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Maximum-norm error analysis of a numerical solution via Laplace transformation and quadrature of a fractional order evolution equation

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In a previous paper, McLean and Thomée (to appear), we studied three numerical methods for the discretization in time of a fractional order evolution equation, in a Banach space framework. Each of the methods applied a quadrature rule to a contour integral representation of the solution in the complex plane, where for each quadrature point an elliptic boundary value problem had to be solved to determine the value of the integrand. The first two methods involved the Laplace transform of the forcing term, but the third did not. We analysed both the quadrature error and the error arising from a spatial discretization by finite elements, measured in the L_2-norm. The present work extends our earlier results by proving error bounds in the technically more complicated case of the maximum-norm. We also establish new regularity properties for the exact solution, needed for our analysis.

Keywords: Laplace tranform, quadrature, resolvent estimate, fractional diffusion.

1. Introduction

We shall consider numerical methods for an initial-value problem of the form

$$\partial_t u + J^\alpha A u = f(t) \quad \text{for } t > 0, \quad \text{with } u(0) = u_0, \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $A = -\Delta$, with Δ the Laplacian under homogeneous Dirichlet boundary conditions in a convex bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary, and $-1 < \alpha < 1$. The fractional order operator J^α is defined by

$$J^\alpha v(t) = \begin{cases} (\omega_\alpha * v)(t) & \text{if } 0 < \alpha < 1, \\ v(t) & \text{if } \alpha = 0, \\ (\omega_{1+\alpha} * v)'(t) & \text{if } -1 < \alpha < 0, \end{cases} \qquad \text{with } \omega_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

where $*$ denotes the Laplace convolution: $(f * g)(t) = \int_0^t f(t-s)g(s)\,ds$. For $\alpha = 0$, the problem (1.1) reduces to a classical initial-boundary value problem for the heat equation, and

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for other values of α, (1.1) models other physical phenomena, cf. McLean and Thomée (to appear). For $\alpha \neq 0$ the evolution equation is non-local in time, and we note the limiting values

$$\lim_{\alpha\uparrow 1} J^\alpha v(t) = \int_0^t v(s)\,ds \quad \text{and} \quad \lim_{\alpha\downarrow -1} J^\alpha v(t) = v'(t).$$

The present paper is a continuation of the paper quoted above in which we proposed three methods for discretization in time of (1.1), and analyzed them a Banach space framework. We then applied our results to fully discrete methods based on a finite element approximation in the spatial variable, and derived bounds for the L_2-norm of the error at time t. Our present work is devoted to the technically somewhat more challenging derivation of error estimates in the maximum-norm.

Denoting the Laplace transform of u with respect to t by

$$\widehat{u}(z) = \mathcal{L}(u)(z) = \int_0^\infty e^{-zt} u(t)\,dt, \tag{1.2}$$

we find easily from (1.1), since $\mathcal{L}(\omega_\alpha)(z) = z^{-\alpha}$ for $\alpha > 0$, that, with I the identity operator,

$$(zI + z^{-\alpha}A)\widehat{u}(z) = g(z) := u_0 + \widehat{f}(z). \tag{1.3}$$

Instead of using time stepping for the numerical solution of (1.1), our approach is to solve (1.3) for $\widehat{u}(z)$, and use this to represent the solution of (1.1) as a modified inverse Laplace transform, which is then approximated by quadrature. In our basic method it is required that $\widehat{f}(z)$ is bounded in a sector in the complex plane containing the real axis and extending into the left half-plane. This method has a $O(e^{-cN})$ convergence rate on any closed time interval bounded away from 0 and ∞, where N is the number of quadrature points. A second method, inspired by Gavrilyuk and Makarov (2005), uses a modification of the inverse Laplace transform and has a $O(e^{-c\sqrt{N}})$ convergence rate, which now holds uniformly down to $t = 0$. A third method has a similar error behavior, but has the additional advantage that it avoids the use of $\widehat{f}$ and so is more generally applicable. Each of the three methods requires the solution of N elliptic problems, which may be solved in parallel. For the first two methods, these elliptic problems are independent of t, and thus give the approximate solution of (1.1) for all t.

In Section 2 below we discuss the initial-boundary value problem (1.1) in an abstract setting, with A a sectorial operator in a Banach space $\mathcal{B}$, and in Section 3 we review the abstract theory from McLean and Thomée (to appear) for our three time discretization methods.

In Section 4 we show error bounds in the maximum-norm for a spatially semidiscrete version of (1.1) based on piecewise linear finite elements. This analysis depends on known resolvent estimates in the maximum-norm, for $-\Delta$ by Stewart (1974), and for its finite element analogue by Bakaev, Thomée & Wahlbin (2003). In Section 5 we use our general theory to derive error bounds in maximum-norm for fully discrete schemes obtained by applying the time discretization methods of Section 3 to the spatially semidiscrete solution. In Section 6 we illustrate our results by some numerical examples, with special focus on the third method, which does not utilize $\widehat{f}$.

In an Appendix we include, for the convenience of the reader, a proof of the maximum-norm resolvent estimate for $-\Delta$, which avoids the technical complexity of the much more general analysis in Stewart (1974), and extends the result from $\mathcal{C}$ to L_∞. The proof has been provided to us by Michel Crouzeix, whose generous help is gratefully acknowledged.

2. The continuous problem

In this section we discuss our initial-boundary value problem (1.1) in an abstract form, with A a closed, sectorial operator in a Banach space $\mathcal{B}$ with norm $\|\cdot\|$. We thus require the spectrum of A to lie in a sector

$$\Sigma_\varphi = \{z : \ |\arg z| < \varphi\}, \quad \text{with } 0 < \varphi < (1-\alpha)\,\frac{\pi}{2}, \tag{2.1}$$

and that for some constant $M \geqslant 1$, the resolvent estimate

$$\|(zI - A)^{-1}\| \leqslant \frac{M}{1+|z|}, \quad \text{for } z \notin \Sigma_\varphi, \tag{2.2}$$

holds in the operator norm induced by the norm $\|\cdot\|$ in $\mathcal{B}$. From (1.3), it follows that

$$\widehat{u}(z) = \widehat{\mathcal{E}}(z)g(z), \quad \text{where } \ \widehat{\mathcal{E}}(z) := z^\alpha(z^{1+\alpha}I + A)^{-1}, \tag{2.3}$$

and from the resolvent estimate (2.2) we obtain

$$\|\widehat{\mathcal{E}}(z)\| \leqslant \frac{M|z|^\alpha}{1+|z|^{1+\alpha}} \leqslant \frac{M}{|z|}, \quad \text{for } z \in \Sigma_{\overline{\beta}}, \ \ \overline{\beta} := \min\Big(\pi, \frac{\pi-\varphi}{1+\alpha}\Big). \tag{2.4}$$

The condition on φ in (2.1) ensures that $\frac{1}{2}\pi < \overline{\beta} \leqslant \pi$.

For any $\omega > 0$, let Γ be a curve in the sector $\Sigma_{\bar\beta}$ which is homotopic with the vertical line $\Gamma_0 = \{z : \operatorname{Re} z = \omega\}$ and extends into the left half-plane. We assume that the Laplace transform $\widehat{f}(z)$, defined according to (1.2), may be continued as an analytic function to the closed subdomain of $\Sigma_{\bar\beta}$ to the right of and including Γ. By the Laplace inversion formula we may then write

$$u(t) = \frac{1}{2\pi i}\int_\Gamma e^{zt} w(z)\,dz, \quad \text{where } w(z) = \widehat{\mathcal{E}}(z)g(z). \tag{2.5}$$

Taking $f \equiv 0$ in (1.3), so that $g(z) = u_0$ in (2.3), we see that the solution operator for the homogeneous case of (1.1) is given by

$$\mathcal{E}(t)u_0 = \frac{1}{2\pi i}\int_\Gamma e^{zt}\widehat{\mathcal{E}}(z)u_0\,dz, \quad \text{for } t > 0. \tag{2.6}$$

We recall that, if A is densely defined, then (2.2) implies that $-A$ generates an analytic semigroup $E(t)$ on $\mathcal{B}$, which may be represented by (2.6) with $\alpha = 0$, i.e.,

$$E(t)v = \frac{1}{2\pi i}\int_\Gamma e^{zt}\,(z\,I + A)^{-1}\,v\,dz, \quad \text{for } t > 0, \tag{2.7}$$

see, e.g., Pazy (1983). Even when A is not densely defined, (2.7) defines an operator with many of the properties of an analytic semigroup, the essential difference being that $E(t)v \to v$ for $t \to 0$ if and only if $v \in \overline{D(A)} \neq \mathcal{B}$, see da Prato and Sinestrari (1987). In this case, the restriction of $E(t)$ to $\mathcal{B}_0 = \overline{D(A)}$ is a standard analytic semigroup on $\mathcal{B}_0$.

Even though the operator in (2.6) is not a semigroup for $\alpha \neq 0$, it still has the stability and smoothing properties of an analytic semigroup described in the following theorem.

