{f}(h) \right|^2 \leq \|f\|_{W_{\alpha,s,\gamma}}^2 \sum_{h \notin A(s, M, \eta)} r_{\alpha,s,\gamma}^2(h) \leq \sum_{h \notin A(s, M, \eta)} r_{\alpha,s,\gamma}^2(h) =: Z(\alpha, \eta, s, M, \gamma). \tag{5}
\]

We therefore need a bound on \( Z(\alpha, \eta, s, M, \gamma) \), which we are going to prove in Lemma 8.

**Lemma 8.** Let \( Z(\alpha, \eta, s, M, \gamma) \) be defined as in (5). Then,

\[
Z(\alpha, \eta, s, M, \gamma) \leq \frac{2}{M^{1/\bar{u}(\eta, \alpha)}} \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j).
\]

**Proof.** We show the result by induction on \( s \). For \( s = 1 \), by the fact that \( 0 \in A(1, M, \eta) \),

\[
\sum_{h \notin A(1, M, \eta)} r_{\alpha,1}^2(h) = \sum_{h \in \mathbb{N}, h \notin A(1, M, \eta)} r_{\alpha,1}^2(h).
\]

By Lemma 4,

\[
\mu_{\alpha}(h) \geq \frac{\mu_{\eta}(h)}{\bar{u}(\eta, \alpha)}.
\]

Hence, by the definition of \( A(1, M, \eta) \), for \( h \in A(1, M, \eta) \),

\[
\gamma_1^{-1} q^{\mu_{\alpha}(h)} \geq \gamma_1^{-1} q^{\mu_{\eta}(h)/\bar{u}(\eta, \alpha)} \geq \left( \gamma_1^{-1} q^{\mu_{\eta}(h)} \right)^{1/\bar{u}(\eta, \alpha)} > M^{1/\bar{u}(\eta, \alpha)}.
\]

Therefore,

\[
\sum_{h \in \mathbb{N}, h \notin A(1, M, \eta)} r_{\alpha,1}^2(h) \leq \sum_{h \in \mathbb{N}, \mu_{\alpha}(h) \geq \log_q(\gamma_1 M^{1/\bar{u}(\eta, \alpha)})} \left( \gamma_1 q^{-\mu_{\alpha}(h)} \right)^2 = \gamma_1^2 \sum_{k=\left[ \log_q(\gamma_1 M^{1/\bar{u}(\eta, \alpha)}) \right]}^{\infty} q^{-2k} \sum_{h \in \mathbb{N}, \mu_{\alpha}(h) = k} 1.
\]

Using Lemma 2, we obtain

\[
\sum_{h \notin A(1, M, \eta)} r_{\alpha,1}^2(h) \leq \gamma_1^2 \sum_{k=\left[ \log_q(\gamma_1 M^{1/\bar{u}(\eta, \alpha)}) \right]}^{\infty} q^{-k},
\]
and therefore
\[
\sum_{h \not\in A(1, M, \eta)} r^2_{\alpha, \gamma_1}(h) \leq \gamma_1^2 \left( \frac{1}{q} \right) \left[ \log_q \left( \gamma_1 M^{1/u(\eta, \alpha)} \right) \right] \sum_{k=0}^{\infty} \frac{1}{q^k} \\
\leq \frac{1}{M^{1/u(\eta, \alpha)}} \frac{q}{q - 1} \\
\leq \frac{2}{M^{1/u(\eta, \alpha)}} (1 + \gamma_1 \xi(\alpha)).
\]

Suppose now that we have shown the result for fixed \( s \). If we would like to show the result for \( s + 1 \), we use the induction assumption, Lemma 4, and Remark 4 to obtain
\[
\sum_{(h, h_{s+1}) \not\in A(s+1, M, \eta)} r^2_{\alpha, s+1, \gamma}((h, h_{s+1})) = \\
\sum_{h \in N_{\eta}^s \setminus A(s, M, \eta)} r^2_{\alpha, s+1, \gamma}((h, 0)) \\
+ \sum_{h_{s+1}=1}^{\infty} \sum_{h \in N_{\eta}^s \setminus A(s+1, M, \eta)} r^2_{\alpha, s+1, \gamma}((h, h_{s+1})) \\
= \sum_{h \in N_{\eta}^s \setminus A(s, M, \eta)} r^2_{\alpha, s, \gamma}(h) + \sum_{h_{s+1}=1}^{\infty} \sum_{h \in N_{\eta}^s \setminus A(s+1, M, \eta)} r^2_{\alpha, s, \gamma}(h) \\
\leq \frac{2}{M^{1/u(\eta, \alpha)}} \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j) \\
+ \sum_{h_{s+1}=1}^{\infty} r^2_{\alpha, \gamma_{s+1}}(h_{s+1}) \frac{2 \mu(q(\gamma_{s+1}))/\mu(\eta, \alpha)}{\gamma_{s+1}} \frac{M^{1/u(\eta, \alpha)}}{1/u(\eta, \alpha)} \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j) \\
\leq \frac{2}{M^{1/u(\eta, \alpha)}} \left( \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j) \right) \left( 1 + \sum_{h_{s+1}=1}^{\infty} r_{\alpha, \gamma_{s+1}}(h_{s+1}) \frac{\mu(q(\gamma_{s+1}))/\mu(\eta, \alpha)}{q^{\mu(\eta, \alpha)(h_{s+1})}} \right) \\
\leq \frac{2}{M^{1/u(\eta, \alpha)}} \left( \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j) \right) \left( 1 + \sum_{h_{s+1}=1}^{\infty} r_{\alpha, \gamma_{s+1}}(h_{s+1}) \right) \\
= \frac{2}{M^{1/u(\eta, \alpha)}} \left( \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j) \right) (1 + \xi(\alpha) \gamma_{s+1}).
\]

The result follows. \( \square \)

We have thus found an upper bound on the first term in (4). Let us now derive an upper bound on the second term in (4). From Lemma 5.2 in [5] and
Lemma 6, we obtain

\[
\left| \int_{[0,1]} f(x) \text{val}_h(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \text{val}_h(x_n) \right| = \left| \sum_{k \in D} \hat{f}(k) \right| \leq \sum_{k \in D} \|f\|_{W,\alpha,s,\gamma}(k) r_{\alpha,s,\gamma}(\hat{k} \oplus h),
\]

where \(D\) is the dual net of the point set used in our algorithm.

