The University of New South Wales
School of Mathematics and Statistics

Mathematics Drop–in Centre

DIFFERENTIATION RULES

Once you know how to differentiate basic functions (see separate worksheet if necessary) you need to be able to combine them into more complicated examples. The fundamental rules are as follows.

sums and differences	$\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$
constant multiple	$\dfrac{d}{dx}(cu) = c\,\dfrac{du}{dx}$
product rule	$\dfrac{d}{dx}(uv) = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$
quotient rule	$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$
chain rule	$\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$

Hopefully the first four of these are easy to understand. Here is an example which uses all four:

$$\begin{aligned}&\frac{d}{dx}\left(x^{23}\sin x - \frac{4\cos x}{x^5}\right)\\ &\qquad = x^{23}\cos x + 23x^{22}\sin x - \frac{x^5(-4\sin x) - (4\cos x)5x^4}{x^{10}}\ .\end{aligned}$$

In writing out the quotient rule we recommend that you write the denominator v^2 first, then begin the numerator with v. This should ensure that you do not get the terms in the numerator the wrong way around.

In the chain rule we are assuming that y is given in terms of u, and u in terms of x. You could find a definite formula for y in terms of x by substitution, then differentiate in the usual way; however it is often easier to use the chain rule. It may be useful to actually define a function u in terms of x. For example, to differentiate $y = \cos(x^2)$ you could write

$$y = \cos u\ , \quad u = x^2$$

and so

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = (-\sin u)(2x) = -2x\sin(x^2)\ .$$

Sometimes we need a longer chain: if $y = \sqrt{1 + e^{\sin x}}$ we can write

$$y = \sqrt{u}\ , \quad u = 1 + e^v\ , \quad v = \sin x\ ,$$

and so we have

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dv}\,\frac{dv}{dx} = \frac{1}{2\sqrt{u}}\,e^v\cos x = \frac{e^{\sin x}\cos x}{2\sqrt{1 + e^{\sin x}}}\ .$$

Here is an example which uses both the product rule and the chain rule:

$$\begin{aligned}\frac{d}{dx}\big((\ln x)\cos(x^2)\big) &= (\ln x)\frac{d}{dx}\big(\cos(x^2)\big) + \cos(x^2)\,\frac{d}{dx}(\ln x)\\ &= -2x(\ln x)\sin(x^2) + \frac{\cos(x^2)}{x}\ .\end{aligned}$$

For the derivative of $\cos(x^2)$ we could work it out separately as above; with practice you should be able to write it down in one go, perhaps making the substitution $u = x^2$ mentally.

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Differentiate
   (a) $x \sin 2x + 3x^4 \cos 5x$;
   (b) $\dfrac{x^7}{1+2e^{5x}}$;
   (c) $e^{-5x} \cos 7x$;
   (d) $\tan^3 x$;
   (e) $\dfrac{\ln x}{x^2}$;
   (f) $x^{-2} \ln x$;
   (g) $\ln(1+\cos^2 x)$;
   (h) $\sec x$;
   (i) $x^3\sqrt{1+x^2}$;
   (j) $(\cos x)\sin(x^2)$;
   (k) $\cos(x \sin(x^2))$.
2. Find a formula for the derivative of uvw by writing it as $u(vw)$ then using the product rule twice. Hence differentiate $x^5(\ln x)(\sin x)$.
3. Differentiate $x^m x^n$ by using the product rule. Then differentiate the same function again by a different (easier!) method, and check that you get the same answer.

ANSWERS.

1. (a) $2x\cos 2x + \sin 2x - 15x^4 \sin 5x + 12x^3 \cos 5x$;
   (b) $\dfrac{(1+2e^{5x})(7x^6) - (10e^{5x})(x^7)}{(1+2e^{5x})^2}$;
   (c) $-7e^{-5x} \sin 7x - 5e^{-5x} \cos 7x$;
   (d) $3 \tan^2 x \sec^2 x$;
   (e) $\dfrac{x - 2x \ln x}{x^4}$;
   (f) $x^{-3} - 2x^{-3} \ln x$; note that this is really the same as (e);
   (g) $-\dfrac{2\cos x \sin x}{1+\cos^2 x}$;
   (h) $(\sin x)/(\cos^2 x)$, or $\sec x \tan x$;
   (i) $\dfrac{x^4}{\sqrt{1+x^2}} + 3x^2\sqrt{1+x^2}$;
   (j) $2x(\cos x)\cos(x^2) - (\sin x)\sin(x^2)$;
   (k) $-\sin(x\sin(x^2))[2x^2\cos(x^2) + \sin(x^2)]$.
2. We have

$$\begin{aligned}
\frac{d}{dx}(u(vw)) &= u\frac{d(vw)}{dx} + (vw)\frac{du}{dx} \\
&= u\left(v\frac{dw}{dx} + w\frac{dv}{dx}\right) + (vw)\frac{du}{dx} \\
&= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}
\end{aligned}$$

   and so the derivative of $x^5(\ln x)(\sin x)$ is

$$x^5(\ln x)(\cos x) + x^4(\sin x) + 5x^4(\ln x)(\sin x) \ .$$

3. The easy way is $\dfrac{d}{dx}(x^m x^n) = \dfrac{d}{dx}(x^{m+n}) = (m+n)x^{m+n-1}$.