THEOREM 2.1 For $-1 < \alpha < 1$ the solution operator (2.6) for the homogenous equation satisfies, for $t > 0$,

$$\|A^\sigma \mathcal{E}^{(q)}(t)v\| \leqslant C_{\bar\beta,q} M t^{-(1+\alpha)\sigma - q}\|v\|, \quad \text{for } q \geqslant 0,\ \sigma = 0,\, 1, \tag{2.8}$$

and

$$\|\mathcal{E}^{(q)}(t)v\| \leqslant C_{\bar\beta,q} M t^{1+\alpha-q}\|Av\|, \quad \text{for } q \geqslant 1. \tag{2.9}$$

Further,

$$\|v - \mathcal{E}(t)v\| \leqslant C_{\bar\beta} M t^{1+\alpha}\|Av\|. \tag{2.10}$$

Proof. From (2.6) we see that

$$\mathcal{E}^{(q)}(t)v = \frac{1}{2\pi i}\int_\Gamma z^q e^{zt}\widehat{\mathcal{E}}(z)v\,dz \quad \text{for } t > 0 \quad \text{and } q \geqslant 0,$$

and since $A\widehat{\mathcal{E}}(z) = z^\alpha\, A(z^{1+\alpha}I + A)^{-1} = z^\alpha(I - z\widehat{\mathcal{E}}(z))$, (2.4) with $M \geqslant 1$ implies

$$\|A\widehat{\mathcal{E}}(z)\| \leqslant 2M\,|z|^\alpha \quad \text{for } z \in \Sigma_{\bar\beta}. \tag{2.11}$$

Hence, for $\sigma = 0,\, 1$, we have

$$\|A^\sigma\widehat{\mathcal{E}}(z)\| \leqslant CM|z|^{(1+\alpha)\sigma-1}. \tag{2.12}$$

We now choose $\Gamma = \Gamma_t^0 \cup \Gamma_t^\infty$ where Γ_t^0 is the circular arc parameterised by $z = t^{-1}e^{i\theta}$ for $|\theta| \leqslant \bar\beta$ and $\Gamma_t^\infty = \{z = r\,e^{\pm i\bar\beta},\ r > 1/t\}$. Since $\operatorname{Re} e^{\pm i\bar\beta} = \cos\bar\beta < 0$, we find using the substitution $r = t^{-1}\rho$ that

$$\begin{aligned}\|A^\sigma\mathcal{E}^{(q)}(t)v\| &\leqslant \frac{M}{2\pi}\bigg(2\int_{1/t}^\infty r^q e^{rt\cos\bar\beta} r^{(1+\alpha)\sigma}\,\frac{dr}{r} + \int_{-\bar\beta}^{\bar\beta} t^{-q}e^{\cos\theta}t^{-(1+\alpha)\sigma}\,d\theta\bigg)\|v\| \\ &= \frac{M}{\pi}\,t^{-q-(1+\alpha)\sigma}\bigg(\int_1^\infty \rho^{q+(1+\alpha)\sigma}e^{\rho\cos\bar\beta}\,\frac{d\rho}{\rho} + \int_0^{\bar\beta} e^{\cos\theta}\,d\theta\bigg)\|v\|,\end{aligned}$$

which shows (2.8). Next, we observe that if we set

$$\widehat{\mathcal{E}}^0(z) = \widehat{\mathcal{E}}(z) - z^{-1}I = -z^{-1-\alpha}\widehat{\mathcal{E}}(z)\,A, \tag{2.13}$$

then, since $(2\pi i)^{-1}\int_\Gamma e^{zt}\,z^{-1}\,dz = 1$, we have

$$\mathcal{E}(t)v - v = \frac{1}{2\pi i}\int_\Gamma e^{zt}\big(\widehat{\mathcal{E}}(z) - z^{-1}\big)v\,dz = \frac{1}{2\pi i}\int_\Gamma e^{zt}\widehat{\mathcal{E}}^0(z)v\,dz. \tag{2.14}$$

Here, by (2.4),

$$\|\widehat{\mathcal{E}}^0(z)v\| \leqslant CM|z|^{-(2+\alpha)}\|Av\|,$$

and hence, for $q \geqslant 1$,

$$\begin{aligned}\|\mathcal{E}^{(q)}(t)v\| &\leqslant CM\bigg(\int_{1/t}^\infty r^q e^{rt\cos\bar\beta}r^{-(1+\alpha)}\,\frac{dr}{r} + \int_{-\bar\beta}^{\bar\beta} t^{-q}e^{\cos\theta}t^{1+\alpha}\,d\theta\bigg)\|Av\| \\ &= CM\,t^{(1+\alpha)-q}\bigg(\int_1^\infty \rho^{q-(1+\alpha)}e^{\rho\cos\bar\beta}\,\frac{d\rho}{\rho} + \int_0^{\bar\beta} e^{\cos\theta}\,d\theta\bigg)\|Av\|,\end{aligned}$$

which shows (2.9). In the same way (2.10) follows using (2.14) □

Note, in particular, that the solution $u(t) = \mathcal{E}(t)u_0$ of the homogeneous case of (1.1) belongs to $D(A)$ for $t > 0$. It also follows from (2.10) that $\mathcal{E}(t)u_0 \to u_0$ as $t \to 0$ if $u_0 \in \overline{D(A)}$. We shall later have reason to apply the following regularity result for the homogeneous problem.

Corollary 2.1 Let $-1 < \alpha < 1$. If $f \equiv 0$ then the solution of (1.1) satisfies

$$\int_0^t \|Au'(s)\|\,ds \leqslant C_{\alpha,\bar\beta} M\, t^{1+\alpha}\|A^2u_0\|, \quad \text{for } t > 0. \tag{2.15}$$

Proof. Theorem 2.1 gives

$$\|Au'(t)\| = \|A\mathcal{E}'(t)u_0\| = \|\mathcal{E}'(t)Au_0\| \leqslant C_{\bar\beta} M\, t^{\alpha}\|A^2u_0\|.$$

□

Using an interpolation space argument, the regularity requirement for u_0 in (2.15) may be somewhat reduced. We shall not insist on the details.

Since the inverse Laplace transform of $\widehat{\mathcal{E}}(z)\widehat{f}(z)$ is the convolution of $\mathcal{E}(t)$ and $f(t)$, one may show the Duhamel formula

$$u(t) = \mathcal{E}(t)u_0 + \int_0^t \mathcal{E}(t-s)f(s)\,ds. \tag{2.16}$$

This expression for $u(t)$ is well defined even without knowledge of $\widehat{f}$, and may be considered to define a mild solution of (1.1). By the case $\sigma = q = 0$ of (2.8), it follows that this mild solution is stable in the sense that

$$\|u(t)\| \leqslant C_{\bar\beta} M\Big(\|u_0\| + \int_0^t \|f(s)\|\,ds\Big), \quad \text{for } t \geqslant 0. \tag{2.17}$$

We now consider the inhomogenous problem (1.1) with zero initial data, $u_0 = 0$. In our applications in the present paper it is desirable to avoid spatial regularity requirements for $f(t)$ for $t > 0$, so as not to impose unnatural restrictions on the boundary values of this function. We shall therefore show two regularity results, for $\alpha < 0$ and $\alpha \geqslant 0$, respectively, for the quantity bounded in Corollary 2.1 and which require no "spatial" regularity of $f(t)$ for $t > 0$.

Theorem 2.2 For $-1 < \alpha < 0$, the solution of the inhomogeneous problem (1.1), with $u_0 = 0$, satisfies

$$\int_0^t \|Au'(s)\|\,ds \leqslant C_{\alpha,\bar\beta} M\, t^{|\alpha|}\Big(\|f(0)\| + \int_0^t \|f'(s)\|\,ds\Big), \quad \text{for } t \geqslant 0.$$

Proof. Since $Au(t) = \int_0^t A\mathcal{E}(s)f(t-s)\,ds$ we see that

$$Au'(t) = A\mathcal{E}(t)f(0) + \int_0^t A\mathcal{E}(s)f'(t-s)\,ds,$$

and hence, by Theorem 2.1,

$$\|Au'(t)\| \leqslant C_{\bar\beta} M\Big(t^{-(1+\alpha)}\|f(0)\| + \int_0^t s^{-(1+\alpha)}\|f'(t-s)\|\,ds\Big).$$

Since $1+\alpha<1$, we find by integration

$$\int_0^t \|Au'(s)\|\,ds \leqslant \frac{C_{\bar\beta}M}{|\alpha|}\,t^{-\alpha}\|f(0)\| + C_{\bar\beta}M\int_0^t \tau^{-(1+\alpha)}\int_\tau^t \|f'(s-\tau)\|\,ds\,d\tau,$$

which is bounded as claimed. □

THEOREM 2.3 Let $u_0=0$ and assume $f(0)\in D(A)$. For $0\leqslant\alpha<1$ the solution of the inhomogeneous problem (1.1) satisfies, for $t\geqslant 0$,

$$\int_0^t \|Au'(s)\|\,ds \leqslant C_{\alpha,\bar\beta}M\,t\,\|Af(0)\| + C_{\alpha,\bar\beta}M\,t^{1-\alpha}\Big(\|f'(0)\| + \int_0^t \|f''(s)\|\,ds\Big).$$

Proof. From the identity $\widehat{f''}(z) = z\widehat{f'}(z) - f'(0) = z^2\widehat{f}(z) - zf(0) - f'(0)$ we see that

$$Au'(t) = \frac{1}{2\pi i}\int_\Gamma e^{zt}A\widehat{\mathcal{E}}(z)\big(f(0)+z^{-1}f'(0)+z^{-1}\widehat{f''}(z)\big)\,dz.$$

Thus, if we define the linear operator

$$B(t)v = \frac{1}{2\pi i}\int_\Gamma e^{zt}z^{-1}A\widehat{\mathcal{E}}(z)v\,dz,$$

then

$$Au'(t) = A\mathcal{E}(t)f(0) + B(t)f'(0) + \int_0^t B(s)\,f''(t-s)\,ds. \tag{2.18}$$

Taking $\sigma=q=0$ and $v=Af(0)$ in (2.8), we see that

$$\int_0^t \|A\mathcal{E}(s)f(0)\|\,ds \leqslant C_{\bar\beta}M\,t\,\|Af(0)\|.$$

Further, using (2.11), we have, by the method of proof used for Theorem 2.1,

$$\|B(t)\| \leqslant CM\Big(\int_{1/t}^\infty e^{rt\cos\bar\beta}r^\alpha\,\frac{dr}{r} + \int_{-\bar\beta}^{\bar\beta} e^{\cos\theta}t^{-\alpha}\,d\theta\Big) \leqslant C_{\bar\beta}Mt^{-\alpha}.$$

This bounds the last two terms in (2.18) as desired. □

COROLLARY 2.2 Assume that $u_0=0$. The solution $u(t)$ of the inhomogeneous equation (1.1) belongs to $D(A)$ for $t\geqslant 0$, in the case $-1<\alpha<0$ if $f'\in L_1(0,t;\mathcal{B})$, and in the case $0\leqslant\alpha<1$ if $f(0)\in D(A)$ and $f''\in L_1(0,t;\mathcal{B})$.