Hence, using Lemma 5 and again the fact that \(f\) is in the unit ball of \(W,\alpha,s,\gamma\), the second term in (4) is bounded by

\[
\|f\|_{W,\alpha,s,\gamma}^2 \sum_{h \in A(s,M,\eta)} \left( \sum_{k \in D} r_{\alpha,s,\gamma}(k \oplus h) \right)^2 \leq \sum_{h \in A(s,M,\eta)} r_{\alpha,s,\gamma}^{-2}(h) \left( \sum_{k \in D} r_{\alpha,s,\gamma}(k) \right)^2.
\]

By Lemma 4, it follows that

\[
r_{\alpha,s,\gamma}^{-1}(h) \leq (r_{\eta,s,\gamma}(h))^{-\hat{u}(\alpha,\eta)},
\]

and hence,

\[
\sum_{h \in A(s,M,\eta)} r_{\alpha,s,\gamma}^{-2}(h) \left( \sum_{k \in D} r_{\alpha,s,\gamma}(k) \right)^2 \leq \sum_{h \in A(s,M,\eta)} (r_{\eta,s,\gamma}^{-1}(h))^{2\hat{u}(\alpha,\eta)} \left( \sum_{k \in D} r_{\alpha,s,\gamma}(k) \right)^2 \leq M^{2\hat{u}(\alpha,\eta)} |A(s,M,\eta)| \left( \sum_{k \in D} r_{\alpha,s,\gamma}(k) \right)^2.
\]

However, by the results in [5],

\[
\left( \sum_{k \in D} r_{\alpha,s,\gamma}(k) \right)^2 = (e_{N,\alpha,s}^{\text{int}}(P))^2,
\]

where \(e_{N,\alpha,s}^{\text{int}}(P)\) denotes the worst case integration error in \(W,\alpha,s,\gamma\) using \(P\).

Summarizing all our results so far, we obtain the following proposition.

**Proposition 1.** Using the same notation as above,

\[
\left( e_{N,\alpha,q,s}^{\text{app}}(P) \right)^2 \leq \left( \prod_{j=1}^s (1 + \xi(\alpha)\gamma_j) \right) \frac{2}{M^{1/\hat{u}(\alpha,\eta)}}
\]

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\[ + \left( \prod_{j=1}^{s} (1 + \xi(\eta)\gamma_j) \right) M^{2\tilde{u}(\alpha,\eta)+1} \left( e_{N,\alpha,s}^{\text{int}}(P) \right)^2. \quad (6) \]

Let us now analyze the error bound in Proposition 1 in more detail. If we would like to practically use our algorithm \( A_{N,\eta,s,M} \), we first need to make a choice of which point set we would like to employ. As described in [5], we can explicitly construct digital \((t, \alpha, \min(1, \alpha/d), dm \times m, s)\)-nets from existing classical digital \((t', m, sd)\)-nets, which is why we will focus on these nets in the following.

We first show that when using a digital \((t, \alpha, \min(1, \alpha/d), dm \times m, s)\)-net over \( \mathbb{F}_q \) in the algorithm \( A_{N,\eta,s,M} \) for approximating a function \( f \in W_{\alpha,s,\gamma} \), choosing \( d = \eta = \alpha \) minimizes the error bound given in (6) in terms of the order of magnitude in \( N \) and \( M \).

**Proposition 2.** Let \( \alpha \geq 2, M \geq q/\gamma_1, m \in \mathbb{N}, s \in \mathbb{N}, \) and let \( q \) be prime. Then, when using a digital \((t, \alpha, \min(1, \alpha/d), dm \times m, s)\)-net in our algorithm \( A_{N,\eta,s,M} \), the error bound (6) is minimized with respect to the order of magnitude in \( M \) and \( N = q^m \) for \( d = \eta = \alpha \).

**Proof.** Note first that we can ignore the terms 
\[ \prod_{j=1}^{s} (1 + \xi(\alpha)\gamma_j), \prod_{j=1}^{s} (1 + \xi(\eta)\gamma_j), \]

since both terms are finite (due to our assumption \( \eta, \alpha \geq 2 \)) and do not depend on \( M \) or \( N \).

We recall from [5] that, if \( P \) is a digital \((t, \alpha, \beta, dm \times m, s)\)-net over \( \mathbb{F}_q \), with \( \beta = \min(1, \alpha/d) \) the worst case integration error in \( W_{\alpha,s,\gamma} \) using \( P, e_{N,\alpha,s}^{\text{int}}(P) \), is of order \( \mathcal{O}(q^tN^{-\min(\alpha,d)}(\beta dm - t + 2)\alpha) \). Choosing \( d = \alpha \) yields the optimal convergence rate as pointed out in [5]. Indeed, it is clear that choosing \( d < \alpha \) would not yield the optimal convergence rate. On the other hand, choosing \( d > \alpha \) might at first sight not be a bad choice; however, this would imply that the \( t \)-value of \( P \) increases, which is undesirable. Therefore it is best to choose \( d = \alpha \).

Let us now deal with the terms 
\[ \frac{2}{M^{1/\tilde{u}(\eta,\alpha)}}, \quad \text{and} \quad M^{2\tilde{u}(\alpha,\eta)+1}. \]

For the choice \( \eta = \alpha \), we get 
\[ \frac{2}{M^{1/\tilde{u}(\eta,\alpha)}} = \frac{2}{M} \text{ and } M^{2\tilde{u}(\alpha,\eta)+1} = M^3. \]

For \( \eta > \alpha \), we obtain 
\[ \frac{2}{M^{1/\tilde{u}(\eta,\alpha)}} > \frac{2}{M} \text{ and } M^{2\tilde{u}(\alpha,\eta)+1} = M^3. \]
Finally, for $\eta < \alpha$, we obtain

$$\frac{2}{M^{1/(\eta, \alpha)}} = \frac{2}{M} \text{ and } M^{2u(\alpha, \eta)+1} > M^3.$$ 

Thus choosing $\alpha = \eta$ yields the smallest values for the terms under consideration, hence the result is shown. \hfill $\square$

We summarize the results from Proposition 1 and Proposition 2 in the following theorem.

