The University of New South Wales
School of Mathematics and Statistics

Mathematics Drop–in Centre

THE BINOMIAL THEOREM

If you need to expand an expression like $(x+y)^5$ you *should not* begin by writing it as

$$(x+y)(x+y)(x+y)(x+y)(x+y)\ .$$

There is a large chance of going wrong if you do it this way, and even if you do get the correct answer it is going to take you a lot of time. Instead you should use the **binomial theorem**. This works as in the following example:

$$(x+y)^5 = 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5\ .$$

The pattern of powers should be easy to understand: we start with x to the highest power (5 in this case) and decrease by 1 at each step; we start with y to the power 0 and increase by 1 at each step. We'll see soon where the coefficients $1, 5, 10, 10, 5, 1$ come from. Of course the above expression can be simplified since $x^0 = 1$ and $x^1 = x$, and the same goes for y; also, there is no need to write 1 times something. Normally we would make these simplifications automatically and we would just write down

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\ . \qquad (*)$$

Using the binomial theorem is all the more important as the power increases, but even for the third power it is better to write

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \qquad (**)$$

than to try to expand $(x+y)(x+y)(x+y)$.

There are many ways to find the coefficients in the above expressions: the easiest is to use *Pascal's triangle*.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

The numbers on the "outside edges" of the triangle are always 1; each number "inside" the triangle is calculated by adding the two numbers above it. For example the 21 in the bottom row comes from $6 + 15$, and the 15 comes from $5 + 10$.

The coefficients for $(x+y)^n$ are found in the row beginning with 1 and n. This is how we got the coefficients $1, 5, 10, 10, 5, 1$ in $(*)$ and $1, 3, 3, 1$ in $(**)$.

We can use the binomial theorem to expand more complicated expressions. For example, replacing x by $2a$ and y by $5b$ gives

$$\begin{aligned}(2a+5b)^3 &= (2a)^3 + 3(2a)^2(5b) + 3(2a)(5b)^2 + (5b)^3\\ &= 8a^3 + 60a^2b + 150ab^2 + 125b^3\ .\end{aligned}$$

Similarly, if we remember that a negative to an even power is positive and a negative to an odd power is negative, we can find

$$\begin{aligned}(3x-y)^5 &= (3x)^5 + 5(3x)^4(-y) + 10(3x)^3(-y)^2\\ &\qquad + 10(3x)^2(-y)^3 + 5(3x)(-y)^4 + (-y)^5\\ &= 243x^5 - 405x^4y + 270x^3y^2 - 90x^2y^3 + 15xy^4 - y^5\ .\end{aligned}$$

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Use Pascal's triangle (on the previous page) to expand
   (a) $(x+y)^4$; (b) $(s+t)^6$; (c) $(a+b)^7$.
2. The binomial theorem works in the same way if x or y is a constant instead of a variable.
   (a) By taking $y=1$, expand $(x+1)^5$.
   (b) Expand $(x-2)^4$.
   (c) Expand $(4+y)^3$.
3. Use the binomial theorem to expand
   (a) $(a+10b)^4$; (b) $(\alpha-2\beta)^5$; (c) $(11x-7y)^2$.
4. More advanced substitutions.
   (a) By taking $y=-4/x^2$, expand $\left(x-\dfrac{4}{x^2}\right)^3$.
   (b) Expand $(x^4+y^5)^2$.
5. Pascal's triangle can be continued to further rows by using the same method of calculation that we have already seen.
   (a) Copy down the last row of Pascal's triangle from the previous page, and calculate the next two rows.
   (b) Expand $(x+y)^9$.

ANSWERS.

1. (a) $x^4+4x^3y+6x^2y^2+4xy^3+y^4$;
   (b) $s^6+6s^5t+15s^4t^2+20s^3t^3+15s^2t^4+6st^5+t^6$;
   (c) $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.
2. (a) $x^5+5x^4+10x^3+10x^2+5x+1$.
   (b) $x^4-8x^3+24x^2-32x+16$.
   (c) $64+48y+12y^2+y^3$.
3. (a) $a^4+40a^3b+600a^2b^2+4000ab^3+10000b^4$;
   (b) $\alpha^5-10\alpha^4\beta+40\alpha^3\beta^2-80\alpha^2\beta^3+80\alpha\beta^4-32\beta^5$;
   (c) $121x^2-154xy+49y^2$.
4. (a) $x^3-12+\dfrac{48}{x^3}-\dfrac{64}{x^6}$.
   (b) $x^8+2x^4y^5+y^{10}$.
5. (a) 1, 8, 28, 56, 70, 56, 28, 8, 1;
   1, 9, 36, 84, 126, 126, 84, 36, 9, 1.
   (b) $x^9+9x^8y+36x^7y^2+84x^6y^3+126x^5y^4+126x^4y^5+84x^3y^6+36x^2y^7+9xy^8+y^9$.