Proof. Since $\|Au(t)\|\leqslant\int_0^t\|Au'(s)\|\,ds$, this follows at once from Theorems 2.2 and 2.3. □

For completeness we remark that conditions for the function $u(t)$ of Corollary 2.2 to belong to $D(A)$ may also be expressed in terms of $\widehat{f}$. Here and below we use the notation

$$\|g\|_Z := \|g\|_{\mathcal{B},Z} := \sup_{z\in Z}\|g(z)\|, \quad \|g\|_{Z,\gamma} := \|g\|_{\mathcal{B},Z,\gamma} := \sup_{z\in Z}\big((1+|z|)^\gamma\|g(z)\|\big), \quad \text{for } \gamma\geqslant 0.$$

With regard to the latter norm, recall that the decay of the Laplace transform $\widehat{f}(z)$ reflects the regularity of f in time, and the order to which $f(t)$ vanishes as $t\to 0$; consider, for instance, $\mathcal{L}\{1\}=z^{-1}$ and $\mathcal{L}\{t\}=z^{-2}$. We cannot expect $\|\widehat{f}\|_{\Sigma_\beta,\gamma}$ to be finite for $\gamma>1$ if $f(0)\neq 0$.

THEOREM 2.4 Setting $\gamma = 1+\alpha$, we have for the solution $u(t)$ of the inhomogeneous equation (1.1) with $u_0 = 0$, for $t \geqslant 0$,

$$\|Au(t)\| \leqslant C_{\bar\beta} M t^{-\gamma} \|\widehat{f}\|_{\Sigma_{\bar\beta}}, \quad \text{and} \quad \|Au(t)\| \leqslant C_{\bar\beta} M \|\widehat{f}\|_{\Sigma_{\bar\beta},\gamma}.$$

Proof. By (2.6) we have

$$Au(t) = \frac{1}{2\pi i}\int_\Gamma e^{zt} A\widehat{\mathcal{E}}(z)\,\widehat{f}(z)\,dz.$$

Hence, in view of (2.11),

$$\|Au(t)\| \leqslant CM \int_\Gamma |z|^{(1-s)\gamma} |e^{zt}|\,\frac{|dz|}{|z|}\,\|\widehat{f}(z)\|_{\Gamma,s\gamma}, \quad s = 0,1.$$

Choosing Γ as in the proof of Theorem 2.1 we may bound the integral over Γ by $Ct^{-(1-s)\gamma}$, which shows the theorem. □

3. Time discretization

In this section we review the analysis from McLean and Thomée (to appear) of our three methods for discretization in time of the initial-value problem (1.1) in the Banach space setting. We formulate the three methods in detail and state the error bounds without proofs, in a form convenient for our applications. Although in our earlier paper we assumed that the operator A was densely defined in $\mathcal{B}$, this was not used in these proofs, so that the results remain valid in our present context.

We select an integration contour Γ in (2.5), such that $\widehat{f}(z)$, and thus also $g(z)$, is analytic on and to the right of Γ, and then apply a quadrature formula to (2.5). Specifically, we assume that $\widehat{f}(z)$ is analytic in $\overline{\Sigma}^\omega_\beta$, where

$$\Sigma^\omega_\beta := \omega + \Sigma_\beta \subset \Sigma_{\bar\beta}, \quad \text{with } \omega \geqslant 0,\ \beta \in (\tfrac12\pi, \bar\beta] \text{ and } \beta < \pi,$$

and choose Γ to be a curve of the form

$$z = z(\xi) := \omega + \lambda(1-\sin(\delta - i\xi)) = \omega + \lambda(1-\sin\delta\,\cosh\xi) + i\lambda\cos\delta\,\sinh\xi,\ \ \xi\in\mathbb{R}, \tag{3.1}$$

where the scaling factor λ is positive and δ will be assumed below to satisfy $\delta \in (0, \beta - \frac12\pi)$. Writing $z = x+iy$ we find that Γ is the left branch of a hyperbola which cuts the real axis at the point $z = \omega + \lambda(1-\sin\delta)$ and has asymptotes $y = \pm(x-\omega-\lambda)\cot\delta$. Thus, the above condition on δ ensures that Γ lies in the sector Σ^ω_β and crosses into the left half-plane. The assumption that $\beta < \pi$ ensures $|z| \geqslant c_\beta |z-\omega|$ for $z \in \Sigma^\omega_\beta$, where $c_\beta = \sin\beta > 0$.

We may therefore represent $u(t)$ as an integral with respect to ξ,

$$u(t) = \int_{-\infty}^{\infty} v(\xi,t)\,d\xi, \quad \text{where } v(\xi,t) = \frac{1}{2\pi i}\,e^{z(\xi)t} w\big(z(\xi)\big) z'(\xi).$$

The factor $e^{z(\xi)t}$ in the integrand has modulus $e^{\omega t} e^{\lambda(1-\sin\delta\cosh\xi)t}$ and so exhibits a double-exponential decay as $|\xi| \to \infty$, for any fixed $t > 0$.

For our first approximation method, we choose a quadrature step k and apply an equal weight quadrature rule

$$Q_N(v) := k \sum_{j=-N}^{N} v(\xi_j) \approx \int_{-\infty}^{\infty} v(\xi)\, d\xi, \quad \text{with } \xi_j := jk. \tag{3.2}$$

In this way, we obtain an approximate solution to (1.1) of the form

$$U_N(t) := Q_N(v(\cdot,t)) = \frac{k}{2\pi i} \sum_{j=-N}^{N} e^{z_j t} w(z_j)\, z_j', \quad \text{where } z_j := z(\xi_j),\ z_j' := z'(\xi_j). \tag{3.3}$$

To compute $U_N(t)$ we must therefore solve the $2N+1$ elliptic equations

$$(z_j^{1+\alpha} I + A) w(z_j) = z_j^{\alpha} g(z_j), \quad \text{for } |j| \leqslant N. \tag{3.4}$$

These equations are independent and hence may be solved in parallel. Note that the $w(z_j) \in \mathcal{B}$ determine the approximate solution (3.3) for all $t > 0$. We remark that the definition (3.3) depends on the choice of the curve Γ, even though the representation (2.5) does not.

To analyze the quadrature error, we extend the parametric representation (3.1) of Γ to a conformal mapping

$$z = \Phi(\zeta) = \omega + \lambda(1 - \sin(\delta - i\zeta)),$$

from $Y_r = \{\, \zeta : |\operatorname{Im}\zeta| \leqslant r \,\}$ onto $S_r = \{\, \Phi(\zeta) : \zeta \in Y_r \,\} \supset \Gamma$. In fact, Φ maps the line $\operatorname{Im}\zeta = \eta$ onto the left branch of the hyperbola (3.1), with δ replaced by $\delta + \eta$, so S_r is bounded on the left and right by the left branches of the hyperbolas corresponding to $\operatorname{Im}\zeta = \pm r$. To ensure that $S_r \subset \Sigma_\beta^\omega$ and that $\operatorname{Re} z \to -\infty$ for $|z| \to \infty$ with $z \in S_r$, we shall require throughout that $0 < \delta - r < \delta + r < \beta - \frac{1}{2}\pi$, or equivalently, see Figure 1, that $0 < r < \min\bigl(\delta, \beta - \frac{1}{2}\pi - \delta\bigr)$.

In (McLean and Thomée, to appear, Theorem 3.1) we showed the following $O(e^{-cN})$ error bound for $t > 0$, with specific $\lambda \propto N$ and $k \propto 1/N$. Here and below we write $\bar{r} = 2\pi r$ and $\ell(s) = \max(1, \log(1/s))$.

THEOREM 3.1 Let $u(t)$ be the solution of (1.1), and let $[t_0, T] \subset (0, \infty)$. Let $0 < \theta < 1$, and define $b > 0$ by $\cosh b = 1/(\theta\tau \sin\delta)$, where $\tau = t_0/T$. Let the scaling factor in Γ be $\lambda = \theta\bar{r}N/(bT)$, and $U_N(t)$ be the approximate solution of (1.1) defined in (3.3), with $k = b/N \leqslant \bar{r}/\log 2$. Then we have

$$\|U_N(t) - u(t)\| \leqslant CM\, e^{\omega t} \ell(\rho_r N) e^{-\mu N} \bigl(\|u_0\| + \|\widehat{f}\|_{\Sigma_\beta^\omega}\bigr), \quad \text{for } t \in [t_0, T],$$

where $\mu = \bar{r}(1-\theta)/b$, $\rho_r = \theta\bar{r}\tau \sin(\delta - r)/b$, and $C = C_{\delta,r,\beta}$.

One drawback of this result is that it says nothing about the error for $0 \leqslant t \leqslant t_0$. Numerical results reported in McLean and Thomée (to appear) confirm that in practice the accuracy of $U_N(t)$ deteriorates as $t \to 0$. For the case $\alpha = 0$ of a parabolic partial differential equation, Gavrilyuk and Makarov (2005) modified the integrand in the representation formula (2.5) in a way that yielded a $O(e^{-c\sqrt{N}})$ convergence rate, uniformly down to $t = 0$, provided the data

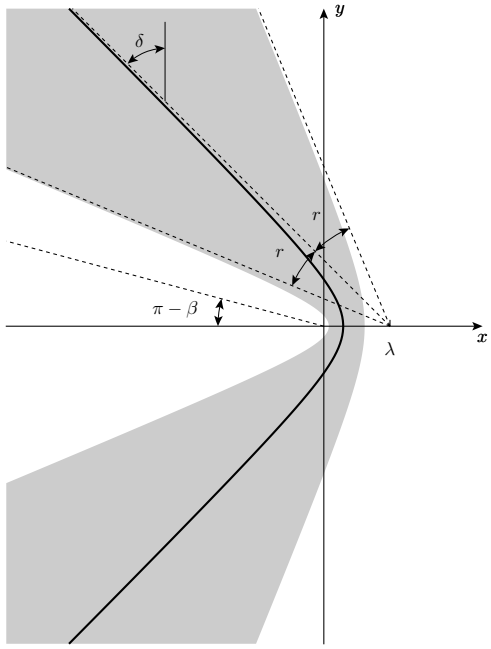

FIG. 1. The region S_r (shaded) and the contour Γ for $\omega = 0$.

possess some "spatial" regularity, and this idea was used in McLean and Thomée (to appear) also when $\alpha \neq 0$. Specifically, we note that

$$\mathcal{L}(u_0 + F(t)) = z^{-1}\big(u_0 + \widehat{f}(z)\big) = z^{-1}g(z), \quad \text{where } F(t) := \int_0^t f(s)\,ds,$$

so that, with $\widehat{\mathcal{E}}^0(z)$ defined in (2.13), we may rewrite (2.5) as

$$u(t) = u_0 + F(t) + \frac{1}{2\pi i}\int_\Gamma e^{zt}\widehat{\mathcal{E}}^0(z)g(z)dz. \tag{3.5}$$

Setting

$$w^0(z) := \widehat{\mathcal{E}}^0(z)g(z) = w(z) - z^{-1}g(z), \tag{3.6}$$

and using the parametric representation (3.1) of Γ, with λ appropriately chosen in the modified integral in (3.5), we now have

$$u(t) = u_0 + F(t) + \int_{-\infty}^{\infty} v^0(\xi, t)\,d\xi, \quad \text{where } v^0(\xi, t) := \frac{1}{2\pi i}e^{z(\xi)t}w^0(z(\xi))\,z'(\xi).$$

Applying again the quadrature rule (3.2) we obtain our second approximate solution to (1.1),

$$U_N^0(t) := u_0 + F(t) + \frac{k}{2\pi i}\sum_{j=-N}^{N} e^{z_j t}w^0(z_j)\,z_j', \quad \text{for } t \geqslant 0. \tag{3.7}$$

Once again, to compute this approximate solution, we must first obtain the values of $w(z_j)$ for $|j| \leqslant N$ by solving the elliptic equations (3.4), and then use (3.6) to find the $w^0(z_j)$. The error bound of (McLean and Thomée, to appear, Theorem 3.2) then contains the following for our modified method (3.7), valid uniformly down to $t = 0$.