**Theorem 1.** Given $s \geq 1$, $\alpha \geq 2$, and $M \geq q/\gamma_1$, the bound for the worst case error of the algorithm $A_{N, \eta, s, M}$ in $W_{\alpha, s, \gamma}$ using a digital $(t, \alpha, \min(1, \alpha/d), dm \times m, s)$-net $P$ over $F_q$, with $N = q^m$ points, is minimized when choosing $d = \eta = \alpha$ and then satisfies

$$\left( e_{\text{app}}^{\alpha, \alpha, s, M}(P) \right)^2 \leq \left( \prod_{j=1}^{s} (1 + \xi(\alpha)\gamma_j) \right) \left( \frac{2}{M^2} + M^3 \left( e_{\text{int}}^{\alpha, s, \gamma}(P) \right)^2 \right),$$

where $e_{\text{int}}^{\alpha, s, \gamma}(P)$ is the worst case integration error of $P$ in $W_{\alpha, s, \gamma}$.

Due to the results in Proposition 2 and Theorem 1 we will in the following always restrict ourselves to the case where $P$ is a digital $(t, \alpha, 1, \alpha m \times m, s)$-net over $F_q$ and where $\eta = \alpha$, i.e., we consider $A_{N, \alpha, s, M}$.

Considering the result in Theorem 1, we need to have good upper bounds on the worst case integration error of a generalized digital net $P$ in $W_{\alpha, s, \gamma}$ in order to obtain a good upper bound on the approximation error. As already mentioned in the proof of Proposition 2, an effective upper bound was shown in [5]. To be more precise, there it was shown (cf. [1]) that the worst case integration error of a digital $(t, \alpha, 1, \alpha m \times m, s)$-net over $F_q$ satisfies

$$e_{\text{int}}^{\alpha, s, \gamma}(P) \leq q^{-\alpha m + t} \sum_{\emptyset \neq u \subseteq S} \gamma_u C_{u, \alpha}(am - t + \alpha + 2)^{|u|\alpha}, \quad (7)$$

where the constants $C_{u, \alpha} = q^{|u|\alpha} \left( q^{-1} + (1 - q^{1/\alpha - 1})^{-|u|\alpha} \right)$, and where $\gamma_u := \prod_{j \in u} \gamma_j$.

Plugging (7) into Theorem 1 gives the following result:

**Corollary 1.** Given $s \geq 1$, $\alpha \geq 2$, and $M \geq q/\gamma_1$, the worst case error of the algorithm $A_{N, \alpha, s, M}$ in $W_{\alpha, s, \gamma}$ using a digital $(t, \alpha, 1, \alpha m \times m, s)$-net $P$ over $F_q$, with $N = q^m$ points, satisfies

$$\left( e_{\text{app}}^{\alpha, \alpha, s, M}(P) \right)^2 \leq \left( \prod_{j=1}^{s} (1 + \xi(\alpha)\gamma_j) \right) \times$$

$$\times \left( \frac{2}{M} + M^3 \left( \sum_{\emptyset \neq u \subseteq S} \gamma_u C_{u, \alpha}(am - t + \alpha + 2)^{|u|\alpha} \right)^2 \right).$$

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Remark 6. In order to obtain the best rate of convergence, one should choose $M$ such that $M^{-1} \approx M^3 N^{-2\alpha}$, which yields $M \approx N^{\alpha/2}$. This yields a convergence of the approximation error $\epsilon_{N,\alpha,s,M}^{\text{app}}(P)$ of $O(N^{-\alpha/4}(\log N)^{\alpha s})$.

It is possible to refine Equation (7), which is going to prove very useful in the following, in particular with respect to Section 5. To be more precise, it is shown in [5, Section 5] that the worst case integration error of a digital $(t, \alpha, 1, \alpha m \times m, s)$-net over $\mathbb{F}_q$ satisfies the equation

$$ e_{N,\alpha,s}^{\text{int}}(P) = \sum_{\emptyset \neq u \subseteq S} \gamma_u \sum_{k_u \in D_u} r_{|u|,\alpha}(k_u), \quad (8) $$

where $D_u$ denotes the dual net of the projection $P_u$ of $P$. Using the notation introduced in Remark 2 we can now refine Lemma 5.2 in [5] to

Lemma 9. Let $P$ be a digital $(t, \alpha, 1, \alpha m \times m, s)$-net over $\mathbb{F}_q$, with $N = q^m$ points, and let $\emptyset \neq u \subseteq S$. Then for $P_u$ it is true that

$$ \sum_{k_u \in D_u} r_{|u|,\alpha}(k_u) \leq q^{-\alpha m + \kappa u \gamma_u C_{u,\alpha}(\alpha m - t_u + \alpha + 2)^{|u|\alpha}}, $$

where $C_{u,\alpha}$ is defined as in Equation (7).

Proof. The result follows by using exactly the same arguments as in the proof of Lemma 5.2 in [5], but taking into consideration the refined quality parameter $t_u$ of the projection $P_u$. \qed

Using Lemma 9 and Equation (8), we can refine Equation (7) to

$$ e_{N,\alpha,s}^{\text{int}}(P) \leq q^{-\alpha m} \sum_{\emptyset \neq u \subseteq S} q^{\kappa u \gamma_u C_{u,\alpha}(\alpha m - t_u + \alpha + 2)^{|u|\alpha}}. \quad (9) $$

Therefore, we obtain the following corollary to Theorem 1.

