Some modular equations in Ramanujan's notebooks

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Presented at the International Conference

"The Legacy of Srinivasa Ramanujan"

Delhi, December 17–22, 2012

§1 Introduction

We examine two claims of Ramanujan.

(1) If

$$u = \cfrac{q^{\frac{1}{5}}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ \cdots}}}}$$

and

$$v = \cfrac{q^{\frac{2}{5}}}{1+\cfrac{q^2}{1+\cfrac{q^4}{1+\cfrac{q^6}{1+\ \cdots}}}}$$

then

$$\frac{v-u^2}{v+u^2} = uv^2$$

and

$$\text{if } u^5 = n\left(\frac{1-n}{1+n}\right)^2 \text{ then } v^5 = n^2\left(\frac{1+n}{1-n}\right)$$

(2) If

$$u = \cfrac{q^{\frac{1}{5}}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}$$

and

$$v = \cfrac{q^{\frac{3}{5}}}{1+\cfrac{q^3}{1+\cfrac{q^6}{1+\cfrac{q^9}{1+\ \cdots}}}}$$

then

$$(v-u^3)(1+uv^3) = 3u^2v^2.$$

In discussing these claims in 15 minutes, I want you to accept the fact that the continued fraction

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ \cdots}}}} = R(q) = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5\right)_\infty.$$

The notation is defined as follows.

$$(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^{k-1}),$$

$$(a_1, a_2, \ \cdots \ , a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty,$$

$$\left(\begin{matrix} a_1, a_2, \ \cdots \ , a_m \\ b_1, b_2, \ \cdots \ , b_n \end{matrix}; q\right)_\infty = \frac{(a_1, a_2, \ \cdots \ , a_m; q)_\infty}{(b_1, b_2, \ \cdots \ , b_n; q)_\infty}.$$

So, in the context of claim (1),

$$u = q^{\frac{1}{5}} R(q), \quad v = q^{\frac{2}{5}} R(q^2).$$

We need only show that

$$R(q)^2 = \left(\begin{matrix} q, q^4, q^6, q^9 \\ q^2, q^5, q^5, q^8 \end{matrix}; q^{10}\right)_\infty - q\left(\begin{matrix} q, q, q^9, q^9 \\ q^3, q^5, q^5, q^7 \end{matrix}; q^{10}\right)_\infty, \tag{1}$$

$$R(q^2) = \left(\begin{matrix} q, q^4, q^6, q^9 \\ q^2, q^5, q^5, q^8 \end{matrix}; q^{10}\right)_\infty + q\left(\begin{matrix} q, q, q^9, q^9 \\ q^3, q^5, q^5, q^7 \end{matrix}; q^{10}\right)_\infty, \tag{2}$$

For then

$$
\begin{aligned}
\frac{v-u^2}{v+u^2} &= \frac{R(q^2)-R(q)^2}{R(q^2)+R(q)^2} \\
&= \frac{2q\left(\begin{matrix} q,q,q^9,q^9 \\ q^3,q^5,q^5,q^7 \end{matrix};q^{10}\right)_\infty}{2\left(\begin{matrix} q,q^4,q^6,q^9 \\ q^2,q^5,q^5,q^8 \end{matrix};q^{10}\right)_\infty} \\
&= q\left(\begin{matrix} q,q^2,q^8,q^9 \\ q^3,q^4,q^6,q^7 \end{matrix};q^{10}\right)_\infty \\
&= q\left(\begin{matrix} q,q^4,q^6,q^9 \\ q^2,q^3,q^7,q^8 \end{matrix};q^{10}\right)_\infty \left(\begin{matrix} q^2,q^8 \\ q^4,q^6 \end{matrix};q^{10}\right)_\infty^2 \\
&= qR(q)R(q^2)^2 \\
&= uv^2.
\end{aligned}
$$