THEOREM 3.2 Let $u(t)$ be the solution of (1.1) and suppose $0 < \gamma \leqslant 1 + \alpha$. Define Γ by (3.1) with $\lambda = \gamma/(\kappa T)$, where $\kappa = 1 - \sin(\delta - r)$. Let $U_N^0(t)$ be the approximate solution of (1.1) from (3.7), with $k = \sqrt{\bar{r}/(\gamma N)} \leqslant \bar{r}/\log 2$. Then we have, with $C = C_{\delta,r,\beta}$,

$$\|U_N^0(t) - u(t)\| \leqslant CMe^{\omega t}\gamma^{-1}T^{\gamma}e^{-\sqrt{\bar{r}\gamma N}}\,(\|A\,u_0\| + \|\widehat{f}\|_{\Sigma_\beta^\omega,\gamma}), \quad \text{for } t \leqslant T.$$

A serious restriction in the application of these two schemes is that they require the Laplace transform $\widehat{f}(z)$ to exist, to be computable for each $z \in \Gamma$, and to be bounded on Σ_β^ω.

We therefore consider a third alternative, based on the application of Duhamel's formula (2.16), which does not have this disadvantage. Defining

$$g(z,t) := e^{zt}u_0 + \int_0^t e^{z(t-s)}f(s)\,ds,$$

we find, by substituting the integral representation (2.6) into (2.16), that

$$u(t) = \frac{1}{2\pi i}\int_\Gamma e^{zt}\widehat{\mathcal{E}}(z)u_0\,dz + \int_0^t \frac{1}{2\pi i}\int_\Gamma e^{z(t-s)}\widehat{\mathcal{E}}(z)f(s)\,dz\,ds = \frac{1}{2\pi i}\int_\Gamma \widehat{\mathcal{E}}(z)g(z,t)\,dz.$$

This means that, compared to (2.5), we have restricted the integration in the definition of $\widehat{f}(z)$ to $(0,t)$, which is consistent with the fact that $u(t)$ only depends on f over this interval. Note that we have included the factor e^{zt} in the definition of $g(z,t)$ to avoid floating-point overflow when $\operatorname{Re} z$ is large and negative. In general, the integrand $\widehat{\mathcal{E}}(z)g(z,t)$ does not exhibit a double-exponential decay for $z = z(\xi)$ and $|\xi| \to \infty$, so we cannot proceed as in our first method but must instead modify the integral representation of $u(t)$ as in our second method. Since

$$\frac{1}{2\pi i}\int_\Gamma z^{-1}g(z,t)\,dz = \operatorname*{res}_{z=0}\frac{g(z,t)}{z} = g(0,t) = u_0 + F(t),$$

it follows that

$$u(t) = u_0 + F(t) + \frac{1}{2\pi i}\int_\Gamma \widehat{\mathcal{E}}^0(z)g(z,t)\,dz,$$

which is similar to (3.5). Setting $w(z,t) := \widehat{\mathcal{E}}(z)g(z,t)$ and

$$\widetilde{w}(z,t) := \widehat{\mathcal{E}}^0(z)g(z,t) = w(z,t) - z^{-1}g(z,t), \tag{3.8}$$

and using once again the representation (3.1) for Γ, we obtain

$$u(t) = u_0 + F(t) + \int_{-\infty}^{\infty}\tilde{v}(\xi,t)\,d\xi, \quad \text{where } \tilde{v}(\xi,t) = \frac{1}{2\pi i}\widetilde{w}\big(z(\xi),t\big)z'(\xi).$$

The quadrature rule (3.2) now gives our third approximate solution

$$\widetilde{U}_N(t) := u_0 + F(t) + \frac{k}{2\pi i}\sum_{j=-N}^{N}\widetilde{w}(z_j,t)z_j', \tag{3.9}$$

where the $\widetilde{w}(z_j,t)$ are obtained by first solving the equations

$$(z_j^{1+\alpha}I + A)w(z_j,t) = z_j^{\alpha}g(z_j,t), \quad \text{for } |j| \leqslant N, \tag{3.10}$$

and then using (3.8). In contrast to the elliptic equations (3.4) arising in the previous two schemes, the right hand sides in (3.10), and hence also the solutions, now depend on t. The equations (3.10) are independent for different j and t, so again we can we solve the $2N+1$ equations in parallel, and also solve these systems for different t in parallel.

The following special case of (McLean and Thomée, to appear, Theorem 3.3) for our third method looks as for our second method, and is also valid uniformly down to $t=0$. Here we may use $\omega = 0$ and $\beta = \overline{\beta}$. We emphasize again that since this method does not depend on $\widehat{f}$ it is more generally applicable than the first two methods.

THEOREM 3.3 Let $u(t)$ be the solution of (1.1) and suppose $\gamma = \min(1, 1+\alpha)$. Define Γ by (3.1) with $\lambda = \gamma/(\kappa T)$, where $\kappa = 1 - \sin(\delta - r)$. Let $\widetilde{U}_N(t)$ be the approximate solution of (1.1) from (3.9), with $k = \sqrt{\bar{r}/(\gamma N)} \leqslant \bar{r}/\log 2$. Then, with $C = C_{\delta,r,\bar{\beta}}$, we have

$$\|\widetilde{U}_N(t) - u(t)\| \leqslant CM\gamma^{-1}T^{\gamma}e^{-\sqrt{\bar{r}\,\gamma\,N}}\Big(\|A\,u_0\| + \|f(0)\| + \int_0^t \|f'\|\,ds\Big) \quad \text{for } t \leqslant T.$$

4. Spatial discretization by finite elements

From now on we suppose that $A = -\Delta$ in a bounded convex domain $\Omega \subset R^d$ with smooth boundary $\partial\Omega$, under homogeneous Dirichlet boundary conditions. More precisely, we shall consider A as the closure of the standard Laplacian on $\dot{\mathcal{C}}^2 = \mathcal{C}^2(\bar{\Omega}) \cap \mathcal{C}_0(\bar{\Omega})$ in the Banach space $\mathcal{B} = L_\infty = L_\infty(\Omega)$, equipped with the norm $\|v\|_{L_\infty} = \text{ess sup}_{x\in\Omega}|v(x)|$. It follows from the estimate (4.6) below of Agmon, Douglis, and Nirenberg that $D(A) \supset W_p^2(\Omega) \cap \mathcal{C}_0(\bar{\Omega})$ for large p, and since it is clear that $D(A) \subset \mathcal{C}_0(\bar{\Omega})$ we have $\overline{D(A)} = \mathcal{C}_0(\bar{\Omega}) \neq L_\infty$. Thus, in order for $\mathcal{E}(t)u_0 \to u_0$ as $t \to 0$, it is required that $u_0 \in \mathcal{C}_0(\bar{\Omega})$.

In the Appendix we show that the resolvent estimate (2.2) holds for A, for arbitrarily small φ, and that $-A$ thus generates a generalized analytic semigroup given by (2.7). In particular, it follows that the stability and smoothness estimates of Section 2 are satisfied in this case. For given Ω the constant M in (2.2) depends only on φ, and, for simplicity, we think of φ as fixed below and let M be implicitly included in the other constant factors in our estimates.

To discretize in the spatial variable only, we use a family of quasiuniform partitions $\mathcal{T}_h = \{K\}$ of Ω into polyhedral elements K, indexed by h, the maximum diameter of the K. Let V_h denote the corresponding space of continuous piecewise linear functions vanishing on $\partial\Omega$,

$$V_h = \{\chi \in \mathcal{C}(\bar{\Omega}) : \ \chi\big|_K \text{ linear in } K, \ \chi = 0 \quad \text{on } \partial\Omega\},$$

and recall the approximation property

$$\inf_{\chi\in V_h}\big\{\|u-\chi\|_{L_\infty} + h\|\nabla(u-\chi)\|_{L_\infty}\big\} \leqslant Ch^2\|u\|_{\dot{W}^2_\infty}.$$

The spatially discrete problem is then to find $u_h(t) \in V_h$ such that

$$(\partial_t u_h, \chi) + J^\alpha(\nabla u_h, \nabla\chi) = (f,\chi), \quad \forall\chi \in V_h, \ t \geqslant 0, \qquad \text{with } u_h(0) = u_{0h}. \tag{4.1}$$

Here, as usual, $(\cdot,\cdot)$ denotes the inner product in $L_2(\Omega)$, and $u_{0h} \in V_h$ is a suitable approximation to u_0.

Introducing the discrete Laplacian $\Delta_h : V_h \to V_h$ defined by

$$-(\Delta_h \psi, \chi) = (\nabla\psi, \nabla\chi), \quad \text{for } \psi,\ \chi \in V_h,$$

the problem (4.1) is equivalent to

$$\partial_t u_h - J^\alpha \Delta_h u_h = P_h f(t), \quad \text{for } t \geqslant 0, \quad \text{with } u(0) = u_{0h},$$

where $P_h : L_2(\Omega) \to V_h$ is the orthogonal projector with respect to $(\cdot,\cdot)$. We shall now consider this as a problem of the form (1.1) in $\mathcal{B} = V_h$, with norm $\|\cdot\|_{L_\infty}$ and with $A = -\Delta_h$.