Corollary 2. Given $s \geq 1$, $\alpha \geq 2$, and $M \geq q/\gamma_1$, the worst case error of the algorithm $A_{N,\alpha,s,M}$ in $W_{\alpha,s}$ using a digital $(t, \alpha, 1, \alpha m \times m, s)$-net $P$ over $\mathbb{F}_q$, with $N = q^m$ points, satisfies

$$ \left( \epsilon_{N,\alpha,s,M}^{\text{app}}(P) \right)^2 \leq \left( \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j) \right) \times \left( \frac{2}{M} + \frac{M^3}{N^{2\alpha}} \sum_{\emptyset \neq u \subseteq S} q^{\kappa u \gamma_u C_{u,\alpha}(\alpha m - t_u + \alpha + 2)^{|u|\alpha}} \right)^2. $$

Corollary 2 enables us to give a result that is similar to Lemma 5 in [7]. The point sets used in the following theorem are constructed from Sobol’ or Niederreiter sequences, which are examples of digital $(t, s)$-sequences (see [19, 20]), and can be constructed explicitly.
Theorem 2. Let $s \geq 1$, $\alpha \geq 2$, $m \geq 1$ be integers and let $q$ be prime. Let $Q \in \{Q_{\text{Sob}}, Q_{\text{Nied}}\}$ be a digital $(t', m, s\alpha)$-net over $\mathbb{F}_q$ obtained from either the Sobol’ sequence ($q = 2$) or the Niederreiter sequence. Furthermore, let $P \in \{P_{\text{Sob}}, P_{\text{Nied}}\}$ be a digital $(t, \alpha, 1, \alpha m \times m, s)$-net over $\mathbb{F}_q$, with $N = q^m$ points, constructed from $Q$ as described in [5, Section 4.4]. Then it is true that

$$e^\text{int}_{N, \alpha, s}(P_{\text{Sob}}) \leq \frac{2^{-am} \sqrt{2}}{\sqrt{2} - 1} \times \prod_{j=1}^{s} \left( 1 + 2^{\alpha(a+1)} \left( \frac{\sqrt{2}}{\sqrt{2} - 1} \right) \alpha \right)^{am + \alpha + 2} C_{\text{Sob}, j, \alpha} \gamma_j,$$

where $C_{\text{Sob}, j, \alpha} = \prod_{k=(j-1)\alpha+1}^{j\alpha} \left( q(k+1) \log_2(k+3) \right)^\alpha$ with $c$ being some constant independent of all parameters, and

$$e^\text{int}_{N, \alpha, s}(P_{\text{Nied}}) \leq \frac{\sqrt{2}}{q^{am}(\sqrt{2} - 1)} \times \prod_{j=1}^{s} \left( 1 + q^{\frac{a(a+1)}{2}} \left( \frac{\sqrt{2}}{\sqrt{2} - 1} \right) \alpha \right)^{am + \alpha + 2} C_{\text{Nied}, j, \alpha} \gamma_j,$$

where $C_{\text{Nied}, j, \alpha} = \prod_{k=(j-1)\alpha+1}^{j\alpha} \left( q(k+1) \log_q(k+q) \right)^\alpha$.

Proof. We begin by considering the case where $Q = Q_{\text{Nied}}$ is a digital $(t', m, s\alpha)$-net over $\mathbb{F}_q$ obtained from a Niederreiter sequence. Starting from $Q_{\text{Nied}}$, we use the algorithm outlined in [5, Section 4.4] to construct a digital $(t, \alpha, 1, \alpha m \times m, s)$-net $P_{\text{Nied}}$ over $\mathbb{F}_q$. Then, according to Equation (9), we have

$$e^\text{int}_{N, \alpha, s}(P_{\text{Nied}}) \leq q^{-am} \sum_{\emptyset \neq u \subseteq S} q^{|u|} \gamma_u C_{u, \alpha} (am - t_u + \alpha + 2)^{|u|\alpha}.$$

It is clear that for $\alpha \geq 2$, $q \geq 2$,

$$(1 - q^{1/\alpha - 1})^{-1} \leq \frac{\sqrt{2}}{\sqrt{2} - 1}.$$

Consequently, it follows that for any $\emptyset \neq u \subseteq S$,

$$C_{u, \alpha} = q^{|u|\alpha} \left( q^{-1} + (1 - q^{1/\alpha - 1})^{-|u|\alpha} \right) \leq q^{|u|\alpha} \left( \frac{1}{2} + \left( \frac{\sqrt{2}}{\sqrt{2} - 1} \right)^{|u|\alpha} \right) \leq q^{|u|\alpha} \left( \frac{\sqrt{2}}{\sqrt{2} - 1} \right)^{|u|\alpha + 1}.$$
Hence we can write, using Lemma 1,
\[ e_{\text{int}}^{p_{\text{Nied}}}(\alpha, s) \leq \frac{1}{q^{\alpha m}} \sum_{\emptyset \neq u \subseteq S} q^{\alpha t'_{\chi, \alpha}(u)} q^{\frac{n(\alpha+1)}{2} \gamma_u} \times \]
\[ \times \left( \frac{\sqrt{2}}{2} \right)^{|u|\alpha+1} (\alpha m + \alpha + 2)^{|u|\alpha} \times \]
\[ \sum_{\emptyset \neq u \subseteq S} \prod_{j \in u} q^{\alpha(t'_{\chi, \alpha}(u))} \leq \prod_{j \in \chi, \alpha(u)} \left( \prod_{k=(j-1)\alpha+1}^{j\alpha} (q k \log_q(k+q)) \right) \times \]
\[ \prod_{j=1}^{s} \left( 1 + q^{\frac{n(\alpha+1)}{2}} \left( \frac{\sqrt{2}}{2} \right)^\alpha \right) (\alpha m + \alpha + 2)^\alpha C_{\text{Nied}, j, \alpha} \gamma_j \].