If we write $uv^2 = n$, then we have

$$\frac{v-u^2}{v+u^2} = n,$$

$$\frac{\frac{v}{u^2}-1}{\frac{v}{u^2}+1} = n,$$

from which we deduce

$$\frac{v}{u^2} = \frac{1+n}{1-n}.$$

From $uv^2 = n$ and $\dfrac{v}{u^2} = \dfrac{1+n}{1-n}$, we deduce

$$u^5 = n\left(\frac{1-n}{1+n}\right)^2, \quad v^5 = n^2\left(\frac{1+n}{1-n}\right).$$

We also deduce

$$v+u^2 = v + \frac{1-n}{1+n}\,v = \left(1+\frac{1-n}{1+n}\right)v = \frac{2v}{1+n},$$

so

$$(v+u^2)(1+uv^2) = 2v,$$

and similarly,

$$(v+u^2)(1-uv^2) = 2u^2,$$
$$(v-u^2)(1+uv^2) = 2uv^3,$$
$$(v-u^2)(1-uv^2) = 2u^3v^3.$$

Ramanujan's claim (1) is correct with the extra condition $n \le \sqrt{5}-2$. Thus

$$\text{if } u^5 = n\left(\frac{1-n}{1+n}\right)^2 \text{ and } n \le \sqrt{5}-2 \text{ then } v^5 = n^2\left(\frac{1+n}{1-n}\right).$$

We now turn to a discussion of Ramanujan's claim (2).
In this context, $u = q^{\frac{1}{5}}R(q)$, $v = q^{\frac{3}{5}}R(q^3)$.

We will show that

$$R(q^3) - R(q)^3 = 3q\left(\begin{matrix} q,q,q^4,q^4,q^6,q^9,q^{11},q^{11},q^{14},q^{14} \\ q^2,q^3,q^5,q^5,q^7,q^8,q^{10},q^{10},q^{12},q^{13} \end{matrix};q^{15}\right)_\infty, \tag{3}$$

$$1+q^2R(q)R(q^3)^3 = \left(\begin{matrix} q^3,q^5,q^5,q^{10},q^{10},q^{12} \\ q^2,q^6,q^7,q^8,q^9,q^{13} \end{matrix};q^{15}\right)_\infty. \tag{4}$$

If we multiply (3) by (4), we readily obtain Ramanujan's claim (2),

$$(v-u^3)(1+uv^3) = 3u^2v^2.$$

§2 Some proofs

The right side of (1) is

$$\left(\begin{matrix} q,q^4,q^6,q^9 \\ q^2,q^5,q^5,q^8 \end{matrix};q^{10}\right)_\infty - q\left(\begin{matrix} q,q,q^9,q^9 \\ q^3,q^5,q^5,q^7 \end{matrix};q^{10}\right)_\infty$$
$$= \left(\begin{matrix} q,q^9 \\ q^2,q^3,q^5,q^5,q^7,q^8,q^{10},q^{10} \end{matrix};q^{10}\right)_\infty$$
$$\times \left\{(q^3,q^4,q^6,q^7,q^{10},q^{10};q^{10})_\infty - q(q,q^2,q^8,q^9,q^{10},q^{10};q^{10})_\infty\right\}$$
$$= \left(\begin{matrix} q,q^9 \\ q^2,q^3,q^5,q^5,q^7,q^8,q^{10},q^{10} \end{matrix};q^{10}\right)_\infty (q,-q^2,-q^3,q^4,q^5,q^5;q^5)_\infty$$

$$= \left(\begin{matrix} q, q, q^4, q^4, q^6, q^6, q^9, q^9 \\ q^2, q^2, q^3, q^3, q^7, q^7, q^8, q^8 \end{matrix} ; q^{10} \right)_\infty$$
$$= R(q)^2,$$

the left side of (1).