It is known from Bakaev, Thomée & Wahlbin (2003) that, when Ω is convex, with a smooth boundary, the maximum-norm resolvent estimate (2.2) holds for $A_h = -\Delta_h$ for arbitrarily small φ, with M independent of h. Thus, with $\Gamma \subset \Sigma_{\bar\beta}$, the results of Section 2 apply, and in particular, the analogue of (2.8) holds for the solution operator of the spatially discrete homogeneous equation,

$$\mathcal{E}_h(t)u_0 = \frac{1}{2\pi i}\int_\Gamma e^{zt}\widehat{\mathcal{E}}_h(z)u_{0h}\,dz, \quad \text{where} \quad \widehat{\mathcal{E}}_h(z) = z^\alpha(z^{1+\alpha}I + A_h)^{-1}.$$

We shall think of the family of approximating spaces V_h as given, and, as in the continuous case, let the constant in (2.2) be implicitly included in with other factors in our estimates. To establish error estimates for u_h we introduce the Ritz projector $R_h : H_0^1(\Omega) \cup \mathcal{C}(\bar\Omega) \to V_h$ defined by

$$(\nabla R_h v, \nabla\chi) = (\nabla v, \nabla\chi) = \sum_{K\in\mathcal{T}_h}\int_{\partial K} v\,\frac{\partial\chi}{\partial n}\,ds, \quad \forall\chi \in V_h.$$

We begin with a nonsmooth data error estimate for the semidiscrete problem, which was shown in (McLean and Thomée, 2004, Theorem 5.1) for $0 < \alpha < 1$. The argument for $-1 < \alpha \leqslant 0$ is the same, but for completeness and later reference we include the proof.

Theorem 4.1 Let $u_h(t)$ and $u(t)$ be the solutions of (4.1) and (1.1), with $\widehat{f}$ analytic in Σ_β^ω, and let $u_{0h} = P_h u_0$. Then we have, with $C = C_{\omega,\beta,T}$ and $\ell_h := \max\bigl(1, \log(1/h)\bigr)$,

$$\|u_h(t) - u(t)\|_{L_\infty} \leqslant \frac{Ct^{-1-\alpha}}{1+\alpha}\,h^2\ell_h^2\bigl(\|u_0\|_{L_\infty} + \|\widehat{f}\|_{L_\infty,\Sigma_\beta^\omega}\bigr), \quad \text{for } 0 < t \leqslant T.$$

Proof. With notation as above we may write, with $\Gamma = \partial\Sigma_\beta^\omega$,

$$u_h(t) - u(t) = \frac{1}{2\pi i}\int_\Gamma e^{tz}\,z^\alpha\,G_h(z)g(z)\,dz, \tag{4.2}$$

where

$$G_h(z) := z^{-\alpha}\bigl(\widehat{\mathcal{E}}_h(z)P_h - \widehat{\mathcal{E}}(z)\bigr) = (z^{1+\alpha}I + A_h)^{-1}P_h - (z^{1+\alpha}I + A)^{-1}.$$

We shall show below that

$$\|G_h(z)v\|_{L_\infty} \leqslant Ch^2\ell_h^2\|v\|_{L_\infty}, \quad \text{for } z \in \Sigma_\beta^\omega. \tag{4.3}$$

Assuming this, we have at once

$$\|u_h(t) - u(t)\|_{L_\infty} \leqslant Ch^2 \ell_h^2 \|g\|_{L_\infty, \Sigma_\beta^\omega} \int_\Gamma |e^{zt}|\, |z|^\alpha |dz|. \tag{4.4}$$

Setting $z = \omega + s\, e^{\pm i\beta}$ on Γ, we have $c\, s \leqslant |z| \leqslant s + \omega$ on Γ, since $c|z - \omega| \leqslant |z|$ for $z \in \Sigma_\beta^\omega$, with $c > 0$. Hence, $|z|^\alpha \leqslant Cs^\alpha$ for $-1 < \alpha \leqslant 0$, whereas $|z|^\alpha \leqslant \omega^\alpha + s^\alpha$ for $0 \leqslant \alpha < 1$, so

$$\int_\Gamma |e^{zt}|\, |z|^\alpha |dz| \leqslant Ce^{\omega t} \int_0^\infty e^{-st\cos\beta} \max\bigl(s^\alpha, \omega^\alpha + s^\alpha\bigr)\, ds \leqslant Ce^{\omega t}\bigl(t^{-1-\alpha} + \omega^\alpha t^{-1}\bigr).$$

Together with (4.4) this shows our claim.

It remains to show (4.3). For this purpose, we write, with $w = z^{1+\alpha}$, $\mathcal{R}(w) = (wI + A)^{-1}$, and $\mathcal{R}_h(w) = (wI + A_h)^{-1}$,

$$G_h(z) = \mathcal{R}_h(w)P_h - \mathcal{R}(w) = G_{h1}(z) + G_{h2}(z),$$

where

$$G_{h1}(z) := (P_h - I)\mathcal{R}(w) \quad \text{and} \quad G_{h2}(z) := \mathcal{R}_h(w)P_h - P_h\mathcal{R}(w).$$

To bound these two operators we use the fact that the piecewise linear interpolant $I_h : \mathcal{C}(\bar{\Omega}) \to V_h$ satisfies

$$\|I_h v - v\|_{L_\infty} \leqslant Ch^{2-d/p}\|v\|_{W_p^2}, \quad \text{for } p \geqslant d/2, \quad \text{if } v = 0 \text{ on } \partial\Omega. \tag{4.5}$$

This follows from the corresponding inequality for an individual element K of the partition $\mathcal{T}_h$, which in turn may be obtained by transforming to a reference element and using the Bramble–Hilbert lemma, together with an obvious estimate for u on $\Omega \setminus \Omega_h$. We also need the regularity estimate

$$\|v\|_{W_p^2} \leqslant Cp\|Av\|_{L_p}, \quad \text{for } 1 < p < \infty, \quad \text{if } v = 0 \text{ on } \partial\Omega, \tag{4.6}$$

from Agmon, Douglis & Nirenberg (1950). These authors state (4.6) without the explicit dependence on p, which may be traced through their proof to the Calderòn–Zygmund lemma. Together (4.5) and (4.6) show, with $p = \ell_h$,

$$\|I_h v - v\|_{L_\infty} \leqslant Ch^{2-d/p}p\|Av\|_{L_p} \leqslant Ch^2\ell_h\|Av\|_{L_\infty}. \tag{4.7}$$

To bound $G_{h1}(z)$ we note that $P_h v - v = (P_h - I)(v - I_h v)$ and recall the stability estimate $\|P_h v\|_{L_\infty} \leqslant C\|v\|_{L_\infty}$ from Douglas, Dupont and Wahlbin (1975). The operator $A\mathcal{R}(w)$ is bounded in L_∞, uniformly for $z \in \Sigma_\beta$, and hence

$$\|G_{h1}(z)v\|_{L_\infty} \leqslant C\bigl\|(I_h - I)\mathcal{R}(w)v\bigr\|_{L_\infty} \leqslant Ch^2\ell_h\|A\mathcal{R}(w)v\|_{L_\infty} \leqslant Ch^2\ell_h\|v\|_{L_\infty}. \tag{4.8}$$

To bound $G_{h2}(z)$, we make use of the identity $P_h A = A_h R_h$ by writing

$$G_{h2}(z) = \mathcal{R}_h(w)(P_h(wI + A) - (wI + A_h)P_h)\mathcal{R}(w) = \mathcal{R}_h(w)A_h(R_h - P_h)\mathcal{R}(w),$$

and deduce that, because $R_h - P_h = R_h(I - P_h)$,

$$G_{h2}(z) = \mathcal{R}_h(w)A_h R_h G_{h1}(z).$$

Since $(wI + A_h)^{-1}A_h$ is uniformly bounded on Σ_β, we have

$$\|G_{h2}(z)\|_{L_\infty} \leqslant C\|R_h G_{h1}(z)\|_{L_\infty}. \tag{4.9}$$

Recalling the maximum-norm stability estimate by Schatz and Wahlbin (1982) for the elliptic projection R_h,

$$\|R_h v\|_{L_\infty} \leqslant C\ell_h \|v\|_{L_\infty}, \quad \text{for } v \in \mathcal{C}(\bar{\Omega}), \tag{4.10}$$

it follows with the help of (4.8), since $G_{h1}(z)v \in \mathcal{C}_0(\bar{\Omega})$, that

$$\|G_{h2}(z)v\|_{L_\infty} \leqslant C\ell_h \|G_{h1}(z)v\|_{L_\infty} \leqslant Ch^2\ell_h^2\|v\|_{L_\infty}, \tag{4.11}$$

which completes the proof of (4.3). □

We now show a smooth data estimate valid uniformly down to $t = 0$.

THEOREM 4.2 Let $u_h(t)$ and $u(t)$ be the solutions of (4.1) and (1.1), with $\widehat{f}$ analytic in Σ_β^ω and $u_{0h} = P_h u_0$. Then we have, with $\gamma = 1 + \alpha$ and $C = C_\beta$,

$$\|u_h(t) - u(t)\|_{L_\infty} \leqslant Ch^2\ell_h^2\, e^{\omega t}\Big(\|\Delta u_0\|_{L_\infty} + \|\widehat{f}\|_{L_\infty,\Sigma_\beta^\omega,\gamma}\Big), \quad \text{for } t \geqslant 0.$$

Proof. We shall again use (4.2), choosing now $\Gamma = \omega + \Gamma_t^0 \cup \Gamma_t^\infty$, with Γ_t^0 and Γ_t^∞ as in the proof of Theorem 2.1. First note that by (4.8) and (4.11), and since $A = -\Delta$ commutes with $\mathcal{R}(w)$, we have

$$\|G_h(z)v\|_{L_\infty} \leqslant Ch^2\ell_h^2\|\mathcal{R}(w)Av\|_{L_\infty} \leqslant \frac{C\,\ell_h^2}{1+|z|^{1+\alpha}}\,\|\Delta v\|_{L_\infty}, \quad \text{for } z \in \Sigma_\beta^\omega.$$

Hence, in the case that $f(t) \equiv 0$, we have from (4.2)

$$\|u_h(t) - u(t)\|_{L_\infty} \leqslant Ch^2\,\ell_h^2 \int_\Gamma |e^{tz}|\,\frac{|dz|}{|z|}\,\|\Delta u_0\|_{L_\infty},$$

and, using $|z| \geqslant c|z-\omega|$ in Σ_β^ω, the result stated follows from

$$\int_{\Gamma_t^0} |e^{zt}|\frac{|dz|}{|z|} \leqslant C\int_{-\beta}^{\beta} d\theta \leqslant C \text{ and } \int_{\Gamma_t^\infty} |e^{zt}|\frac{|dz|}{|z|} \leqslant C\int_{1/t}^{\infty} e^{-st\cos\beta}\,\frac{ds}{s} \leqslant C.$$

To treat the term in $\widehat{f}$, we use (4.3) to find, for $z \in \Sigma_\beta^\omega$,

$$|z|^\alpha\|G_h(z)\widehat{f}(z)\|_{L_\infty} \leqslant \frac{Ch^2\ell_h^2|z|^\alpha}{(1+|z|)^\gamma}\,\|\widehat{f}\|_{L_\infty,\Sigma_\beta^\omega,\gamma} \leqslant \frac{Ch^2\ell_h^2}{|z|}\,\|\widehat{f}\|_{L_\infty,\Sigma_\beta^\omega,\gamma}.$$

In the same way as above this shows the result stated for $u_0 = 0$. □

For the purpose of application to the analysis of our third time discretization method we next show a classical type smooth data estimate that does not use $\widehat{f}(z)$.