The construction of the Niederreiter sequence makes use of monic irreducible polynomials in base \( q \), one polynomial \( p_j \) for each dimension \( j \), with non-decreasing degrees as the dimension increases. It is known (see [29]) that \( t'_{\chi, \alpha}(u) = \sum_{j \in \chi, \alpha(u)} (\deg(p_j) - 1) \) and \( \deg(p_j) \leq \log_q j + \log_q \log_q(j+q) + 2 \). Thus,
\[ q^{t'_{\chi, \alpha}(u)} \leq \prod_{j \in \chi, \alpha(u)} (q j \log_q(j+q)) = \prod_{j \in \alpha} \left( \prod_{k=(j-1)\alpha+1}^{j\alpha} (q k \log_q(k+q)) \right) \times \]
\[ \prod_{j=1}^{s} \left( 1 + q^{\frac{n(\alpha+1)}{4}} \left( \frac{\sqrt{2}}{2} \right)^\alpha \right) (\alpha m + \alpha + 2)^\alpha C_{\text{Nied}, j, \alpha} \gamma_j \].

Let \( C_{\text{Nied}, j, \alpha} = \prod_{k=(j-1)\alpha+1}^{j\alpha} (q k \log_q(k+q))^\alpha \). Consequently,
\[ e_{\text{int}}^{p_{\text{Nied}}}(\alpha, s) \leq \frac{\sqrt{2}}{q^{\alpha m}} \left( \frac{\sqrt{2}}{2} - 1 \right) \times \]
\[ \sum_{\emptyset \neq u \subseteq S} \prod_{j \in u} q^{\alpha(t'_{\chi, \alpha}(u))} \left( \frac{\sqrt{2}}{2} - 1 \right)^\alpha (\alpha m + \alpha + 2)^\alpha C_{\text{Nied}, j, \alpha} \gamma_j \leq \frac{\sqrt{2}}{q^{\alpha m}} \left( \frac{\sqrt{2}}{2} - 1 \right) \times \]
\[ \prod_{j=1}^{s} \left( 1 + q^{\frac{n(\alpha+1)}{2}} \left( \frac{\sqrt{2}}{2} - 1 \right) \right)^\alpha (\alpha m + \alpha + 2)^\alpha C_{\text{Nied}, j, \alpha} \gamma_j \].

The proof for \( P_{\text{Sob}} \) follows by using the fact that the Sobol’ sequence makes use of primitive polynomials \( p_j \) in base \( q = 2 \), \( t'_{\chi, \alpha}(u) = \sum_{j \in \chi, \alpha(u)} (\deg(p_j) - 1) \), and \( \deg(p_j) \leq \log_2 j + \log_2 \log_2(j+1) + \log_2 \log_2 \log_2(j+3) + c \), where \( c \) is a constant independent of \( j \). \( \square \)

5. Tractability

Let us now summarize a few facts about tractability and strong tractability, two concepts widely used in the research field of Information Based Complexity. For more detailed information on tractability, we refer to [23, 27, 30, 31].

Let us suppose we approximate a function \( f \) in a general function space \( W \) of \( s \)-variate functions by an algorithm \( A_{N, s} \) which is a suitable linear combination of \( N \) function values. We consider the worst case error of the algorithm \( A_{N, s} \) in the \( L_2 \) norm,
\[ e_{N, s}^{\text{app}} := \sup_{f \in W} \| f - A_{N, s}(f) \|_{L_2} \].
The initial error, associated with \( A_{0,s} \equiv 0 \), is
\[
e^{\text{app}}_{0,s} := \sup_{f \in W} \| f \|_{L^2}.
\]

For \( \varepsilon \in (0, 1) \) and \( s \geq 1 \), we define
\[
N^\text{wor}(\varepsilon, W) := \min(N : \exists A_{N,s} \text{ such that } e^{\text{app}}_{N,s} \leq \varepsilon e^{\text{app}}_{0,s}).
\]

We speak of tractability of the \( L^2 \) approximation problem in \( W \) if there exist non-negative numbers \( C, p, \) and \( a \) such that
\[
N^\text{wor}(\varepsilon, W) \leq C\varepsilon^{-p}s^a, \forall \varepsilon \in (0, 1), \forall s \geq 1.
\]

We say that the approximation problem is strongly tractable if (10) holds with \( a = 0 \).

Note that in \( W_{\alpha,s,\gamma} \) we conveniently have that the initial error equals 1. We are now going to show that we can achieve strong tractability in \( W_{\alpha,s,\gamma} \) by using the point sets and results from Theorem 2.

For this purpose we need the following lemma, which is also in analogy to Lemma 5 in [7].

**Lemma 10.** Let \( P \in \{ P_{\text{Sob}}, P_{\text{Nied}} \} \) be defined as in Theorem 2. If
\[
\begin{align*}
\sum_{j=1}^{\infty} \left( \prod_{k=(j-1)\alpha+1}^{j\alpha} (k \log k \log \log k) \right)^{\frac{1}{\alpha}} \gamma_j^{\frac{1}{\gamma}} < \infty & \quad \text{when } P = P_{\text{Sob}}, \\
\sum_{j=1}^{\infty} \left( \prod_{k=(j-1)\alpha+1}^{j\alpha} (k \log k) \right)^{\frac{1}{\alpha}} \gamma_j^{\frac{1}{\gamma}} < \infty & \quad \text{when } P = P_{\text{Nied}},
\end{align*}
\]

we have
\[
e^{\text{int}}_{N,\alpha,s}(P_{\text{Sob}}) \leq C_{\text{Sob},\delta} 2^{(-\alpha+\delta)m}
\]
and
\[
e^{\text{int}}_{N,\alpha,s}(P_{\text{Nied}}) \leq C_{\text{Nied},\delta} q^{(-\alpha+\delta)m},
\]
for any \( \delta > 0 \), where \( C_{\text{Sob},\delta}, C_{\text{Nied},\delta} \) are independent of \( m \) and \( s \) but depend on \( \delta, q, \alpha \) and \( \gamma \).