The crucial step is the following.

$$(q, -q^2, -q^3, q^4, q^5, q^5; q^5)_\infty = \sum_{r,s=-\infty}^{\infty} (-1)^r q^{(5r^2-3r+5s^2-s)/2}.$$

Split this sum in two, acoording as $r \equiv s \pmod 2$ $(r+s = 2t,\ r-s = 2u,\ rt+u,\ s = t-u)$, or $r \not\equiv s \pmod 2$ $(r+s = 2t+1,\ r-s = 2u+1,\ r = t+u+1, s = t-u)$, and we obtain

$$\begin{aligned}&(q, -q^2, -q^3, q^4, q^5, q^5; q^5)_\infty \\ &= (q^3, q^4, q^6, q^7, q^{10}, q^{10}; q^{10})_\infty - q(q, q^2, q^8, q^9, q^{10}, q^{10}; q^{10})_\infty.\end{aligned}$$

The proof of (2) is similar to that of (1), so I omit it.

Now we prove (3).

From the quintuple product identity (which is a simple corollary of Jacobi's triple product identity), we obtain

$$\left(\begin{matrix} a^{-2}q^4, a^2q \\ a^{-1}q^2, aq^3 \end{matrix} ; q^5 \right)_\infty = \left(\begin{matrix} a^3q^4, a^{-3}q^{11} \\ q^5, q^{10} \end{matrix} ; q^{15} \right)_\infty - a^2q \left(\begin{matrix} a^3q, a^{-3}q^{14} \\ q^5, q^{10} \end{matrix} ; q^{15} \right)_\infty.$$

If we set $a = 1$, we find

$$R(q) = C(q) - qD(q),$$

while if we set $a = 1,\ a = \omega,\ a = \omega^2$ (where $\omega \neq 1$ is a cube root of 1) and multiply the three results, we obtain

$$R(q^3) = C(q)^3 - q^3D(q)^3,$$

where

$$C(q) = \left(\begin{matrix} q^4, q^{11} \\ q^5, q^{10} \end{matrix} ; q^{15} \right)_\infty, \quad D(q) = \left(\begin{matrix} q, q^{14} \\ q^5, q^{10} \end{matrix} ; q^{15} \right)_\infty.$$

It follows that

$$\begin{aligned} R(q^3) - R(q)^3 &= C(q)^3 - q^3 D(q)^3 - (C(q) - qD(q))^3 \\ &= 3qC(q)D(q)\,(C(q) - qD(q)) \\ &= 3qC(q)D(q)R(q), \end{aligned}$$

and since $C(q)$, $D(q)$ and $R(q)$ are all products, it is trivial to show (3) holds.

So now we need only prove (4). Our proof is truly miraculous. I have to say, I extracted it from a paper by Chadwick Gugg.

It is fairly straightforward to show that

$$\begin{aligned} &1 + q^2 R(q) R(q^3)^3 \\ &= \frac{(q^3;q^3)_\infty}{(q^2,q^3;q^5)_\infty (q^6,q^9;q^{15})_\infty (q^{15};q^{15})_\infty^4} \\ &\times \Big((q^2,q^6,q^9,q^{13},q^{15},q^{15};q^{15})_\infty \\ &\qquad\qquad \times (q^6,q^7,q^8,q^9,q^{15},q^{15};q^{15})_\infty \\ &\qquad\qquad + q^2 (q,q^3,q^{12},q^{14},q^{15},q^{15};q^{15})_\infty \\ &\qquad\qquad\qquad \times (q^3,q^4,q^{11},q^{12},q^{15},q^{15};q^{15})_\infty \Big) \end{aligned}$$

The next step is to split each of the four products into two, as we saw before.

If we do that, and then expand the resulting complicated expression, whereas we might have expected a sum of eight (long) products, four miraculously cancel, and we are left with an expression involving four products.

This expression factors as the product of two terms, each of which is the sum of two products.

Each of these sums of two products collapses to a single product, just as we saw in the proof of (1).

Thus we are left with a single product, which is the right side of (4), and we are done!

Hooray!

And Happy Birthday, Ramanujan!

Thank you!