THEOREM 4.3 Let $u_h(t)$ and $u(t)$ be the solutions of (4.1) and (1.1), respectively, with $u_{0h} = P_h u_0$. Then, with $C = C_{\bar{\beta}}$,

$$\|u_h(t) - u(t)\|_{L_\infty} \leqslant Ch^2\ell_h^2\Big(\|\Delta u_0\|_{L_\infty} + \int_0^t \|\Delta u_t(s)\|_{L_\infty}\,ds\Big) \quad \text{for } t \geqslant 0. \tag{4.12}$$

Proof. We write $u_h - u = (u_h - R_h u) + (R_h u - u) = \vartheta + \varrho$ and note that, by (4.10) and (4.7),

$$\|R_h v - v\|_{L_\infty} \leqslant C\ell_h \|I_h v - v\|_{L_\infty} \leqslant Ch^2 \ell_h^2 \|\Delta v\|_{L_\infty}.$$

Hence

$$\|\varrho(t)\|_{L_\infty} \leqslant Ch^2\ell_h^2 \|\Delta u(t)\|_{L_\infty} \leqslant Ch^2\ell_h^2 \Big(\|\Delta u_0\|_{L_\infty} + \int_0^t \|\Delta u_t(s)\|_{L_\infty}\, ds \Big).$$

Further, one easily finds

$$\partial_t \vartheta - J^\alpha \Delta_h \vartheta = -P_h \varrho_t, \quad \text{for } t > 0,$$

and hence, using Duhamel's principle (2.16) and the stability of $\mathcal{E}_h(t)$ and P_h in L_∞,

$$\|\vartheta(t)\|_{L_\infty} \leqslant C\Big(\|\vartheta(0)\|_{L_\infty} + \int_0^t \|\varrho_t(s)\|_{L_\infty} ds \Big) \leqslant Ch^2\ell_h^2 \Big(\|\Delta u_0\|_{L_\infty} + \int_0^t \|\Delta u_t(s)\|_{L_\infty}\, ds \Big).$$

Together these estimates show the theorem. □

Using the results of Corollary 2.1 and Theorems 2.2 and 2.3 for $A = -\Delta$, $u' = u_t$, etc., with $\|\cdot\| = \|\cdot\|_{L_\infty}$, immediately bounds the right hand side of (4.12) in the respective cases in terms of the data of (1.1), and thus provide a maximum-norm error estimate for the semidiscrete problem in (4.1).

5. Discretization in both time and space

In this section we present some error bounds for the fully discrete methods obtained by applying our three time discretization methods to the spatially semidiscrete problem (4.1). The fully discrete solution $U_{N,h}(t)$ obtained by application of our first method (3.3) to (4.1), with $u_{0h} = P_h u_0$, is thus defined by

$$U_{N,h}(t) := \frac{k}{2\pi i} \sum_{j=-N}^{N} e^{z_j t} w_h(z_j)\, z_j', \quad \text{with } w_h(z) = \widehat{\mathcal{E}}_h(z) P_h g(z). \tag{5.1}$$

To find $U_{N,h}(t)$ for a range of values of t it is now required to solve the discrete elliptic problems to find $w_h(z_j) \in V_h$, for $|j| \leqslant N$, such that

$$z_j^{1+\alpha}\big(w_h(z_j), \chi\big) + \big(\nabla w_h(z_j), \nabla \chi\big) = z_j^\alpha \big(g(z_j), \chi\big), \quad \forall\, \chi \in V_h. \tag{5.2}$$

As before, these problems may be solved in parallel.

Combining Theorems 3.1 and 4.1 immediately shows the following error estimate for our fully discrete method.

THEOREM 5.1 Let $u(t)$ be the solution of (1.1), and let $U_{N,h}(t)$ be defined by (5.1). Then, under the assumptions of Theorem 3.1, we have, with $C = C_{\delta,r,\beta,\omega,t_0,T}$ and $[t_0, T] \subset (0, \infty)$,

$$\|U_{N,h}(t) - u(t)\|_{L_\infty} \leqslant \frac{C}{1+\alpha} \big(h^2\ell_h^2 + \ell(\rho_r N) e^{-\mu N} \big) \big(\|u_0\|_{L_\infty} + \|\widehat{f}\|_{L_\infty, \Sigma_\beta^\omega} \big), \quad \text{for } t_0 \leqslant t \leqslant T.$$

Applying the modified time discretization method (3.7) to the spatially discrete problem (4.1), again with $u_{0h} = P_h u_0$, we obtain a different fully discrete solution, namely,

$$U^0_{N,h}(t) = P_h\big(u_0 + F(t)\big) + \frac{k}{2\pi i}\sum_{j=-N}^{N} e^{z_j t} w^0_h(z_j)\, z'_j, \quad \text{with } w^0_h(z) = w_h(z) - z^{-1}P_h g(z). \tag{5.3}$$

Notice that since $w^0_h(z) = P_h\big(w_h(z) - z^{-1}g(z)\big)$, we may write

$$U^0_{N,h}(t) = P_h\bigg(u_0 + F(t) + \frac{k}{2\pi i}\sum_{j=-N}^{N} e^{z_j t}\big(w_h(z_j) - z_j^{-1}g(z_j)\big)\, z'_j\bigg),$$

which avoids computing $P_h g(z_j)$ for each j. Using Theorem 3.2 we now have the following estimate for the error in the discretization in time of the spatially discrete problem (4.1). This estimate may then be combined with Theorem 4.2 to obtain a complete $O(h^2\ell_h^2 + e^{-\sqrt{\bar{r}\gamma N}})$ error estimate.

THEOREM 5.2 Let $u_h(t)$ be the solution of (4.1), let $U_{N,h}(t)$ be defined by (5.3). Then, under the assumptions of Theorem 3.2 we have, with $C = C_{\delta,r,\beta,\omega,T}$,

$$\|U^0_{N,h}(t) - u_h(t)\|_{L_\infty} \leqslant Ce^{-\sqrt{\bar{r}\gamma N}}\big(\ell_h^2\|\Delta\, u_0\|_{L_\infty} + \|\widehat{f}\|_{L_\infty,\Sigma^\omega_\beta,\gamma}\big) \quad \text{for } t \leqslant T.$$

Proof. This follows at once from Theorem 3.2 and the inequality

$$\|A_h P_h v\|_{L_\infty} \leqslant C\ell_h^2\|Av\|_{L_\infty}, \quad \text{for } v \in D(A). \tag{5.4}$$

To show (5.4) we write

$$\|A_h P_h v\|_{L_\infty} \leqslant \|A_h R_h v\|_{L_\infty} + \|A_h P_h(R_h - I)v\|_{L_\infty} = I + II.$$

Since $A_h R_h = P_h A$, we find $I \leqslant C\|Av\|$. For II we note that since $\{\mathcal{T}_h\}$ is quasiuniform, we have the inverse estimate

$$\|A_h\chi\|_{L_\infty} \leqslant Ch^{-2}\|\chi\|_{L_\infty}, \quad \forall \chi \in V_h. \tag{5.5}$$

In fact, for any $\varphi \in L_1$, since P_h is stable in L_1,

$$\begin{aligned}(A_h\chi,\varphi) = (A_h\chi, P_h\varphi) = (\nabla\chi, \nabla P_h\varphi) &\leqslant \sum_{\tau\in\mathcal{T}_h}\|\nabla\chi\|_{L_\infty(\tau)}\|\nabla P_h\phi\|_{L_1(\tau)}\\ &\leqslant Ch^{-2}\|\chi\|_{L_\infty}\,\|P_h\varphi\|_{L_1} \leqslant Ch^{-2}\|\chi\|_{L_\infty}\,\|\varphi\|_{L_1},\end{aligned}$$

which shows (5.5). Thus, using (4.7),

$$II \leqslant Ch^{-2}\|(R_h - I)v\|_{L_\infty} \leqslant Ch^{-2}\ell_h\|(I_h - I)v\|_{L_\infty} \leqslant C\ell_h^2\|Av\|_{L_\infty}.$$

This shows (5.4) and thus the theorem. □

We finally turn to our third method, using (3.9). With $w_h(z,t) = \widehat{\mathcal{E}}^0(z)P_h g(z,t)$, so that the $w_h(z_j,t)$ satisfy the obvious modifications of the elliptic finite element equations (5.2), we define $\widetilde{w}_h(z,t) = w_h(z,t) - z^{-1}P_h g(z,t)$ and put

$$\widetilde{U}_{N,h}(t) := P_h\big(u_0 + F(t)\big) + \frac{k}{2\pi i}\sum_{j=-N}^{N}\widetilde{w}_h(z_j,t)z'_j. \tag{5.6}$$

As with (5.3), since $\widetilde{w}_h(z,t) = P_h\big(w_h(z,t) - z^{-1}g(z,t)\big)$ we can avoid computing $P_h g(z_j,t)$ for each j by moving P_h outside the sum.

For this method we have the following, now based on Theorem 3.3.