**Proof.** We set
\[
\tilde{\gamma}_j := q^{\frac{\alpha(j+1)}{2}} \left( \frac{2^{j+1}}{2^{j+1} - 1} \right)^{\alpha} C_{\text{Nied},j,\alpha \gamma_j}.
\]

From Theorem 2, it now follows that
\[
e^{\text{int}}_{N,\alpha,s}(P_{\text{Nied}}) \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \frac{1}{q^{\alpha m}} \prod_{j=1}^{s} (1 + (\alpha m + \alpha + 2)\gamma_j)
\]
\[
\leq \frac{\sqrt{2}}{\sqrt{2} - 1} \frac{1}{q^{\alpha m}} \prod_{j=1}^{s} (1 + C_{\alpha} m^\alpha \tilde{\gamma}_j),
\]

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where $\bar{C}_\alpha$ is a constant only dependent on $\alpha$. Let
\[
T(\tilde{\gamma}, m, \alpha) := \prod_{j=1}^{\infty} (1 + \bar{C}_\alpha m^{\tilde{\gamma}_j})^\alpha.
\]

Then,
\[
T(\tilde{\gamma}, m, \alpha) = \prod_{j=1}^{\infty} \left(1 + \left(\bar{C}_\alpha \tilde{\gamma}_j\right)^\frac{1}{\alpha} m\right)^\alpha \leq \prod_{j=1}^{\infty} \left(1 + \left(\bar{C}_\alpha \tilde{\gamma}_j\right)^\frac{1}{\alpha} m\right)^\alpha.
\]

Hence, defining $\sigma_d := \sum_{j=d+1}^{\infty} (\bar{C}_\alpha \tilde{\gamma}_j)^\frac{1}{\alpha}$, $d = 0, 1, \ldots$, where we can assume without loss of generality that all $\sigma_d > 0$, we use the same argument as in the proof of Lemma 3 in [10] to obtain,
\[
\log(T(\tilde{\gamma}, m, \alpha)) \leq \sum_{j=1}^{\infty} \log((1 + \left(\bar{C}_\alpha \tilde{\gamma}_j\right)^\frac{1}{\alpha} m)^\alpha)
\]
\[
= \alpha \sum_{j=1}^{\infty} \log(1 + \left(\bar{C}_\alpha \tilde{\gamma}_j\right)^\frac{1}{\alpha} m)
\]
\[
\leq \alpha (d \log(1 + \sigma_d^{-1}) + \sigma_d (\sigma_0 + 1)m).
\]

Consequently,
\[
T(\tilde{\gamma}, m, \alpha) \leq (1 + \sigma_d^{-1}d \alpha q^{\sigma_d (\sigma_0 + 1)\frac{m}{\sigma_0 + 1}}). \quad \text{Let } \delta > 0, \text{ then we choose } d \text{ large enough to make } \sigma_d \leq \frac{\delta \log q}{\sigma_0 + 1}. \text{ The desired result is obtained by choosing } C_{\text{Nied}, \delta} = \left(\frac{\sqrt{2}}{\sqrt{2} - 1}\right) (1 + \sigma_d^{-1})^{\delta \alpha}. \text{ The result for the Sobol' sequence can be derived using the same chain of arguments.}
\]

We therefore have shown the following lemma.

**Lemma 11.** Let $s \geq 1$, $\alpha \geq 2$, $m \geq 1$ be integers, and let $q$ be prime. Let $Q \in \{Q_{\text{Sob}}, Q_{\text{Nied}}\}$ be a digital $(t', m, s\alpha)$-net over $\mathbb{F}_q$ obtained from either the Sobol' sequence ($q = 2$) or the Niederreiter sequence. Furthermore, let $P \in \{P_{\text{Sob}}, P_{\text{Nied}}\}$ be a digital $(t, \alpha, 1, \alpha m \times m, s)$-net over $\mathbb{F}_q$, with $N = q^m$ points, obtained from $Q$ as described in [5, Section 4.4]. If Condition (11) holds, we have, for any $\delta > 0$,
\[
e_{N, \alpha, \alpha, s, M}(P_{\text{Sob}}) \leq \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j)^{1/2} \left(\frac{2}{M} + \frac{M^3}{N^{2\alpha - \delta}} C_{\text{Sob}, \delta}^2\right)^{1/2},
\]
and
\[
e_{N, \alpha, \alpha, s, M}(P_{\text{Nied}}) \leq \prod_{j=1}^{s} (1 + \xi(\alpha) \gamma_j)^{1/2} \left(\frac{2}{M} + \frac{M^3}{N^{2\alpha - \delta}} C_{\text{Nied}, \delta}^2\right)^{1/2},
\]
where $N = q^m$ and $C_{\text{Sob}, \delta}, C_{\text{Nied}, \delta}$ are defined as in Lemma 10.
Let us now analyze the error bound in Lemma 11 in more detail and with respect to tractability. Note first that Condition (11) obviously implies \( \sum_{j=1}^{\infty} \gamma_j < \infty \). Using the property

\[
\prod_{j=1}^{s} (1 + \xi(\gamma_j)) \leq \exp \left( \xi(\alpha) \sum_{j=1}^{s} \gamma_j \right),
\]

and the latter expression can be bounded by a constant \( \tilde{C}_{\alpha} \) that is independent of \( N \) and \( s \) if \( \sum_{j=1}^{\infty} \gamma_j < \infty \). Consequently, assuming that Condition (11) holds and choosing \( M = O(N^{\alpha/2}) \), we achieve that the worst case approximation error using \( P_{\text{Sob}} \) or \( P_{\text{Nied}} \) is of order \( O(N^{-\alpha/4+\delta}) \), with the implied constant independent of \( s \).

If, on the other hand, we would like to achieve that the error bound in Lemma 11 is at most \( \varepsilon \) for some small \( \varepsilon > 0 \), we choose \( \varepsilon' = \varepsilon / \left( \tilde{C}_{\alpha}^{1/2} \right) \) and try to choose \( M \) and \( N = q^m \) such that

\[
\left( \frac{2}{M} + \frac{M^3}{N^{2\alpha - \delta} C_{\delta}^2} \right)^{1/2} \leq \varepsilon',
\]

for \( C_{\delta} \in \{ C_{\text{Sob},\delta}, C_{\text{Nied},\delta} \} \). This can be achieved by choosing \( M = 4(\varepsilon')^{-2} \) and \( N \) such that the second term in the above equation is no more than the first term. Hence it is sufficient to take \( N = q^m \) with

\[
m = \left\lceil \log_q \left( \left( C_{\delta}^2 M^4 / 2 \right)^{1/2} \right) \right\rceil,
\]

i.e., \( N = O \left( \varepsilon^{-\frac{8\alpha}{2\alpha - s}} \right) \).

