The University of New South Wales
School of Mathematics and Statistics
Mathematics Drop–in Centre

BASIC INEQUALITIES

Solving inequalities is broadly similar to solving equations: you can perform algebraic operations on the inequality, always doing the same to both sides, in order to isolate the variable x and hence solve the inequality. However inequalities are more difficult than equations because there are more restrictions on the sort of operations that are valid. The first, however, is very easy:

$$\text{if} \quad x < y \quad \text{then} \quad x + z < y + z \ .$$

That is, you can add anything to both sides of the inequality. The same goes for subtraction. Already with multiplication you have to be more careful:

$$\begin{aligned} &\text{if} \quad x < y \text{ and } z > 0 \quad \text{then} \quad xz < yz \ ; \\ &\text{if} \quad x < y \text{ and } z < 0 \quad \text{then} \quad xz > yz \ . \end{aligned}$$

So, you can multiply both sides by a positive number, but *if you multiply by a negative number, you must reverse the direction of the inequality*. Consequently, you must **never** multiply an inequality by any quantity if you don't know whether it is positive or negative. The same goes for division.

Example. To solve the inequality

$$2x + 3 < 4x - 5$$

we could begin by subtracting $4x$ from both sides, which gives $-2x + 3 < -5$; then subtract 3 from both sides, $-2x < -8$; then divide both sides by -2, remembering to reverse the inequality, to obtain the solution

$$x > 4 \ .$$

An error. Suppose we have to solve

$$x^2 + 2x \geq 3x^2 + 4x \ .$$

We might see that the inequality will be greatly simplified if we divide both sides by x to obtain $x + 2 \geq 3x + 4$. However this is **incorrect**, because we don't know what x is; in particular, we don't know if it is positive or negative; so we don't know whether we should have reversed the direction of the inequality. The easiest way to solve the given problem is to treat it as a polynomial inequality – see another revision worksheet for this.

Inequalities involving **absolute values** can be solved by first eliminating the absolute value. If a is positive, we have

$$|x| < a \quad \Leftrightarrow \quad -a < x < a$$

and

$$|x| > a \quad \Leftrightarrow \quad x < -a \ \text{ or } \ x > a \ .$$

For example, to solve the inequality $|5x - 2| > 1$ we have

$$\begin{aligned} |5x - 2| > 1 \quad &\Leftrightarrow \quad 5x - 2 < -1 \ \text{ or } \ 5x - 2 > 1 \\ &\Leftrightarrow \quad 5x < 1 \ \text{ or } \ 5x > 3 \\ &\Leftrightarrow \quad x < \tfrac{1}{5} \ \text{ or } \ x > \tfrac{3}{5} \ . \end{aligned}$$

Be sure to write the final answer precisely: "$x < \frac{1}{5}$ and $x > \frac{3}{5}$" is different, and is **wrong**. And don't be lazy: if you omit words entirely and write "$x < \frac{1}{5}$, $x > \frac{3}{5}$" then the comma means "and", so this is also **wrong**.

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Solve
   (a) $4x - 11 < -31$;
   (b) $3x + 4 \leq 5x - 2$;
   (c) $x > 6x - 11$;
   (d) $2 - 7x \geq x - 4$;
   (e) $11x - 3 < 5x + 2$;
   (f) $x + 7 \geq 3x + 1$.
2. Solve
   (a) $|x| < 5$;
   (b) $|x| \geq 2$;
   (c) $|4x - 1| < 7$;
   (d) $|5x + 3| \leq 4$;
   (e) $|3x + 8| \geq 1$;
   (f) $|6x - 13| < -5$;
   (g) $|7 - 2x| > 6$.
3. Explain why the following is wrong: "to solve the inequality
   $$\frac{3}{x} \leq \frac{4}{x-5}$$
   we cross–multiply and then solve $3(x - 5) \leq 4x$."

ANSWERS.

1. (a) $x < -5$;
   (b) $x \geq 3$;
   (c) $x < \frac{11}{5}$;
   (d) $x \leq \frac{3}{4}$;
   (e) $x < \frac{5}{6}$;
   (f) $x \leq 3$.
2. (a) $-5 < x < 5$;
   (b) $x \leq -2$ or $x \geq 2$;
   (c) $-\frac{3}{2} < x < 2$;
   (d) $-\frac{7}{5} \leq x \leq \frac{1}{5}$;
   (e) $x \leq -3$ or $x \geq -\frac{7}{3}$;
   (f) no solution (an absolute value cannot be negative!);
   (g) $x < \frac{1}{2}$ or $x > \frac{13}{2}$.
3. In this case to "cross–multiply" means to multiply both sides by $x(x - 5)$, and we don't know if this quantity is positive or negative, so we don't know whether or not we should have reversed the direction of the inequality.