THEOREM 5.3 Let $u_h(t)$ be the solution of (4.1), and let $\widetilde{U}_{N,h}(t)$ be defined by (5.6). Then, under the assumptions of Theorem 3.3, we have with $C = C_{\delta,r,\bar{\beta},\omega,T}$,

$$\|\widetilde{U}_{N,h}(t) - u_h(t)\|_{L_\infty} \leqslant Ce^{-\sqrt{\bar{r}\gamma N}}\Big(\|\Delta\, u_0\|_{L_\infty} + \|f(0)\|_{L_\infty} + \int_0^t \|f_t(s)\|_{L_\infty}\, ds\Big) \quad \text{for } t \leqslant T.$$

6. Numerical Examples

We illustrate our results using two numerical examples. In both cases, $A = -\triangle$ on the square $\Omega = (0,4)\times(0,4)$ with homogenous Dirichlet boundary conditions, and we choose $T = 5$. For the initial data we take a simple linear combination of eigenfunctions of A,

$$u_0(x) = \tfrac{1}{2}\sin\big(\tfrac{1}{4}\pi x_1\big)\sin\big(\tfrac{1}{4}\pi x_2\big) + \tfrac{3}{10}\sin\big(\tfrac{2}{4}\pi x_1\big)\sin\big(\tfrac{3}{4}\pi x_2\big). \tag{6.1}$$

Since the spectrum of A is a subset of the positive real half-line, the angle φ in (2.1) may be arbitrarily small, and we choose $\delta = \frac{\pi}{4}\min\big(1,(1-\alpha)/(1+\alpha)\big)$, so that our assumptions are satisfied for $0 < r < \delta$; see Figure 1 and the discussion preceding Theorem 3.1. In our numerical computations we use $r = 0.9\times\delta$. For our first method, we choose $t_0 = 0.5$ and $\theta = 0.1$, and for our second and third methods we set $\gamma = \min(1, 1+\alpha)$. In the experiments described below we use a fixed triangulation of Ω and increase N until the error from the time discretization is neglible in comparison to that from the spatial discretization.

We start with a uniform 60×60 grid and then bisect each square along its north-west to south-east diagonal, thereby triangulating Ω. Since analytic solutions are not available for our example problems, in both cases we compute a high accuracy, reference solution using one of our time discretization methods with a sufficiently large choice of $N = N^*$. In addition, our reference solution employs a finer 240×240 spatial grid, and for each elliptic problem we incorporate one step of Richardson extrapolation (requiring a further elliptic solve on a 480×480 grid), so that the spatial error is $O(h^4)$. This reference solution serves as our "exact" solution for $0 < t \leqslant T$, but at $t = 0$ we use the known initial data (6.1). The errors shown in the tables are for the discrete maximum-norm using the nodes of the 240×240 grid, which means we sample the error at 15 points in each triangle of the original 60×60 grid. To solve the linear systems we use the sparse direct solver for complex matrices provided by UMFPACK, see Davis (2004).

To simplify our implementation, when assembling the load vector for the finite element equations (5.2) we approximate $g(z_j)$ by $I_h g(z_j)$, where I_h is the interpolation projection that takes $v \in \mathcal{C}(\overline{\Omega})$ to the continuous, piecwise-linear function $I_h v$ satisfying $I_h v(x_p) = v(x_p)$ for all vertices x_p in the triangulation (including the vertices on the boundary, so $I_h v \notin V_h$ if $v(x_p) \neq 0$ for any $x_p \in \partial\Omega$). In effect, we compute a numerical solution to a modified continuous problem, with u_0 and $f(t)$ in (1.1) replaced by $I_h u_0$ and $I_h f(t)$. The stability estimate (2.17) for the continuous problem implies that the associated additional error is of order $\|u_0 - I_h u_0\| + \int_0^t \|f(s) - I_h f(s)\|\, ds$, which will be $O(h^2)$ for our choices of u_0 and f.

EXAMPLE 6.1 We begin by revisiting a calculation from our previous paper, McLean and Thomée (to appear), but with the error measured in the maximum-norm instead of the L_2-

TABLE 1 *Errors in* $U_{N,h}(t)$, $U^0_{N,h}(t)$, $\widetilde{U}_{N,h}(t)$ *at* $t = 2.0$ *for a* 60×60 *grid with* $f(x,t) = e^{-t/4}$ *and* $\alpha = -1/2$.

N	method 1	method 2	method 3
5	3.0464e-02	4.7060e-01	1.1820e-01
10	3.4677e-03	3.1097e-02	2.6197e-02
20	3.4677e-03	8.2922e-03	5.2769e-03
30	3.4677e-03	4.6244e-03	3.4677e-03
40	3.4677e-03	3.4677e-03	3.4677e-03

TABLE 2 *Errors in* $\widetilde{U}_{N,h}(t)$ *for* f *given by* (6.2) *and* $\alpha = -1/2$.

N	$t = 0.0$	$t = 1.0$	$t = 3.0$	$t = 5.0$
5	1.6655e-01	8.2316e-02	1.1979e-01	9.3548e-02
10	5.5016e-02	1.7858e-02	2.2820e-02	2.0539e-02
20	1.0004e-02	3.9879e-03	3.7614e-03	3.4283e-03
30	3.5077e-03	3.2388e-03	1.7528e-03	1.3733e-03
40	2.6674e-03	3.2388e-03	1.7296e-03	1.2651e-03
50	2.8163e-03	3.2388e-03	1.7296e-03	1.2651e-03
60	2.8741e-03	3.2388e-03	1.7296e-03	1.2651e-03
80	2.9043e-03	3.2388e-03	1.7296e-03	1.2651e-03

TABLE 3 *Errors in* $\widetilde{U}_{N,h}(t)$ *for* f *given by* (6.2) *and* $\alpha = 0$.

N	$t = 0.0$	$t = 1.0$	$t = 2.0$	$t = 3.0$
5	5.2112e-02	2.1288e-02	2.1385e-02	2.3927e-02
10	7.9343e-03	4.5562e-03	2.6404e-03	2.6243e-03
20	2.0079e-03	3.1085e-03	4.6697e-04	1.6832e-04
30	1.6319e-03	3.1085e-03	3.5982e-04	6.4468e-05
40	1.5893e-03	3.1085e-03	3.5100e-04	5.6079e-05
50	1.5829e-03	3.1085e-03	3.4990e-04	5.4931e-05
60	1.5814e-03	3.1085e-03	3.4971e-04	5.4739e-05
80	1.5810e-03	3.1085e-03	3.4966e-04	5.4694e-05

TABLE 4 *Errors in* $\widetilde{U}_{N,h}(t)$ *for* f *given by* (6.2) *and* $\alpha = 1/2$.

N	$t = 0.0$	$t = 1.0$	$t = 3.0$	$t = 5.0$
5	1.7901e-01	3.2777e-01	7.1592e-02	2.3052e-02
10	3.6276e-02	8.3494e-02	1.9718e-03	3.2560e-03
20	3.6340e-03	1.0699e-02	6.9519e-04	7.5804e-04
30	2.2102e-03	2.7341e-03	6.9519e-04	6.4513e-04
40	2.5698e-03	2.2754e-03	6.9519e-04	6.4710e-04
50	2.6234e-03	2.2754e-03	6.9519e-04	6.5022e-04
60	2.6310e-03	2.2754e-03	6.9519e-04	6.4920e-04
80	2.6336e-03	2.2754e-03	6.9519e-04	6.4935e-04

norm. For $\alpha = -1/2$, we make the very simple choice $f(x,t) = e^{-t/4}$, which has the Laplace transform $\widehat{f}(x,z) = (z + \frac{1}{4})^{-1}$. Note that $f(x,t)$ does not vanish for $x \in \partial\Omega$.

Table 1 shows the errors in the maximum-norm for each method when $t = 2$ (using method 1 with $N^* = 60$ to generate the "exact" solution). We see that for the first method, nothing is gained by choosing N larger than 10, whereas for the second and third methods we should use N in the range from 20 to 30.

Example 6.2 We now choose

$$f(x,t) = \begin{cases} t(2-t)^2, & \text{for } 0 < t < 2, \\ 0, & \text{for } t > 2, \end{cases} \tag{6.2}$$

and find that $\widehat{f}(x,z) = 6z^{-4} - 8z^{-3} + 4z^{-2} - (6z^{-4} + 4z^{-3})e^{-2z}$. Since $\|\widehat{f}\|_{L_\infty, \Sigma^\omega_\beta, \gamma} = \infty$ for all $\gamma \geqslant 0$, we cannot apply the error bounds for our first or second method. (In practice, our code failed for these methods due to floating-point overflow in the evaluation of $\widehat{f}$.) Note that the regularity results of Theorems 2.2 and 2.3 apply because $f(x,0) = 0$ and $\int_0^T \|f_{tt}\|_{L_\infty}\, dt < \infty$.

Tables 2, 3 and 4 show, for $\alpha = -1/2$, 0 and $+1/2$, respectively, the maximum-norm errors for our third method at four different times (using method 3 with $N^* = 120$ to generate the "exact" solution). As before, modest values of N suffice to reduce the quadrature error to a level comensurate with the accuracy of the spatial discretization. Of course, in practice there is no point in reconstructing the initial data, but the errors at $t = 0$ give an indication of the accuracy for small $t > 0$. Observe that the accuracy is best in the middle of the time interval $[0,T] = [0,5]$; cf. our discussion in (McLean and Thomée, to appear, §6.1).

To conclude, we remark that for our methods to be competitive, one needs to develop iterative methods suitable for the elliptic problems occurring. We envisage future work addressing this question; a preliminary study appears in the final section of Sheen, Sloan and Thomée (2003).

A. Proof of the resolvent estimate in maximum-norm for the Laplacian

In this appendix we give a simple direct proof of the resolvent estimate (2.2) in the case when A is the closure of $-\Delta$, under homogeneous Dirichlet boundary conditions, in $\mathcal{B} = L_\infty = L_\infty(\Omega)$. For $\mathcal{B} = \mathcal{C}(\bar{\Omega})$, the corresponding result is contained in the technically rather complicated paper Stewart (1974), for more general elliptic operators. The approach in the present proof is to first show that the semigroup $E(t)$ in L_∞, generated by Δ is, in fact, analytic and bounded in the variable t, in any sector in the right half plane. The proof is based on material from Arendt (2005–2006) and Ouhabaz (1995), as put together in Crouzeix (2008).