We summarize these results in the following theorem.

**Theorem 3.** Let \( s \geq 1, \alpha \geq 2, m \geq 1 \) be integers, and let \( q \) be prime. Let \( Q \in \{ Q_{\text{Sob}}, Q_{\text{Nied}} \} \) be a digital \((t', m, s\alpha)\)-net over \( \mathbb{F}_q \) obtained from either the Sobol’ sequence \((q = 2)\) or the Niederreiter sequence. Furthermore, let \( P \in \{ P_{\text{Sob}}, P_{\text{Nied}} \} \) be a digital \((t, \alpha, 1, \alpha m \times m, s)\)-net over \( \mathbb{F}_q \), with \( N = q^m \) points, obtained from \( Q \) as described in [5, Section 4.4]. If Condition (11) holds and if we choose \( M = O(N^{\alpha/2}) \), we have, for any \( \delta > 0 \),

\[
e_{N,\alpha,\alpha,\varepsilon,M}(P) = O(N^{-\alpha/4+\delta}).
\]

Furthermore, we achieve strong tractability and the error bound \( e_{N,\alpha,\alpha,\varepsilon,M}(P) \leq \varepsilon \) is achieved using \( N = O \left( \varepsilon^{-\frac{8\alpha}{2\alpha - s}} \right) \) function values. The implied factors in the \( O \)-notation are independent of \( \varepsilon \) and \( s \).
6. Computational complexity of constructing the set $\mathcal{A}$

In this section we discuss the computational complexities involved in computing the approximation (3). In order to compute (3), one needs to construct the set $\mathcal{A}(s,M,\eta)$ for some given values of $s$, $M$, and $\eta$. Before we investigate the construction cost we give a remark about some properties of $\mathcal{A}(s,M,\eta)$.

**Remark 7.** As $M \geq q/\gamma_1$ and $r_{\eta,s,\gamma}(0) = 1$, it follows that $h = 0$ is always in the set $\mathcal{A}(s,M,\eta)$.

Further, let $h \in \mathbb{N}$ with $h = h^{(1)}q^{\mu_1}^{-1} + \cdots + h^{(v)}q^{\mu_v}^{-1}$ with $h^{(1)}, \ldots, h^{(v)} \in \{1, \ldots, q - 1\}$ and $a_1 > a_2 > \cdots > a_v \geq 1$. If $r_{\eta,1,\gamma}(h) \leq M$, then $\mu_\eta(h) \leq \log_q(\gamma M)$. As $1 \leq a_1 = \mu_\eta(h) \leq \mu_\eta(h) \leq \log_q(\gamma M)$, it follows that $h < q^{a_1} \leq \gamma M$. Hence if $\gamma M < q$, then there is no $h \in \mathbb{N}$ such that $r_{\eta,1,\gamma}(h) \leq M$ and only $h = 0$ satisfies this inequality.

In dimension $s$, let $u \subseteq S$ and consider projections of $h \in \mathbb{N}_0^s$, where $h_j = 0$ for $j \in S \setminus u$ and $h_j > 0$ for $j \in u$. If $M \prod_{j \in u} \gamma_j < q^{u}$, then no such $h$ with $h_j > 0$ for $j \in u$ and $0$ otherwise satisfies $r_{\eta,s,\gamma}(h) \leq M$. Thus, we have that for a vector $h \in \mathcal{A}(s,M,\eta)$ the set of non-zero coordinates of $h$, $u$, satisfies $M \geq \prod_{j \in u}(q/\gamma_j)$.

The above shows that if $M < q/\gamma_1$, then it follows that $\mathcal{A}(s,M,\eta) = \{0\}$. In this case the approximation $A_{N,\alpha,s,M}(f)$ is a constant function.

In the following proposition we show that the set $\mathcal{A}(s,M,\eta)$ can be constructed in $\mathcal{O}(M \log M)$ operations, where the implied constant is allowed to depend on $s$. The algorithm which achieves this order is described in the proof of Proposition 3.

**Proposition 3.** Let $s, \eta \in \mathbb{N}$, and $M \geq q/\gamma_1$ be a given real number. The set $\mathcal{A}(s,M,\eta)$ can be constructed in $\mathcal{O}(M \log M)$ operations. Further, if Condition (11) is satisfied then the number of operations needed to construct $\mathcal{A}(s,M,\eta)$ is independent of the dimension $s$.

**Proof.** Firstly, for $l = 1, \ldots, \lfloor \log_q(\gamma_1 M) \rfloor$ compute the sets $U_l := \{h \in \mathbb{N} : \mu_\eta(h) = l\}$. Note that $U_l \cap U_{l'} = \emptyset$ for $l \neq l'$, $\{q^{-1}\} \subseteq U_l$, and $\bigcup_{l=1}^{\lfloor \log_q(\gamma_1 M) \rfloor} U_l \subseteq \{1, \ldots, \lfloor \gamma_1 M \rfloor\}$. Computing the sets $U_l$ can be done by calculating $\mu_\eta(h)$ for $h = 1, \ldots, \lfloor \gamma_1 M \rfloor$ and assigning $h$ to the set $U_{\mu_\eta(h)}$. As $\mu_\eta(h)$ can be calculated in $\mathcal{O}(\log M)$ operations for $1 \leq h \leq \gamma_1 M$, the sets $U_l$ for $l = 1, \ldots, \lfloor \log_q(\gamma_1 M) \rfloor$ can be computed in $\mathcal{O}(M \log M)$ operations. Set $U_0 = \{0\}$.