Theorem A.1 Let $E(t) = e^{\Delta t}$ be the semigroup in L_∞ generated by Δ. Then $E(t)$ extends to an analytic semigroup for $\operatorname{Re} t > 0$, and for any $\varphi \in (0, \frac{1}{2}\pi)$ we have

$$\|E(t)v\|_{L_\infty} \leqslant C_\varphi \|v\|_{L_\infty}, \quad \text{for } t \in \Sigma_\varphi, \quad \text{with } C_\varphi = \Big(\frac{2}{1 - \sin\varphi}\Big)^{d/2}.$$

Proof. We shall use that $u(t) = E(t)v$ is the solution of the initial-value problem

$$\partial_t u + Au = 0, \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \quad \text{with } u(0) = v. \tag{A.1}$$

With $\{\lambda_j\}_{j=1}^\infty, \{\varphi_j\}_{j=1}^\infty$ the eigenvalues and orthonormal eigenfunctions of $-\Delta$, we have

$$E(t)v = \sum_{j=0}^\infty e^{-\lambda_j t}\,(v,\varphi_j)\,\varphi_j, \quad \text{for } t \geqslant 0, \quad v \in L_2, \tag{A.2}$$

from which we see at once that $E(t)$ is a contraction semigroup in L_2, which extends to an analytic semigroup in the right half-plane $\operatorname{Re} t \geqslant 0$. By Parseval's relation $E(t)$ is an isometry on the imaginary axis. The Sobolev imbedding theorem shows that $\|\varphi_j\|_{L_\infty} \leqslant C\|A^s\varphi_j\|_{L_2} = \lambda_j^s$ for $s > d/2$, so we see that $E(t)v$ is defined by (A.2) also for $v \in L_\infty$ and $\operatorname{Re} t > 0$.

The solution of (A.1) may thus be written in the form

$$(E(t)v)(x) = \int_\Omega k(x,y,t)\,v(y)\,dy, \quad \text{for } x \in \Omega,\ t > 0, \quad k(x,y,t) = \sum_{j=0}^\infty e^{-\lambda_j t}\varphi_j(x)\,\varphi_j(y). \tag{A.3}$$

We remark that $E(t)v \in \mathcal{C}_0(\bar\Omega)$ for $t > 0$, but that $E(t)v$ converges to v in L_∞ as $t \to 0$ only when $v \in \mathcal{C}_0(\bar\Omega)$.

For the pure initial-value problem on $\mathbb{R}^d$, the representation (A.3) holds with $k(x,y,t)$ replaced by the Gaussian kernel $G(x,y,t) = (4\pi t)^{-d/2}e^{-|x-y|^2/(4t)}$, and, since both $k(x,\cdot,\cdot)$ and $G(x,\cdot,\cdot)$ satisfy the heat equation, the maximum-principle on Ω shows

$$0 < k(x,y,t) \leqslant G(x,y,t), \quad \text{for } x,y \in \Omega,\ t > 0.$$

From this we deduce, with $\|A\|_{L_p,L_q}$ denoting the operator norm of $A:\ L_p \to L_q$,

$$\|E(t)\|_{L_2,L_\infty} = \|E(t)\|_{L_1,L_2} \leqslant \|G(\cdot,0,t)\|_{L_2(\mathbb{R}^d)} = (8\pi t)^{-d/4}, \quad \text{for } t > 0. \tag{A.4}$$

Writing $t = \tau + i\sigma$ we have by the semigroup property that $E(t) = E(\frac12\tau)E(i\sigma)E(\frac12\tau)$ for $\tau > 0$ and hence

$$\|E(t)\|_{L_1,L_\infty} \leqslant \|E(\tfrac12\tau)\|_{L_2,L\infty}\|E(i\sigma)\|_{L_2,L_2}\|E(\tfrac12\tau)\|_{L_1,L_2} \leqslant (4\pi \operatorname{Re} t)^{-d/2}.$$

We now note that (A.3) holds for $t \in \Sigma_\varphi$, for any $\varphi \in (0,\frac12\pi)$, and that the kernel $k(\cdot,\cdot,t)$ is analytic there. To show the desired result it therefore suffices to demonstrate that

$$\|k(x,\cdot,t)\|_{L_1} \leqslant C_\varphi, \quad \text{for } x \in \Omega,\ t \in \Sigma_\varphi. \tag{A.5}$$

For this we shall need the following consequence of the Phragmen–Lindelöf theorem.

Lemma A.1 Let $f(z)$ be analytic in Σ_θ, with $\theta \in (0,\frac12\pi)$, and assume

$$|f(z)| \leqslant M, \quad \text{for } z \in \Sigma_\theta \quad \text{and} \quad |f(x)| \leqslant M\,e^{-b/x}, \quad \text{for } x \in \mathbb{R}_+.$$

Then, for $\varphi \in (0,\theta)$ we have

$$|f(z)| \leqslant M\,e^{-\kappa\,b/|z|}, \quad \text{for } z \in \Sigma_\varphi, \quad \text{where } \kappa = \frac{\sin(\theta-\varphi)}{\sin\theta}.$$

Proof. We set $g^+(z) = f(z^{-1})\,\exp(i\,b\,e^{i\theta}\,z/\sin\theta)$. After simple calculations one finds

$$|g^+(z)| \leqslant M \quad \text{for } \arg z = -\theta,\, 0, \quad \text{and } |g^+(z)| \leqslant M\,e^{b|z|/\sin\theta}, \quad \text{for } \arg z \in [-\theta, 0].$$

From the Phragmen–Lindelöf theorem it follows that $|g^+(z)| \leqslant M$ for $\arg z \in [-\theta, 0]$. Hence, for $\arg z \in (0, \varphi)$,

$$|f(z)| = |g^+(z^{-1})\,\exp\big(-i\,b\,e^{i\theta}z^{-1}/\sin\theta\big)| \leqslant M\,\exp\big(-b\,|z|^{-1}\sin(\theta-\omega)/\sin\theta\big) \leqslant M\,e^{-\kappa\,b/|z|}.$$

Using instead $g^-(z) = f(z^{-1})\,\exp(-i\,b\,e^{-i\theta}\,z/\sin\theta)$, the analogous argument shows the result for $\arg z \in (-\varphi, 0)$. □

In the proof of (A.5) we shall use a result of Dunford and Pettis concerning the norm of an operator $K : L_1 \to L_\infty$, defined by a kernel $k(x, y)$, which states that

$$\|K\|_{L_1,L_\infty} = \|k\|_{L_\infty(\Omega\times\Omega)} \quad \text{if} \quad (Kv)(x) = \int_\Omega k(x,y)v(y)\,dy.$$

Using this it follows from (A.4) that

$$\|k(\cdot,\cdot,t)\|_{L_\infty(\bar\Omega\times\bar\Omega)} = \|E(t)\|_{L_1,L_\infty} \leqslant (4\pi\,\mathrm{Re}\,t)^{-d/2}, \quad \text{for } \mathrm{Re}\,t > 0,$$

and it thus follows that

$$|(4\pi t)^{d/2}k(x,y,t)| \leqslant \begin{cases} (\cos\theta)^{-d/2}, & \text{for } t \in \Sigma_\theta,\ 0 < \theta < \frac{1}{2}\pi, \\ e^{-|x-y|^2/(4t)}, & \text{for } t \in \mathbb{R}_+,\ x, y \in \Omega. \end{cases}$$

Applying Lemma A.1 to the function $f(t) = (4\pi t)^{d/2}\,k(x,y,t)$ gives, with $\varphi = \arg t \in [0,\theta]$,

$$|(4\pi t)^{d/2}\,k(x,y,t)| \leqslant (\cos\theta)^{-d/2}\,e^{-\kappa\,|x-y|^2/(4|t|)} \leqslant \Big(\frac{4\pi|t|}{\kappa\cos\theta}\Big)^{d/2} G(x,y,|t|/\kappa),$$

and hence

$$\begin{aligned} \|E(t)\|_{L_\infty,L_\infty} &\leqslant \sup_{x\in\Omega}\|k(x,\cdot,t)\|_{L_1} \\ &\leqslant (4\pi|t|)^{-d/2}\Big(\frac{4\pi|t|}{\kappa\cos\theta}\Big)^{d/2}\int_{\mathbb{R}^d} G(x,y,|t|/\kappa)\,dy = \Big(\frac{\tan\theta}{\sin(\theta-\varphi)}\Big)^{d/2}. \end{aligned}$$

Choosing $\theta = \frac{1}{2}(\frac{1}{2}\pi + \varphi)$ we find easily

$$\frac{\tan\theta}{\sin(\theta-\varphi)} = \frac{2\sin\theta}{1-\sin\varphi} \leqslant \frac{2}{1-\sin\varphi},$$

which completes the proof of the theorem. □

We are now in a position to show the desired resolvent estimate (2.2).

THEOREM A.2 For any $\varphi \in (0, \frac{1}{2}\pi)$ there is a constant C_φ such that

$$\|(zI+\Delta)^{-1}v\|_{L_\infty} \leqslant \frac{C_\varphi}{1+|z|}\,\|v\|_{L_\infty}, \quad \text{for } z \notin \Sigma_\varphi, \quad v \in L_\infty.$$

Proof. For $z = |z|\, e^{i\omega}$ with $\varphi \leqslant \omega \leqslant \frac{1}{2}\pi$ (negative ω may be treated analogously), we have with $\psi = \frac{1}{2}\omega - \frac{1}{2}\pi$, after changing the contour of integration from $\mathbb{R}_+$ to $e^{i\psi}\mathbb{R}_+$,

$$(z\,I + \Delta)^{-1} = -e^{i\psi} \int_0^\infty e^{zte^{i\psi}} E(te^{i\psi})\, dt.$$

In view of Theorem A.1 this shows

$$\|(z\,I + \Delta)^{-1}\|_{L_\infty} \leqslant \Big(\frac{2}{1-\sin\varphi}\Big)^{d/2} \int_0^\infty e^{-\cos(\omega/2)|z|\,t}\, dt \leqslant \frac{1}{|z|\cos(\frac{1}{2}\varphi)} \Big(\frac{2}{1-\sin\varphi}\Big)^{d/2}$$

Since $(-\Delta)^{-1}$ is bounded in L_∞, the resolvent is bounded for small $|z|$, which completes the proof. □

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