For $l = (l_1, \ldots, l_s) \in \prod_{j=1}^{s} \{0, 1, \ldots, \lfloor \log_q \gamma_j M \rfloor\}$ define

$$V_l := \{h = (h_1, \ldots, h_s) \in \mathbb{N}_0^s : h_j \in U_{l_j} \text{ for } j = 1, \ldots, s\}.$$ 

Further let $\|l\|_1 = \sum_{j=1}^{s} l_j$ and $u(l) := \{1 \leq j \leq s : l_j > 0\}$. Then we have

$$\mathcal{A}(s,M,\eta) = \bigcup_{l \in \mathbb{N}_0^s} V_l.$$ 

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Thus, to construct the set $\mathcal{A}(s, M, \eta)$, we need to find all $l \in \mathbb{N}_0^s$ such that $\|l\|_1 \leq \log_q(\gamma_{u(t)}M)$. Note that each set $V_l$ has at least one element, as $U_l \neq \emptyset$, and $V_l \cap V_{l'} = \emptyset$ for $l \neq l'$ as $U_l \cap U_{l'} = \emptyset$ for $l \neq l'$. Hence the number of $l \in \mathbb{N}_0^s$ with $\|l\|_1 \leq \log_q(\gamma_{u(t)}M)$ is bounded by the number of elements in the set $\mathcal{A}(s, M, \eta)$, which is itself bounded by $M \prod_{j=1}^s (1 + \xi(\eta)\gamma_j)$.

Hence the set of $l \in \mathbb{N}_0^s$ with $\|l\|_1 \leq \log_q(\gamma_{u(t)}M)$ can be generated in $O(M)$ operations and therefore the complexity of generating the set $\mathcal{A}(s, M, \eta)$ is $O(M \log M)$ operations.

If Condition (11) is satisfied the number of elements in $\mathcal{A}(s, M, \eta)$ is independent of the dimension and hence the number of operations needed is independent of the dimension. $\square$

We end this section with two remarks concerning the evaluation of the algorithm $A_{N,\eta,s,M}(f)$ for a fixed $f \in W_{\alpha,s,\gamma}$.

**Remark 8.** For a fixed $x \in [0,1]^s$ we require $\mathcal{O}(NM \log M)$ operations to evaluate $A_{N,\eta,s,M}(f)(x)$. Here the factor $\log M$ stems from the evaluation of $\text{val}_h(x)$ for $h \in \mathcal{A}(s, M, \eta)$ (note that the total number of non-zero digits of all the $h_j$, $j = 1, \ldots, s$ is bounded by $\log_q M$ and hence $\text{val}_h(x)$ can be evaluated in $O(\log_q M)$ operations).

**Remark 9.** As our approximation is constant over boxes of the form

$$\prod_{j=1}^s [B_j q^{-d_j}, (B_j + 1) q^{-d_j})],$$

$0 \leq B_j < q^{d_j}$, where we can choose $d_j = \max(0, \lfloor \log_q \gamma_j M \rfloor)$, the approximation takes on only a finite number of different values. Hence, in principle, one could create a table of all possible function values together with the boxes in which these function values are attained, thereby saving time if the approximation is evaluated frequently. The number of different $x$ at which $A_{N,\eta,s,M}(f)(x)$ needs to be evaluated is bounded by $q^{\sum_{j=1}^s d_j} \leq \prod_{j=1}^s (\gamma_j M)$. If Condition (11) is satisfied, then $\gamma_j$ converges to 0 as $j$ increases, hence for some $j_0 \in \mathbb{N}$ we will have $\gamma_j M < q$. Then the number of $x$ at which $A_{N,\eta,s,M}(f)$ needs to be evaluated is bounded by $\prod_{j=1}^{j_0-1} (\gamma_j M)$, which is independent of the dimension $s$. Further, for $h = (h_1, \ldots, h_s) \in \mathcal{A}(s, M, \eta)$ we have $h_j = 0$ for $j \geq j_0$, hence the evaluation of $\text{val}_h(x)$ and $\text{val}_h(x_t)$ depends only on $j_0$ and not on $s$.

As also the number of elements in the set $\mathcal{A}(s, M, \eta)$ is bounded independently of $s$ if Condition (11) is satisfied, it follows that when constructing a table containing complete information about $A_{N,\eta,s,M}(f)$, only the evaluation of $f(x_n)$, $n = 0, 1, \ldots, N - 1$, depends on the dimension, whereas all other components of the algorithm are independent of the dimension.

7. Numerical Experiments

In this section, we illustrate the algorithm $A_{N,\eta,s,M}$ when applied to three test problems, each of them on $[0,1]^2$: 

\[\text{24}\]
• $f(x_1, x_2) := x_1 x_2$,
• $g(x_1, x_2) := \sin(x_1) \cos(x_2)$,
• $h(x_1, x_2) := \sin(4\pi x_1) \cos(4\pi x_2)$.

Note that these functions lie in $W_{\alpha,2,\gamma}$ for all $\alpha \geq 2$.

For the approximation of the Walsh coefficients of our test functions we use a digital $(t, \alpha, \min(1, \alpha/d), d6 \times 6, 2)$-net over $\mathbb{F}_2$ with $d = 2$, constructed as in [5] from a digital $(1, 6, 4)$-net, which itself is obtained by the construction principle by Niederreiter and Xing [21], using an implementation by Piracic [24]. Furthermore, we set $M = N^{d/2} = 64$, choose $\eta = d = 2$, and $\gamma_1 = \gamma_2 = 1$.

The graphs of the function and the approximation $A_{64,2,64}$ for the functions $f$, $g$ and $h$ are shown in Figures 1 and 3, and 4, respectively. Figure 2 illustrates, by zooming in on the approximation of the function $f$, that the algorithm indeed returns a piecewise constant function. Regarding the approximation, we evaluate the algorithm at the points

$$(x_i, x_j), \ i, j \in \{0, 1, \ldots, 255\},$$

where

$$x_i = \frac{1}{256} + i \left(1 - \frac{1}{25600} - \frac{1}{256}\right).$$

Figure 1: Approximation of the function $f(x_1, x_2) = x_1 x_2$. 
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References


Figure 3: Approximation of the function $g(x_1, x_2) = \sin(x_1) \cos(x_2)$.


Figure 4: Approximation of the function $h(x_1, x_2) = \sin(4\pi x_1) \cos(4\pi x_2)$.


